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Symmetry Approach in the Problem of Gas Expansion into Vacuum

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Abstract—A brief review of the results on the expansion of quantum and classical gases into vacuum based on the use of symmetries is presented. For quantum gases in the Gross–Pitaevskii (GP) approximation, additional symmetries arise for gases with a chemical potential μ that depends on the density n powerfully with exponent $\nu = 2/D$, where D is the space dimension. For gas condensates of Bose atoms at temperatures $T \to 0$, this symmetry arises for two-dimensional systems. For $D = 3$ and, accordingly, $\nu = 2/3$, this situation is realized for an interacting Fermi gas at low temperatures in the so-called unitary limit (see, for example, L. P. Pitaevskii, Phys. Usp. **51**, 603 (2008)). The same symmetry for classical gases in three-dimensional geometry arises for gases with the adiabatic exponent $\gamma = 5/3$. Both of these facts were discovered in 1970 independently by Talanov [V. I. Talanov, JETP Lett. **11**, 199 (1970).] for a two-dimensional nonlinear Schrödinger (NLS equation, which coincides with the Gross–Pitaevskii equation), describing stationary self-focusing of light in media with Kerr nonlinearity, and for classical gases, by Anisimov and Lysikov [S. I. Anisimov and Yu. I. Lysikov, J. Appl. Math. Mech. **34**, 882 (1970)]. In the quasiclassical limit, these GP equations coincide with the equations of the hydrodynamics of an ideal gas with the adiabatic exponent $\gamma = 1 + 2/D$. Self-similar solutions in this approximation describe the angular deformations of the gas cloud against the background of an expanding gas by means of Ermakov-type equations. Such changes in the shape of an expanding cloud are observed in numerous experiments both during the expansion of gas after exposure to powerful laser radiation, for example, on metal, and during the expansion of quantum gases into vacuum.

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1. INTRODUCTION: BACKGROUND

Symmetries in physics have always played a key role in obtaining exact relationships and the results based on them. It hardly makes sense to list many different physical examples where symmetries are used. It seems enough to us to refer to the Landau and Lifshitz, *Course of Theoretical Physics* [4], where the numerous symmetries are widely used.

In this brief review, we will consider how symmetries can be applied to the problem of expansion into vacuum of quantum and classical gases within the framework of the Gross–Pitaevskii equations and gas dynamics equations, namely, the continuity equation and the Euler equation for monatomic gases with the adiabatic exponent $\gamma = 5/3$.

For the quantum gases there will be considered the case when the chemical potential μ depends on the density in a power-law fashion with the exponent $\nu = 2/D$ where D is the space dimension. Only for these values of the exponent ν an additional symmetry arises in the problem. Note that in the GP approximation at a temperature $T \to 0$ for the condensate of a weakly non-ideal Bose gas the main contribution to the interaction between atoms is s-wave scattering.

For a positive value of the scattering length a_s the interaction between the Bose atoms corresponds to repulsion and such a condensate is stable in magneto-optical traps. In this case, the chemical potential is $\mu = gn$, where $g = 4\pi\hbar^2 a_s/m$. Thus, an additional symmetry which we consider in the review arises only for the 2D Bose gas. For a negative scattering length, an attraction arises between Bose atoms. In nonlinear optics, this kind of attraction leads to the self-focusing of light for media with Kerr nonlinearity. Repulsion, in turn, leads to defocusing of the light beam, which works in the same transverse direction as diffraction to the beam. In the case $a_s < 0$, Bose condensates turn out to be unstable, which leads to the formation of contracting gas regions, i.e. collapse (see [5] and references there) observed in experiment [6, 7]. To obtain an unstable value of the scattering length in experiments, the Feshbach resonance is used [8, 9], allowing to change a_s within a wide range: from very large positive values to strongly negative values. If for Bose atoms with negative a_s a collapse occurs, then for Fermi gases s-attraction provides the formation of Cooper pairs, which at $T \to 0$ form a superfluid Bose condensate. By varying the wave scattering length using the Feshbach resonance, it is possible to create a Bose condensate in the so-called unitary limit [1], which is realized under the condition $(|a_s|k_F)^{-1} \to 0$, where $p_F = \hbar k_F$ is the Fermi momentum. In this regime, the positive chemical potential is given by $\mu(n) = 2(1 + \beta)\varepsilon_F$, where according to [10–13] $\beta = -0.63$ is a universal constant, and

$$\varepsilon_F = \frac{\hbar^2}{2m}(6\pi^2 n)^{2/3}$$

is the local Fermi energy while m is twice the mass of the Fermi atom. In the unitary limit, the chemical potential thus has a power-law dependence on density with exponent $\nu = 2/3$.

The simplest symmetry in the case of a power-law dependence of the chemical potential on density is the symmetry with respect to the dilatation of both spatial coordinates and time of the form:

$$\mathbf{r} \to \alpha\mathbf{r}, \quad t \to \alpha^2 t, \tag{1}$$

where α is a scaling parameter.

This fact can be easily verified based on the Gross–Pitaevskii equation [14] for the wave function of the Bose condensate ψ:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \mu(n)\psi, \tag{2}$$

where $n = |\psi|^2$. The conservation of the number of particles $N = \int|\psi|^2 d\mathbf{r}$ and the power dependence $\mu \propto n^{2/D}$ provide the same dependencies on the scaling parameter α for both the kinetic term and the chemical potential (defining nonlinearity in (2)): $\propto\alpha^{-2}$.

