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Quantum Scattering of the Bound Pair on the Third Particle in One-Dimensional Case

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Abstract—In this paper, the problem of scattering $2 \to 2(3)$ of three 1D quantum particles with pair short-range attraction potentials is considered in the framework of the diffraction approach. The solution to the scattering problem is constructed in terms of the solution to the model nonhomogeneous boundary-value problem in a circle of a large radius with radiation conditions on the boundary. Possible physical applications of the constructed model are studied.

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INTRODUCTION

The ideas of the diffraction approach to the problem of scattering of three 1D quantum particles were proposed in the research works [1–3] and got further development in publications [4–7]. In the above mentioned works the problem of scattering of three particles with pair repulsion potentials was considered.

The present study is dedicated to the case of pair attraction potentials supporting bound states in each pair. There will be two stages in the construction of the solution to the problem of scattering $2 \to 2(3)$. At the first stage we propose an algorithm to establish a relation between the scattering amplitudes of processes $2 \to 2$ and $2 \to 3$. The relation will be obtained in terms of a certain external object – the patch function, the influence of which will be neutralized at the second stage. At this (second) stage a numerical mechanism of the reconstruction of the complete solution to the scattering problem in the whole configuration space will be proposed. A boundary-value problem for a nonhomogeneous differential equation in second order partial derivatives in a circle of a large radius with a radiation condition on the boundary will be formulated. In the case of the problem of scattering $3 \to 3$ a similar algorithm of the solution was proposed in [4] and realized numerically in [5, 6].

It should be noted that the general algorithm of construction of the solution to the problem of scattering $2 \to 2(3)$ for the case of 3D charged particles was proposed in [8] with respect to reactions connected with accumulation of antiprotons [9–11]. The proposed study can be regarded as the first step necessary for the realization of this algorithm in the simplest situation of 1D particles and finite pair attraction potentials. On the other hand, the proposed model is a completely finished one and has its own value for example for describing scattering of nucleons in parallel beams.

FORMULATION OF THE PROBLEM

We consider a system of three quantum particles, the dynamics of which is described by the Schrödinger operator

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \sum_{i=1}^{3} v_i(x_i). \quad (1)$$

We assume that pair potentials v_i, $i = 1,2,3$ are finite, even, nonpositive and support one bound state. We rely here on the Calogero criterion [12] (and its generalization for potentials set on the axis), defining the number of bound states in the system of two bodies. We assume as well that pair (x, y) – is a pair of Jacobi coordinates corresponding to the system of three bodies. Here $x \in \mathbb{R}$, $y \in \mathbb{R}$. We assume that the particles masses and pair potentials are identical.

We consider the problem of scattering $2 \to 2(3)$ of three particles on the axis, i.e., the coordinate of each particle is characterized by a real number. To be more precise, we consider scattering of a bound pair on the third particle using formalism of the diffraction approach described in detail in the research works [1, 2, 4]. In the framework of formalism the configuration space of the problem is plane Γ, each of three pairs of Jacobi coordinates forms Γ oriented system of coordinates, these systems of coordinates (just as the pairs of

Jacobi coordinates corresponding to two arbitrary different pair subsystems) are connected by the rotation transformation. The complete support of potential (the union of three pair supports of potential) is a combination of three intersecting at one point beams – "screens" l_j with neighborhoods. In the given case of finite pair potentials the carrier of the total potential is the union of three oriented bands in the plane, the width of each band being defined by the carrier of the corresponding pair potential. Each of the screens with index j defines a domain in the configuration space, in which particles in pair j coincide, i.e. equality $x_j = 0$ holds. Thus, along the "screen" with index j the Jacobi coordinate y_j changes, while orthogonally to the screen so does the Jacobi coordinate x_j. The sign of coordinate x_j is defined by the evenness of particles permutation in pair j, while the sign of coordinate y_j is defined by the evenness of particle j permutation and of the center of mass of particles pair k and l. We assume here that the tripod of indices (j, k, l) is formed by the permutation of numbers (1, 2, 3).

We also assume that the asymptotic form of the solution to the Schrödinger equation

$$(H - E)\Psi = 0,$$

satisfying radiation conditions at infinity in the configuration space, is structured as follows

$$\Psi \sim e^{-ip_1y_1}\varphi_1^-(x_1) + \sum_{j=1}^{3}\sum_{\tau_j\in\{+,-\}} a_j^{\tau_j} e^{ip_jy_j}\varphi_j^{\tau_j}(x_j) + A(\hat{\mathbf{X}},\mathbf{P})\frac{e^{i\sqrt{E}X}}{\sqrt{X}}. \quad (2)$$

Here

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \quad \mathbf{P} = \begin{pmatrix} k \\ p \end{pmatrix} \in \mathbb{R}^2,$$
$$X = \sqrt{x^2 + y^2}, \quad \hat{\mathbf{X}} = \frac{\mathbf{X}}{X}.$$

We are using notations $k \in \mathbb{R}$ and $p \in \mathbb{R}$ for momenta Fourier conjugate to the Jacobi coordinates x and y.