In nonlinear optics and plasma physics, the GP Eq. (2) is usually called the nonlinear Schrödinger equation (NLS). The standard form of the NLS equation results from Eq. (2) rewritten in dimensionless variables:

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\Delta\psi - (\nu+1)|\psi|^{2\nu}\psi = 0, \tag{3}$$

which can be represented in the Hamiltonian form (see, e.g. [5])

$$i\frac{\partial\psi}{\partial t} = \frac{\delta H}{\delta\psi^*}$$

with the Hamiltonian

$$H = \int\left[\frac{1}{2}|\nabla\psi|^2 + |\psi|^{2(\nu+1)}\right]d\mathbf{r}. \tag{4}$$

It should be noted that the scaling symmetry (1) is also in the Schrödinger equation for the motion of a quantum mechanical particle in the potential $U = \beta r^{-2}$, independently on the sign of β, for any dimension D. The same symmetry is also present for the hydrodynamics of an ideal gas with the adiabatic exponent $\gamma = 5/3$ for potential three-dimensional flows [3], in the case of two-dimensional flows—with $\gamma = 2$. The latter, in particular, follows from the fact that the GP Eq. (2) in the quasiclassical limit (in the Thomas–Fermi approximation) coincides with the hydrodynamic equations for potential gas flows (see [15, 16]) and therefore this symmetry is preserved in the case of gas dynamics with $\gamma = 1 + 2/D$. This circumstance was first drawn to the attention of the authors of the paper [3] by Dzyaloshinskii [17].

However, the scaling transformation does not exhaust all symmetries of Eqs. (2) and (3). More general is the symmetry with respect to the Talanov transforms [2]—transformations of the conformal type, which include both amplitude scaling transformations and phase changes of the wave function ψ. This transformation was found for the two-dimensional nonlinear Schrödinger equation (NLS) describing stationary self-focusing of light in a medium with Kerr nonlinearity, in which the role of time in (3) is played by the coordinate z along the direction of propagation of the light beam.

Under the Talanov transformations of the general form (for all $\nu = 2/D$), Eq. (3) remains invariant under the change of the wave function ψ, coordinates $\mathbf{r}$ and time t to the new wave function $\tilde{\psi}$ and new coordinates $\mathbf{r}'$ and time t' [18]:

$$\begin{gathered}\psi(\mathbf{r},t) = [\tau/(\tau+t)]\exp\left[\frac{ir^2}{4(\tau+t)}\right]\tilde{\psi}(\mathbf{r}',t'),\\ \mathbf{r}' = \mathbf{r}\tau/(\tau+t), \quad t' = t\tau/(\tau+t).\end{gathered} \tag{5}$$

In linear optics, these relationships are known as lens transforms.

It is important to note that the superposition of transforms with $\lambda_1 = \tau_1^{-1}$ and $\lambda_2 = \tau_2^{-1}$ represents a transform (5) with $\lambda_3 = \lambda_1 + \lambda_2$. Thus, the transformations (5) form the abelian group [18].

A direct consequence of this symmetry is the virial theorem obtained by Vlasov, Petrishchev, Talanov [19] for the two-dimensional nonlinear Schrödinger equation (NLS) with cubic nonlinearity:

$$\frac{d^2}{dt^2}\int r^2|\psi|^2 d\mathbf{r} = 4H. \tag{6}$$

This theorem was first established in [19] for focusing nonlinearity. It is easy to see that (6) is also true in the case of repulsion (defocusing nonlinearity) [15, 16]. It is important that the virial theorem (6) is true for any value of $\nu = 2/D$ [18, 20]. Note that from the classification point of view, the NLS with $\nu = 2/D$ belongs to the so-called critical models (see, for example, [5, 18, 20]).

Simple integration of (6) gives that the mean square of the cloud size $R^2 = \int r^2|\psi|^2 d\mathbf{r}/N$ changes in time quadratically:

$$NR^2 = 2Ht^2 + C_1 t + C_2. \tag{7}$$

In the case of repulsion (defocusing nonlinearity), the Hamiltonian H is positive. Therefore, for large times, $t \to \infty$, the average size of R grows linearly with time. The relation (7) contains two constants C_1 and C_2, which are new in comparison with H and N integrals of motion. However, they differ from the Hamiltonian and the number of particles by the presence of an explicit dependence on the time t:

$$\begin{aligned} C_1 &= \frac{d}{dt}\int r^2|\psi|^2 d\mathbf{r} - 4Ht, \\ C_2 &= \int r^2|\psi|^2 d\mathbf{r} - 2Ht^2 - C_1 t. \end{aligned} \tag{8}$$

Integrals of this kind are non-autonomous, which, as will be seen below, does not allow one to establish complete integrability in the case of self-similar reduction of the quasiclassical equations. An example of such nonautonomous integrals of motion is the law of conservation of the center of mass, which explicitly contains the time t.

In the case of gas dynamics, to our best knowledge, this symmetry was first found by Ovsyannikov [21]. It was effectively used by Anisimov and Lysikov in [3] to construct an exact axisymmetric self-similar solution describing nonlinear angular deformations of a gas cloud against the background of its expansion into vacuum. Subsequently, it was found that such deformations are observed in various physical systems, e.g. for the action of the powerful laser radiation on the solid substance. As a result its original shape in the form of a disk is converted into a sigar-shape against the background of the expanding gas (see the monograph [22] and references therein).

For quantum gases expanding into vacuum, such transformations are also typical for both Bose gases and Fermi gases (see, respectively, [23–25] and references therein). It should be noted that the first scale-invariant time-dependent solutions for Bose condensates in the hydrodynamic regime for an anisotropic trap were found by Kagan, Surkov and Shlyapnikov [26]; in particular, they found a spectrum of oscillating breathing modes. Later, self-similar regimes were observed in experiments by Thomas' group [24, 25] with anisotropic expansion from an optical trap into vacuum of a strongly interacting degenerate Fermi gas of atoms ^{6}Li.

We should note that in nineteen sixties this problem, namely the problem of the gas expansion into vacuum was a very popular one. The first classical results in this field were obtained in 1956 by Ovsyannikov [21] and in 1968 by Dyson [27]. These works had a lot of different applications not only in hydrodynamics but in astrophysics as well (see e.g. the original paper by Zel'dovich [28]).

In this brief review, we mainly restrict ourselves to considering the quasiclassical expansion of quantum gases into vacuum, which coincides with the expansion of an ideal gas with the adiabatic exponent $\gamma = 1 + 2/D$. It will be shown, using a self-similar solution, how the form of an expanding cloud evolves for quantum gases in the quasiclassical limit and, accordingly, for the expansion of an ideal gas. The dynamics of self-similar expansion in this case is described within the framework of a system of ordinary differential equations of the Ermakov type. It should be noted that the considered symmetry was first used by Ermakov in 1880 [29] when constructing solutions for a number of mechanical systems, including the motion of a classical particle in a potential representing the sum of the oscillatory potential and $V(r) = \beta/r^2$.

Note that the same symmetry is helpful also in finding the spectrum in the quantum case for the system of N particles moving in a plane and interacting with each other with the potential $V(r_{ij}) = \beta/r_{ij}^2$ (see [30] and also [31]). It is worth noting also that in nighteen seventies the results of Ermakov were rediscovered by Ray and Reid [32]. Nowadays this type of equations are usually called as Ermakov–Ray–Reid systems of equations (see e.g. [33] and references therein).

In the conclusion, we will discuss the difference between the expansion of a quantum gas and a classical one, as well as experimental data on the expansion of quantum gases into vacuum.

2. BASIC EQUATIONS AND QUASICLASSICS

Consider the Gross–Pitaevskii equations for the wave function of the Bose condensate ψ [14]:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \mu(n)\psi. \tag{9}$$

Here $\hbar$ is the Planck constant, m is the boson mass. In the case of Bose atoms, m is the mass of the atom; for Fermi atoms, m is the mass of the Cooper pair, i.e. equal to twice the mass of an atom. The chemical potential μ for Bose atoms is

$$\mu = gn, \quad \text{where} \quad g = 4\pi\hbar^2 a_s/m \tag{10}$$

is the interaction constant proportional to the s-wave scattering length a_s. For positive values of the scattering length between the bosons, repulsion takes place, and for $a_s < 0$—attraction. In the latter case, the condensate is unstable—the development of this instability leads to a collapse (see, for example, the review [5]).

For Fermi atoms, the negative value of a_s with decreasing temperature, $T \to 0$, first promotes the formation of Cooper pairing of atoms, which subsequently form a Bose condensate. As noted in the Introduction, the limit $(|a_s|k_F)^{-1} \to 0$, where $p_F = \hbar k_F$ is the Fermi momentum, corresponds to the so-called unitary regime, for which

$$\mu(n) = 2(1+\beta)\varepsilon_F, \quad (11)$$

where $\beta = -0.63$ and

$$\varepsilon_F = \frac{\hbar^2}{2m}(6\pi^2 n)^{2/3}$$

is the local Fermi energy. In this case, the equation of motion for the wave function of the condensate ψ ($T \to 0$) is the generalized Gross–Pitaevskii equation (2) with $\mu(n)$ (11).

Transition to the dimensionless variables in the GP equation for $\nu = 2/D$ leads to the standard form of the nonlinear Schrödinger equation (3) with the Hamiltonian (4), in which the first term coincides with the total kinetic energy, and the second term is responsible for the repulsion between Bose particles.

Introducing by the standard way the amplitude and phase $\psi = A\exp(i\varphi(r,t))$ ($n = A^2$), NLS Eq. (3) is rewritten as two equations, namely, the continuity equation for n and the eikonal equation for the phase φ:

$$\frac{\partial n}{\partial t} + (\nabla \cdot n\nabla\varphi) = 0, \quad (12)$$

$$\frac{\partial \varphi}{\partial t} + \frac{(\nabla\varphi)^2}{2} + \mu(n) + T_{QP} = 0, \quad (13)$$

where $\mathbf{v} = \nabla\varphi$ represents the velocity (we assume the absence of the vortices $\nabla \times \mathbf{v} = 0$). In the second equation the term

$$T_{QP} = -\frac{\Delta\sqrt{n}}{2\sqrt{n}} \quad (14)$$

is responsible for the quantum pressure.

The equations for the density n and the phase φ preserve the Hamiltonian form:

$$\frac{\partial n}{\partial t} = \frac{\delta H}{\delta \varphi}, \quad \frac{\partial \varphi}{\partial t} = -\frac{\delta H}{\delta n}, \quad (15)$$

where the Hamiltonian coincides with (4). In terms of n and φ H has the form

$$H = \int\left[\frac{n(\nabla\varphi)^2}{2} + \frac{(\nabla\sqrt{n})^2}{2} + n^{\nu+1}\right]d\mathbf{r}. \quad (16)$$

As noted in the Introduction, the nonlinear Schrödinger equation (3) for $\nu = 2/D$ has an additional symmetry relative to Talanov transformations [2]. Talanov transforms contain scaling transformation of the form $t \to \alpha^2 t$ and $\mathbf{r} \to \alpha\mathbf{r}$ due to the conservation of the total number of particles $N = \int|\psi|^2 d\mathbf{r}$ as well as phase transformations. Both of these symmetries are of the Noether type and lead to the appearance of two additional integrals of motion C_1 and C_2 (7), following from the integration of the virial relation (6).