We assume that in the initial state the particles of pair $j = 1$ are in bound state φ_1^- with energy $\kappa_j < 0$. Functions $\varphi_j^{\tau_j}$ satisfy normalization condition

$$\int_{\mathbb{R}} |\varphi_j^{\tau_j}(x)|^2 dx = 1, \quad j = 1,2,3, \quad \tau_j \in \{+,-\}. \quad (3)$$

It should be noted that the first term in expression (2) corresponds to the incident wave. We can emphasize that the following relations are valid

$$y_1 < 0, \quad p_1 > 0.$$

The second term in expression (2) corresponds to the superposition of diverging waves (processes $2 \to 2$) with amplitudes $a_j^{\tau_j}$. Here index $j = 1, 2, 3$ denotes the number of pair subsystem, while index $\tau \in \{+, -\}$ defines evenness of the coordinate permutation of particle j and of the center of mass of the subsystem with index j. In other words, index τ_j corresponds to the sign of Jacobi coordinate y_j and thus defines "semi-screen" $l_j^{\tau_j}$. For diverging waves the following relations are observed:

$$y_j > 0, \quad p_j > 0 \quad \text{or} \quad y_j < 0, \quad p_j < 0.$$

In should be noted that each diverging wave with amplitude $a_j^{\tau_j}$ is defined only on the semi-screen, which corresponds to index τ_j. On the semi-screen with index $-\tau_j$ it continues by zero.

Finally, the third term in expression (2) corresponds to the process of breakup $2 \to 3$ and describes a diverging wave with amplitude $A(\hat{\mathbf{X}}, \mathbf{P})$. Now we seek to construct a set of equations relating amplitudes $a_j^{\tau_j}$ of scattering processes $2 \to 2$ and amplitude $A(\hat{\mathbf{X}}, \mathbf{P})$ of process $2 \to 3$. It should be emphasized that we know the solution to the Schrödinger equation only in the asymptotic domain of the configuration space for $X \gg 1$. We introduce a patch function, which will cut o the solution for bounded and small values of X. Multiplying the exact solution to the problem of scattering (or its certain part) by such a patch function, we obtain a new function which will remain an exact solution to the Schrödinger equation for large X, while for bounded and small values of X it, although not being an exact solution to the Schrödinger equation anymore, will generate a known discrepancy different from zero in the bounded domain of the configuration space.

This fact allows us to realize the following algorithm of the solution to the problem of scattering. At the first stage we use the Green formula in a plane and establish, even if in terms of a certain patch function, a relation between the scattering amplitudes of processes $2 \to 2$ and $2 \to 3$. It should be emphasized that we single out the part of the solution corresponding to the singular part of the scattering matrix. In other words, we relate cluster solutions to the problem of scattering (corresponding to processes $2 \to 2$) with a diverging circular wave (corresponding to processes $2 \to 3$). At the second stage of the solution to the scattering problem we construct a nonhomogeneous boundary-value problem for the part of the solution, which is a compliment for the full set of cluster solutions, leading up to the complete solution of the scattering problem. The asymptotic form of this unknown part of the solution behaves as a diverging circular wave with a smooth amplitude and satisfies the radiation conditions at infinity. In its turn, cluster waves "cut o" by limited and small values of X will generate nonhomogeneity of the boundary-value problem, which in all cluster solutions with exception of incident wave turns out to be connected to the amplitude of a diverging wave. Now let us realize the algorithm described above.

CONSTRUCTION OF EQUATIONS CONNECTING THE AMPLITUDES OF SCATTERING PROCESSES 2 → 2 AND 2 → 3

We introduce radial patch function $\zeta(X) \in C^2_{[0,\infty)}$ as follows

$$\zeta(X) = \begin{cases} 0, \quad X < R_1, \quad R_1 \gg 1, \\ \text{increases from 0 to 1}, \quad R_1 < X < R_2, \\ 1, \quad X > R_2. \end{cases} \qquad (4)$$

We also introduce notation

$$\psi_j^{\tau_j}(\mathbf{X}) = e^{ip_j y_j}\varphi_j^{\tau_j}(x_j);$$

for the scattered wave corresponding to process 2 → 2, and notation

$$\tilde{\psi}_j^{\tau_j}(\mathbf{X}) \equiv \psi_j^{\tau_j}(\mathbf{X})\zeta(X),$$

for the scattered wave multiplied by patch function.