The quasi-classical approximation (the non-stationary Thomas–Fermi approximation) for the Gross–Pitaevskii equation corresponds to neglecting the quantum pressure in (13):

$$\mu(n) \gg |T_{QP}|. \quad (17)$$

As a result the equations of motion transform into the hydrodynamic equations for the potential flow of an ideal gas with the adiabatic exponent $\gamma = 1 + 2/D$:

$$\frac{\partial n}{\partial t} + (\nabla \cdot n\nabla\varphi) = 0, \quad (18)$$

$$\frac{\partial \varphi}{\partial t} + \frac{(\nabla\varphi)^2}{2} + \mu(n) = 0. \quad (19)$$

It should be emphasized that all symmetries in these equations are preserved. The virial theorem also remains valid for this system; in this case, in the Hamiltonian (16), it is necessary to neglect the second term responsible for the quantum pressure.

Basically, further we will neglect quantum pressure. Neglecting quantum pressure implies faster spatial and temporal phase changes (large phase gradients and derivatives with respect to time) compared to space-time variations of the modulus of the ψ-function in the GP equation. Let us emphasize that all the indicated symmetries do not depend on the nature of the interaction—repulsion or attraction. The same applies to the virial theorem (6). For repulsion, it follows from the virial theorem that asymptotically at large t the average size of a quantum gas cloud expanding into vacuum, independently on whether or not quantum pressure is taken into account, grows linearly with time, i.e. the ballistic regime is reached [34, 35].

Let us consider how the expansion of the quantum gases into vacuum occurs in the quasi-classical approximation. In this case the equations of motion, as already mentioned, coincide with the hydrodynamic equations for the ideal gas with $\gamma = 1 + 2/D$. Thus, in a three-dimensional case we deal with the expansion of the monoatomic gas into vacuum (note that for the ideal gas $\gamma = (i+2)/i$ where i is the number of the degrees of freedom).

In 1970 Anisimov and Lysikov [3] discovered a very interesting phenomenon associated with the nonilinear deformations of the shape of a gas cloud expanding into vacuum. This behavior follows directly from the solution which they found for a gas with $\gamma = 5/3$ (see also [22, 36, 37]). In this section we will use the virial theorem and construct an anisotropic quasiclassical solution which coincides with the result of Anisimov and Lysikov for $D = 3$.

We will look for a solution to Eqs. (12) and (13) in a self-similar form:

$$n = \frac{1}{V(\mathbf{a})} f(\xi), \tag{20}$$

where the scaling parameters $\mathbf{a}$ are functions of time t, $\xi_l = x_l/a_l$ are self-similar variables, and $V(\mathbf{a}) = \prod_{l=1}^{D} a_l$ is the volume in the space of scaling parameters. Note that the ansatz (20) preserves the total number of particles.

Substitution (20) into the continuity Eq. (12) allows integration, as a result of which the phase φ is found up to an arbitrary function $\varphi_0(t)$:

$$\varphi = \varphi_0(t) + \sum_l \frac{\dot{a}_l a_l}{2} \xi_l^2. \tag{21}$$

The function $\varphi_0(t)$ is determined from the eikonal equation. Substitution (21) in (13) gives D equations of motion for the parameters a_l:

$$\ddot{a}_l a_l = \frac{\lambda}{V(a)^{2/D}}, \tag{22}$$

where λ is an arbitrary positive constant determined from the initial conditions. For $f(\xi)$, as a result, we have

$$f(\xi) = \left[1 - \frac{D\lambda}{2(2+D)}\xi^2\right]^{D/2}. \tag{23}$$

Thus, the density in terms of the variables ξ depends only on the modulus $|\xi|$. For

$$|\xi| > |\xi|_{\max} = \sqrt{\frac{2(2+D)}{D\lambda}}$$

the density n should be taken equal to zero (see Fig. 1 for $D = 3$). In accordance with (22) the dynamics of the parameters $a_i(t)$ ($i = 1, \ldots, D$) is described by the Newton equations for a particle motion in the D-dimensional space

$$\ddot{a}_i = -\frac{\partial U}{\partial a_i} \tag{24}$$

in the potential

$$U(a) = \frac{D\lambda}{2V(a)^{2/D}}. \tag{25}$$

Note that these equations correspond to the so-called systems of the Ermakov-type [29] (see also [33] and references therein).

It is evident that Eq. (24) should have the same symmetry as the initial GP Eq. (2). First, Eqs. (24) conserve the energy

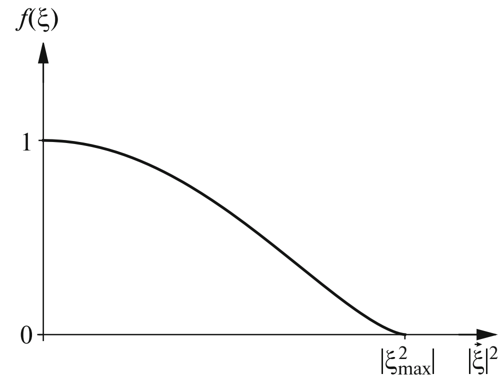

Fig. 1. Dependence of $f(\xi)$ for Fermi gas in the unitary limit (arbitrary units).

$$E = \frac{1}{2}\sum_{l=1}^{D} \dot{a}_l^2 + U(a). \tag{26}$$

Second, for Eqs. (24), a virial relation (6) written in terms of a_i is easily established by direct computation. For $\sum a_i^2$ we have

$$\frac{d^2}{dt^2}\sum_i a_i^2 = 2\sum_i \left[\left(\frac{da_i}{dt}\right)^2 + a_i \frac{d^2 a_i}{dt^2}\right].$$

Substitution of (24) into this relation gives

$$\frac{d^2}{dt^2}\sum_i a_i^2 = 2\sum_i \left(\frac{da_i}{dt}\right)^2 + \frac{2D\lambda}{V(a)^{2/D}} = 4E,$$

which is the same as (6). Integrating twice, we obtain two integrals C_1 and C_2:

$$\sum_i a_i^2 = 2Et^2 + C_1 t + C_2. \tag{27}$$

It would seem that the presence of three integrals for the system (24), i.e. E, C_1 and C_2, guarantees its integrability for all physical dimensions, including $D = 3$. However, this is not so due to the fact that the integrals C_1 and C_2, as functions of a_i, explicitly depend on time:

$$C_1 = \frac{d}{dt}\sum_i a_i^2 - 4Et, \tag{28}$$

$$C_2 = \sum_i a_i^2 - 2Et^2 - C_1 t, \tag{29}$$

for this reason, they are non-autonomous, although they are in involution with other integrals of motion, cf. with (8).

2.1. Two-Dimensional Gas Expansion

Let us first consider the expansion of a two-dimensional gas in more detail (see [16]). In a cylindrical coordinate system with $a^2 = a_1^2 + a_2^2$ and polar angle ϕ the energy integral is written as

$$E = \frac{\dot{a}^2}{2} + \frac{a^2\dot{\phi}^2}{2} + \frac{2\lambda}{a^2\sin(2\phi)}.$$

Next multiplying E by $a^2 = 2Et^2 + C_1t + C_2$, it is easy to obtain that the combination

$$\tilde{E} = Ea^2 - \frac{1}{2}a^2\dot{a}^2 = EC_2 - C_1^2/8$$

is a constant (Ermakov's integral). As a result, we come to the law of conservation of the new "energy"

$$\tilde{E} = \frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + U_{\rm eff}(\phi), \tag{30}$$

with new time τ:

$$d\tau = \frac{dt}{a^2}, \quad \tau = \int_0^t \frac{dt'}{2Et'^2 + C_1t' + C_2}, \tag{31}$$

where

$$U_{\rm eff}(\phi) = \frac{2\lambda}{\sin 2\phi} \tag{32}$$

plays the role of potential energy. $U_{\rm eff}$ is always positive and tends to infinity as $\phi \to 0$ and $\phi \to \pi/2$. Minimum $U_{\rm eff} = 2\lambda$ for $\phi = \pi/4$ corresponds to the isotropic case. In this case, only a^2 changes, which is determined from the virial relation: $a^2 = 2Et^2 + C_1t + C_2$.

New time τ (31) can easily be expressed in terms of t,

$$\sqrt{2\tilde{E}}\tau = \arcsin\frac{\sqrt{2E}(t+t_0)}{\chi} - \arctan\frac{\sqrt{2E}t_0}{\chi},$$

where $\chi^2 = \tilde{E}/E$ and $t_0 = \dfrac{C_1}{4E}$, so $\tau = 0$ at $t = 0$. If the initial gas velocity is zero (which is typical for an experiment), then the constant $C_1 = 0$ and

$$\sqrt{2\tilde{E}}\tau = \arctan\frac{\sqrt{2\tilde{E}}t}{C_2}.$$

In this case, asymptotically for $t \to \infty$

$$\tau \to \tau_\infty = \frac{\pi}{2\sqrt{2\tilde{E}}}. \tag{33}$$

The trajectory $\phi(\tau)$ is found by integrating Eq. (30):

$$\tau = \int \frac{d\phi}{\sqrt{2[\tilde{E} - U_{\rm eff}(\phi)]}}.$$

Hence, the τ-period in the potential $U_{\rm eff}(\phi)$ (32) is expressed through the integral

$$T = 2\int_{\phi^{(-)}}^{\phi^{(+)}} \frac{d\phi}{\sqrt{2[\tilde{E} - U_{\rm eff}(\phi)]}},$$

where $\phi^{(\pm)}$ are the roots of the equation $\tilde{E} = U_{\rm eff}(\phi)$ (turning points). For large values of $\tilde{E}$ the oscillations weakly depend on the details of the function $U_{\rm eff}(\phi)$. Asymptotically the angular "velocity" $\dfrac{d\phi}{d\tau} \to \pm\sqrt{2\tilde{E}}$ while the τ-period $T \to \pi/\sqrt{2\tilde{E}}$, i.e. in this limit T is twice τ_∞ given by (33). Note also that the dependence $T(\tilde{E})$ is monotonic for $U_{\rm eff}(\phi)$ with a maximum corresponding to the potential minimum $U_{\rm eff}(\pi/4)$. This means that for $C_1 = 0$ the second turning point is unreachable when $t \to \infty$.