Now we act by operator $H - E$ upon Hermite conjugate exact solution Ψ^* and upon function $\tilde{\psi}_j^{\tau_j}$:

$$\begin{cases} (H - E)\Psi^* = 0, \\ (H - E)\tilde{\psi}_j^{\tau_j} = -Q_j^{\tau_j}. \end{cases} \qquad (5)$$

Here notation $Q_j^{\tau_j}$ is used for the discrepancy of function $\tilde{\psi}_j^{\tau_j}$ in the Schrödinger equation. We multiply the first of the equations of system (5) by function $\tilde{\psi}_j^{\tau_j}$, multiply the second equation in system (5) by function Ψ^*, and subtract the second equation from the first one and integrate the result in circle B_R of radius $R > R_2$. Having used the Green formula, we arrive at the following relation

$$\int_{\partial B_R} \left(\frac{\partial \tilde{\psi}_j^{\tau_j}}{\partial n}\Psi^* - \frac{\partial \Psi^*}{\partial n}\tilde{\psi}_j^{\tau_j} \right) dl = \int_{B_R} Q_j^{\tau_j}\Psi^*. \qquad (6)$$

It should be noted that the set of indices $\{j, \tau_j\}$ describes one of six semi-screens, on each of them equation (6) being realized. We emphasize that the semi-screen corresponding to the incident wave is the selected one. Let us consider these two cases separately.

(1) First we consider the case: $\{j, \tau_j\} \neq \{1, -\}$.

Relation (6) with account for explicit form of asymptotics (2) and of normalization conditions (3) takes form

$$\begin{aligned} &\int_{d_j^{\tau_j}} A^*(\theta, \mathbf{P}) \frac{e^{-i\sqrt{E}R}}{\sqrt{R}} ip_j e^{iRp_j\cos(\theta-\theta_j^{\tau_j})}\varphi_j^{\tau_j}(R\sin(\theta - \theta_j^{\tau_j}))Rd\theta \\ &+ \int_{d_j^{\tau_j}} A^*(\theta, \mathbf{P}) \frac{e^{-i\sqrt{E}R}}{\sqrt{R}} i\sqrt{E}e^{iRp_j\cos(\theta-\theta_j^{\tau_j})} \\ &\times \varphi_j^{\tau_j}(R\sin(\theta - \theta_j^{\tau_j}))Rd\theta + 2ip_j a_j^{\tau_j*} \\ &= \int_{B_R} Q_j^{\tau_j}(\mathbf{X}) \left(a_j^{\tau_j*} e^{-ip_j y_j}\varphi_j^{\tau_j}(x_j) + A^*(\hat{\mathbf{X}}, P)\frac{e^{-i\sqrt{E}X}}{\sqrt{X}} \right) d\mathbf{X}. \end{aligned} \qquad (7)$$

Notation $d_j^{\tau_j}$ is introduced for a narrow arc of the circle of radius R in the neighborhood of intersection of the circle and semi-screen $l_j^{\tau_j}$. At the ends of arc $d_j^{\tau_j}$ function $\varphi_j^{\tau_j}$ vanishes.

It should be noted that discrepancy $Q_j^{\tau_j}$ differs from zero in the neighborhood of semi-screen $l_j^{\tau_j}$, in which patch function $\zeta(X)$ changes from $\zeta = 0$ to $\zeta = 1$, and precisely, $R_1 < X < R_2$. The width of the neighborhood is defined by the width of the carrier of pair potential v_j. Thus $Q_j^{\tau_j} \sim O\left(\frac{1}{R_2 - R_1}\right)$, while the area of integration domain in the integral in the right-hand side of equation (7) also has order $R_2 - R_1$. The integral value in the left-hand side of equation (7) is defined by the method of stationary phase.

Finally,

$$\begin{aligned} &i\sqrt{\frac{2\pi}{|p_j|}}A^*(\theta_j^{\tau_j}, \mathbf{P})(p_j + \sqrt{E})e^{iR(p_j-\sqrt{E})}\varphi_j^{\tau_j}(0) \\ &+ 2ip_j a_j^{\tau_j*} = a_j^{\tau_j*}\int_{B_R} Q_j^{\tau_j}(\mathbf{X})e^{-ip_y y_j}\varphi_j^{\tau_j}(x_j)d\mathbf{X}. \end{aligned} \qquad (8)$$

Angular variable θ defining the point at the boundary of circle B_R, is changed in the interval $[0, 2\pi)$. Stationary points $\theta_j^{\tau_j}$, from which the main contribution to the integrals along the boundary of circle B_R comes, coincide with six values defining the intersections of semi-screens $l_j^{\tau_j}$ with circumference ∂B_R. We have also taken into account that in the sense of the above analysis the second term under the integral in the right-hand side of equation (7) has the next order of smallness compared to the main contribution.