In particular, if we start from the left turning point, then the system does not reach the right turning point. And vice versa: if the starting point is right, then the left is unreachable. This situation, as we will see below, is also typical for the three-dimensional cylindrically symmetric case.

2.2. Expansion of Three-Dimensional Gas

For the expansion of the Fermi gas in the unitary limit, when $\nu = 2/3$, the equations for the scaling parameters are integrated in the same way as in the two-dimensional case. Foe $D = 3$, the energy (26) should be written by introducing a spherical coordinate system (a, θ, ϕ):

$$E = \frac{1}{2}\left[\left(\frac{da}{dt}\right)^2 + a^2\left(\frac{d\theta}{dt}\right)^2 + a^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right] + \frac{3\lambda}{2^{1/3}a^2}\frac{1}{(\sin^2\theta\cos\theta\sin 2\phi)^{2/3}}.$$

Accordingly, we introduce again the energy $\tilde{E} = Ea^2 - \frac{1}{2}a^2\dot{a}^2 = EC_2 - C_1^2/8$ (the Ermakov integral), the conservation of which is a consequence of the symmetry with respect to dilatations, and new time τ using the same formula (31). As a result, we get

$$\tilde{E} = \left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 + U_{\rm eff}, \tag{34}$$

where the effective potential is expressed in terms of spherical angles,

$$U_{\rm eff} = \frac{3\lambda}{2^{1/3}(\sin^2\theta\cos\theta\sin 2\phi)^{2/3}}. \tag{35}$$

Thus, we obtain a system with two degrees of freedom specified by the angles θ and ϕ.

For the cylindrically symmetric solutions, when $\cos\phi = \sin\phi = \sqrt{2}/2$ (i.e., for $\phi = \pi/4$), we can write $\tilde{E}$ in the form:

$$\tilde{E} = \left(\frac{d\theta}{d\tau}\right)^2 + \frac{3\lambda}{2^{1/3}(\sin^2\theta\cos\theta)^{2/3}}, \tag{36}$$

analogous to (30) for the two-dimensional case. The only difference is in the form of the effective potential U_{eff}. Integration of this equation leads to the Anisimov–Lysikov result [3].

In the general case, when we take into account the dependence from the both angles, the knowledge of the only one integral $\tilde{E}$ is not enough. As it was shown by Gaffet [38], the given system has one more additional integral of motion (besides $\tilde{E}$), which follows from the Painleve test. The existence of two integrals of motion already guarantees the complete integrability of this system. It is important to note, as in the previous case, that the motion in the potential (35) retains its nonlinear quasi-oscillatory character.

2.3. *Account of the Quantum Pressure*

Let us now discuss the role of quantum pressure in the expansion of quantum gases into vacuum. Note that for the solutions obtained above, the quasi-classical criterion (17) is violated at this point at the point $\xi = \xi_{\max}$. Namely, the second derivative of the amplitude A with respect to ξ becomes infinite at this point, and accordingly the term of the quantum pressure becomes infinitely large. This is a typical situation for quasiclassical solutions in quantum mechanics, when it is required to solve the problem of matching the solutions at the turning point (see [39]). In our case, the role of the turning point plays $\xi = \xi_{\max}$. In the neighborhood of the point $\xi = \xi_{\max}$ we should match the solution constructed for $\xi < \xi_{\max}$ (the internal region) with the solution in the external region for $\xi > \xi_{\max}$.

In the internal region far from $\xi_{\max}$ the solution should transform into the quasi-classical one which we found before. In the same time in the external region the function ψ should be governed by the linear Schrödinger equation. It should be said that this problem was discussed in details in [40] for the regime of strong three-dimensional collapse in the cubic NLSE ($\nu = 1$). In this case, the matching problem can be considered in a similar way.

In what follows, we will assume that $\Delta\xi \ll \xi_{\max}$ representing the transition region $\Delta\xi$ to be rather narrow. It is easy to understand that the problem then can be considered as a one-dimensional one in the direction normal to the surface $\xi = \xi_{\max}$.

Let us start with the isotropic expansion of the two-dimensional Bose gas when the chemical potential $\mu(n) = n$. Turn to the equation (13) for the phase φ. For the self-similar quasiclassical solution, the recalled phase can be found from the integration of the continuity equation, i.e., the phase φ is not sensitive to the variation of the amplitude in the matching region. This means that everywhere in the transition region we can assume that

$$\frac{\partial\varphi}{\partial t} + \frac{(\nabla\varphi)^2}{2} \approx -2A_0^2,$$

where $A_0^2 = a^{-2}(1 - \lambda\xi^2/4)$ is the solution of the quasi-classical equations. As a result, the equation for the amplitude A in the matching region can be written as

$$\frac{1}{2}\nabla^2 A - 2(A^2 - A_0^2)A = 0.$$

Since $A_0(\xi_{\max}) = 0$ ($\xi_{\max} = 2/\sqrt{\lambda}$), in this equation it is necessary to keep in A_0^2 linear deviations in $\chi = \xi - \xi_{\max}$: $A_0^2 \approx -\chi a^{-2}\sqrt{\lambda}$, and in the Laplace operator ∇^2, due to the narrowness of the transition layer, leave the second derivative with respect to χ. As a result, we come to the following differential equation for the function $g = A/a$

$$\frac{1}{2}\frac{d^2 g}{d\chi^2} - 2(g^2 + \sqrt{\lambda}\chi)g = 0$$

with the boundary conditions $g \to 0$ for $\chi \to \infty$ and $g \to \sqrt{-\lambda^{1/2}\chi}$ for $\chi \to -\infty$. This equation represents the Painleve II equation (see, e.g. [40]). For large positive χ this equation turns into the Airy equation with an exponentially decaying solution. For smaller $|\chi|$ the solution will be close to the Airy function and will have an oscillatory character. As we move further from the boundary $\xi = \xi_{\max}$ into the internal region, oscillations will remain in the solution, however their amplitude will be decreased. The solution itself for $\xi \to -\infty$ will approach the required asymptotic form.

The appearance of these oscillations is the main manifestation of the quantum nature of the Bose condensate during its expansion into vacuum. These oscillations are of the diffraction origin and are similar to the Newton's rings.

We would like to note that in the one-dimensional problem of the expansion of a Bose gas into vacuum, as shown in the work [41] (see Fig. 3f in this paper), there are no oscillations at all at the edge. The reason is that the density in this case in the vicinity of the extreme point behaves quadratically and therefore no violation of the quasi-classical approximation at this point is observed. However, under different conditions [42] oscillations at the edge are observed. We emphasize that the matching problem in the limit of small $\Delta\xi \ll \xi_{\max}$ is reduced to a one-dimensional problem, but by construction it differs significantly from the expansion problem within the one-dimensional NLS equation, integrable by means of the inverse scattering transform [43].

In the anisotropic case, the oscillations will obviously be preserved. First, in the Laplace operator, the largest derivative will be along the normal to the surface $\xi = \xi_{\max}$. Secondly, angular velocity $v_\Omega = r\dot{\Omega}$ for sufficiently long times $t \to \infty$, when the size of the gas

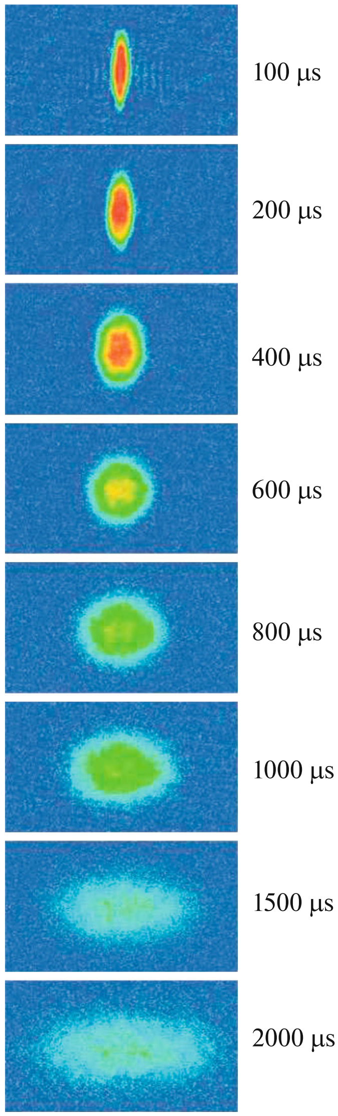

Fig. 2. Images of an expanding strongly interacting Fermi gas in time (in μs). Initial form is a cigar.

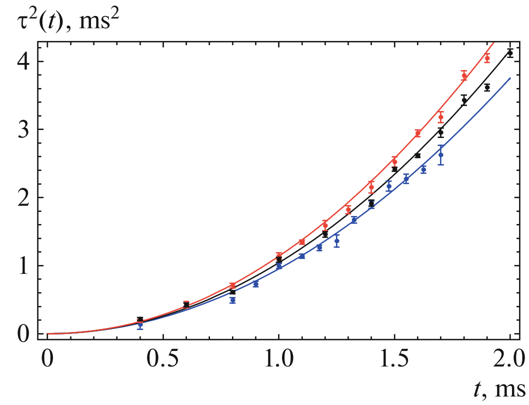

Fig. 3. Dependencies of $\tau^2(t)$. On the vertical axis the experimental values of $\tau^2(t) \equiv m[\langle \mathbf{r}^2\rangle - \langle \mathbf{r}^2\rangle_{t=0}]/\langle \mathbf{r} \cdot \nabla U\rangle_{t=0}$ are measured for expansion of strongly interacting Fermi gas as a function of time t, $U(\mathbf{r})$ is the initial trap potential value. Black markers correspond to the gas at resonance $1/(k_Fa_s) = 0$, red and blue $1/(k_Fa_s) = 0.59$ and $1/(k_Fa_s) = -0.61$, while the solid curves are the calculation results [25].

cloud R significantly exceeds the initial size of R_0, turns out to be much less than $\dot{R}$. According to the virial theorem (6), $\dot{R} \approx v_\infty$, and

$$v_\Omega = r|\dot{\Omega}| = |d\Omega/d\tau|/r \le 2\sqrt{\tilde{E}}/(v_\infty t).$$

This means that the main changes will occur along the normal to the boundary $\xi = \xi_{max}$, and temporal changes in the angle can be neglected. If at the initial moment the condition of quasiclassical approximation is satisfied, then it will be satisfied everywhere except for the narrow region $\delta\xi \gg |\xi|_{max}$. In this case, the ratio of $\delta\xi$ and $|\xi|_{max}$, due to self-similarity, can be considered unchanged, which is true at least for large t. Thus, asymptotically, the matching problem should be considered one-dimensional in $|\xi|$ with a locally frozen direction of the normal. Hence, it becomes clear that a belt of diffraction-like density oscillations is formed around the surface $\xi = \xi_{max}$.

The behavior of a quantum Fermi gas in the unitary limit in the vicinity of the surface $\xi = \xi_{max}$ is analyzed in a similar way.