A relation between amplitudes $a_j^{\tau_j}$ of scattering processes 2 → 2 and amplitude $A(\hat{\mathbf{X}}, \mathbf{P})$ of scattering process 2 → 3 in accordance with equation (8) takes form

$$a_j^{\tau_j *} = A^*(\theta_j^{\tau_j}, \mathbf{P}) \frac{i\sqrt{\frac{2\pi}{|p_j|}}(p_j + \sqrt{E})e^{iR(p_j - \sqrt{E})}\varphi_j^{\tau_j}(0)}{\int\limits_{B_R} Q_j^{\tau_j}(\mathbf{X})e^{-ip_jy_j}\varphi_j^{\tau_j}(x_j)d\mathbf{X} - 2ip_j}. \quad (9)$$

(2) Now let us consider the case: $\{j, \tau_j\} = \{1, -\}$.

Repeating the computations described in the previous section, it is not too di cult to see that constraint equation (9) will be modified. And precisely, relation (6) with account for the explicit asymptotic form (2) and normalization condition (3) leads in this case to the following equation

$$\begin{aligned} &\int\limits_{d_1^-} A^*(\theta, \mathbf{P})\frac{e^{-i\sqrt{E}R}}{\sqrt{R}} ip_1 e^{iRp_1\cos(\theta - \theta_1^-)} \\ &\quad\times \varphi_1^-(R\sin(\theta - \theta_1^-))Rd\theta \\ &+ \int\limits_{d_1^-} A^*(\theta, \mathbf{P})\frac{e^{-i\sqrt{E}R}}{\sqrt{R}} i\sqrt{E}e^{iRp_1\cos(\theta - \theta_1^-)} \\ &\quad\times \varphi_1^-(R\sin(\theta - \theta_1^-))Rd\theta + 2ip_1 a_1^{-*} \\ &= \int\limits_{B_R} Q_1^-(\mathbf{X})\Bigg(a_1^{-*}e^{-ip_1y_1}\varphi_1^-(x_1) + e^{ip_1y_1}\varphi_1^-(x_1) \\ &\quad + A^*(\hat{\mathbf{X}}, \mathbf{P})\frac{e^{-i\sqrt{E}X}}{\sqrt{X}}\Bigg)d\mathbf{X}. \end{aligned} \quad (10)$$

Now in the integral term in the right-hand side of equation an additional term has appeared, corresponding to the incident wave. Repeating the computations held in case $\{j, \tau_j\} \neq \{1, -\}$ and taking into account the additional term, we obtain the final expression

$$\begin{aligned} a_1^{-*} &= A^*(\theta_1^-, \mathbf{P}) \frac{i\sqrt{\frac{2\pi}{|p_1|}}(p_1 + \sqrt{E})e^{iR(p_1 - \sqrt{E})}\varphi_1^-(0)}{\int\limits_{B_R} Q_1^-(\mathbf{X})e^{-ip_1y_1}\varphi_1^-(x_1)d\mathbf{X} - 2ip_1} \\ &\quad - \frac{\int\limits_{B_R} Q_1^-(\mathbf{X})e^{ip_1y_1}\varphi_1^-(x_1)d\mathbf{X}}{\int\limits_{B_R} Q_1^-(\mathbf{X})e^{-ip_1y_1}\varphi_1^-(x_1)d\mathbf{X} - 2ip_1}. \end{aligned} \quad (11)$$

It should be noted that the contribution of the second term turns out to be negligibly small due to estimation

$$\begin{aligned} &\left|\int\limits_{B_R} Q_1^-(\mathbf{X})e^{ip_1y_1}\varphi_1^-(x_1)d\mathbf{X}\right| \\ &\leq \frac{C}{R_2 - R_1}\left|\int\limits_{R_1}^{R_2} e^{2ip_1y}dy\right| = O\left(\frac{1}{R_2 - R_1}\right). \end{aligned}$$

Thus with an accuracy to the values of the next order of smallness, the following constraint equation holds

$$a_1^{-*} = A^*(\theta_1^-, \mathbf{P}) \frac{i\sqrt{\frac{2\pi}{|p_1|}}(p_1 + \sqrt{E})e^{iR(p_1 - \sqrt{E})}\varphi_1^-(0)}{\int\limits_{B_R} Q_1^-(\mathbf{X})e^{-ip_1y_1}\varphi_1^-(x_1)d\mathbf{X} - 2ip_1}. \quad (12)$$

The obtained constraint equations (9) and (12) turn out to be sufficient for constructing the boundary-value problem to completely solve the problem of scattering.