3. CONCLUSIONS

Thus, we have shown how the symmetries for the GPE, when the chemical potential has a power-law dependence on the density n with exponent $\nu = 2/D$ (where D is the space dimension), affect the expansion of quantum gases into vacuum. As a consequence of the virial theorem, independently of the relation between the quantum pressure and the chemical potential, the average size of the expanding cloud grows asymptotically at large times linearly with t, so that the expansion rate tends to a constant value $v_\infty = (2H/N)^{1/2}$. The most general symmetry of the GPE corresponds to the Talanov transformations which form the abelian group. The same symmetry takes place for potential gas flows with the adiabatic exponent $\gamma = 1 + 2/D$, described using the equations of gas dynamics: the continuity equation and the Euler equation. It is important that this hydrodynamic system coincides with the Gross–Pitaevskii equation in the quasiclassical limit and thus inherits the symmetries of the GP equation.

We showed also that, in the quasiclassical approximation, the GP equation has self-similar anisotropic solutions describing nonlinear angular quasi-oscillations of the cloud shape against the background of the quantum gas expansion. In the three-dimensional case, these solutions coincide with those found by Anisimov and Lysikov for the expansion of a classical monoatomic gas with the adiabatic exponent $\gamma = 5/3$ into vacuum excluding the region where the density vanishes in the quasiclassical theory. This is a whole surface which plays the same role as a turning point in the quasiclassical approximation in standard quantum mechanics. The problem of matching the solutions in the internal and external regions shows that spatial density oscillations of a diffraction nature arise in the neighborhood of this surface. This is what distinguishes quantum and classical gases as they expand.

In conclusion, let us discuss to what extent the experimental data correspond to the obtained analytical results. The self-similar expansion of a strongly interacting Fermi gas from a trap of a cigar-shape was observed in experiments [24]. Figure 2 taken from [24] shows the expanding Fermi gas. At the initial moment of time, the gas cloud had the shape of a strongly elongated ellipsoid in the form of a cigar (time $t = 100$ μs), then at $t = 600$ μs was almost spherical and at the final stage the cloud had the shape of a disk. The total observation time was 2000 μs, which can be taken as a half-period (or less) of angular oscillations of the gas cloud shape, $t \leq t_{\rm osc}/2$, according to the results of the previous section. Thus, all these stages qualitatively correspond to the self-similar solution.

Figure 3 presents the results of the measurements of the average size as a function of time for three values of $1/(k_F a_s)$ [25].

All three dependencies $\tau^2(t)$ represent with good accuracy parabolic dependencies, in full accordance with the relation (7) following from the virial theorem (6). Strictly in resonance at $1/(k_F a_s) = 0$ the mean size of the cloud $\langle \mathbf{r}^2 \rangle$ can be expressed via the initial trapping potential $U(\mathbf{r})$ in the following form [25]:

$$\langle \mathbf{r}^2 \rangle = \langle \mathbf{r}^2 \rangle_{t=0} + \frac{t^2}{m} \langle \mathbf{r} \cdot \nabla U(\mathbf{r}) \rangle_{t=0}. \qquad (37)$$

The calculations presented in [44] show that the expansion law (37) coincides with the quasiclassical dependence of $\langle \mathbf{r}^2 \rangle$ (27) in the unitary limit. Note that in accordance with (27), the mean size $\langle \mathbf{r}^2 \rangle$ indeed linearly depends on energy that was verified in the experiments [25].

We emphasize that these dependencies are based on the quasiclassical theory, which is not different from the hydrodynamics of an ideal gas with the adiabatic exponent $\gamma = 5/3$. As we have shown, the difference between a quantum gas and a classical one in the problem of expansion into vacuum consists in taking into account the quantum pressure, which leads to the appearance of density oscillations at the boundary of the expanding cloud. Apparently in the experiments [24] and [25], we think that the expansion of the normal Fermi gas rather than the superfluid gas was observed.

Concerning the expansion of the Bose atoms note that in the experiments [23], qualitatively the same sequence of the shape variations of the gas cloud was observed (as in the three-dimensional Anisimov-Lysikov solution [3]). This fact serves as an evidence that in the experiment the normal component plays more essential role than the superfluid one.

Recall that one of the key experiments for the discovery of Bose–Einstein condensates in gases of alkali elements ^{7}Li, ^{23}Na, ^{87}Rb [45–47] was the determination of the distribution function of Bose atoms during the gas expansion into vacuum when the trapping potential was switched off. The distribution function had a bimodal form, which corresponded to normal and superfluid components. For the normal component, the velocity distribution was wide—thermal—of the Maxwellian type, and the superfluid component had a narrower distribution with a width determined by the interaction parameter (in the sense of Gross–Pitaevskii).

At low but finite temperatures, the Bose condensation temperature drops due to the density decrease during expansion, which must inevitably lead to an increase in the number of atoms of the normal component. For this reason the shape of the cloud should be determined by the normal component, which can be considered as a monoatomic gas. Cold superfluid component will be concentrated inside the expanding cloud. For Fermi gases, this situation, apparently, also occurs. In contrast to Bose gases, the transition to the normal component upon expansion of the Fermi gas will also be accompanied by the destruction of Cooper pairs. Thus, the expansion of a quantum gas should lead to the appearance of Newton's rings, which was not observed in the experiments [23], as well as in [24, 25]. Observation of such oscillations, at least at the initial stage of expansion, would be an evidence that the gas is in a quantum state.

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