CONSTRUCTION OF THE BOUNDARY-VALUE PROBLEM FOR A DESCRIPTION OF THE COMPLETE SOLUTION

Now let us construct the boundary-value problem for complete solution Ψ of scattering problem $2 \to 2(3)$, based on the obtained above system of bonds (9) between scattering amplitudes $a_j^{\tau_j}$ of scattering processes $2 \to 2$ and amplitude $A(\hat{\mathbf{X}}, \mathbf{P})$ of scattering process $2 \to 3$.

Let us write the complete solution of scattering problem Ψ as follows

$$\Psi(\mathbf{X}, P) = \chi(\mathbf{X}, P) + \Phi(\mathbf{X}, P), \quad (13)$$

where the first term in expression (13) contains an incident wave and a set of scattered waves corresponding to scattering processes $2 \to 2$, cut o by function $\zeta(X)$ (4) on exterior of a circle of large radius with the center in the origin of coordinates

$$\begin{aligned} \chi(\mathbf{X}, \mathbf{P}) &\equiv \Bigg[e^{-ip_1y_1}\varphi_1^-(x_1) \\ &\quad + \sum_{j=1}^{3}\sum_{\tau_j \in \{+,-\}} a_j^{\tau_j}e^{ip_jy_j}\varphi_j^{\tau_j}(x_j)\Bigg]\zeta(X). \end{aligned}$$

Unknown function $\Phi(\mathbf{X}, \mathbf{P})$ is a complement for the first term $\xi(\mathbf{X}, \mathbf{P})$ to the complete solution. An analogous approach was applied, for example, in [5]. We will follow similar ideas. According to expression (2), the asymptotic form of function $\Phi(\mathbf{X}, \mathbf{P})$ for $X \to \infty$ takes form

$$\Phi(\mathbf{X}, \mathbf{P}) \sim A(\hat{\mathbf{X}}, \mathbf{P})\frac{e^{i\sqrt{E}X}}{\sqrt{X}}, \quad (14)$$

where $A(\hat{\mathbf{X}}, \mathbf{P})$ is a smooth function on the circumference.

Since function Ψ is an exact solution to the Schrödinger equation, for function $\Phi(\mathbf{X}, \mathbf{P})$ we obtain equation

$$\begin{aligned} (H - E)\Phi(\mathbf{X}, \mathbf{P}) &= -S(\mathbf{X}, \mathbf{P}), \\ S(\mathbf{X}, \mathbf{P}) &= (H - E)\chi(\mathbf{X}, \mathbf{P}). \end{aligned} \quad (15)$$

It should be noted that the right-hand side of equation (15) $S(\mathbf{X}, \mathbf{P})$ is defined with an accuracy to six coefficients $a_j^{\tau_j}$. In its turn, the coefficients are determined in accordance with (9) and (14) in terms of determined values of function $\Phi(\mathbf{X}, \mathbf{P})$ on the circumference of radius R. It should also be noted that equation (15) on function Φ is nonhomogeneous, since the incident wave does not depend on coefficients $a_j^{\tau_j}$ and thus does not depend on Φ.

Now we consider nonhomogeneous equation (15) in a circle of large radius R in plane Γ with boundary conditions of form

$$\left(\frac{\partial \Phi}{\partial n} - i\sqrt{E}\Phi\right)\bigg|_{X=R} = O(R^{-3/2}). \qquad (16)$$

The solution to the formulated boundary-value problem in summation with function χ gives, according to (13), a complete solution to the initial problem of scattering.

CONCLUSIONS

The constructed model can be considered as a realization of the method proposed in the research work [8] for a description of the mechanism of accumulation of antiprotons [13, 14] in a simpler situation of 1D particles and short range pair potentials. Being a necessary step for the realization of a complete situation described in [8], the proposed model has a value of its own. In the framework of the proposed model one can consider for example a problem of scattering in parallel beams [15–17]. In the situation when the angle of scattering of break-up products turns out to be small, scattering of a two-body cluster on the third particle in the main order is determined precisely by the above described algorithm.

It should be also noted that the proposed algorithm of solution to the scattering problem of three particles on the axis offers scope for analytical (and numerical) research of the problem of one-dimensional scattering $3 \to 2(3)$ with pair potentials of attraction, developing results of the study [6].

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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