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On Times and Speeds of Time-Dependent Quantum and Electromagnetic Tunneling

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Abstract—This is a review of studies of quantum tunneling, which is described by the one-dimensional Schrödinger equation, and electromagnetic tunneling, where "superluminal" velocities and times of tunneling are considered. Integral and integrodifferential equations have been presented to describe tunneling. According to these equations, superluminal motion is impossible. The paradoxical Hartman effect has been discussed and explained. It has been shown that the velocity of passage of a particle in a beam through a barrier in the case of steady-state and time-dependent quantum tunneling is equal to the velocity of its incidence on the barrier and quasiphotons inside any layer of matter carry the energy always at a subluminal velocity. However, the tunneling time of a single particle or a photon is meaningless.

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1. INTRODUCTION

This work concerns the simplest one-dimensional time-dependent tunneling problems for electromagnetic waves and quantum particles that are described by one-dimensional Maxwell's equations and one-dimensional time-dependent Schrödinger equation, respectively. These problems include the passage of wave packets through regions where propagation is impossible if they are infinitely wide but passage is possible for finite regions. Such regions produce a finite damping, which is radiative for electromagnetic waves and is related to reflection. In particular, stopbands in photonic crystals are formed because of Bragg reflections. However, electromagnetic waves also allow dissipative damping, which affects tunneling. Dissipative damping for the Schrödinger equation is due to the presence of drains, i.e., to a decrease in the number of particles in the beam; but this effect, as well as the multiparticle Schrödinger equation with a variable number of particles, is not considered in this work. Highly dissipative tunneling often occurs for electromagnetic waves and should be considered. Damping for quantum tunneling is due to the interference of de Broglie waves. Since a wave packet (or train or pulse) is an unsteady wave, it is necessary to solve the problem of propagation of the pulse or wave packet [1, 2] in the presence of interfaces and inhomogeneity of the medium (correspondingly, a local inhomogeneity of the potential for the Schrödinger equation). The propagation time and speed of wave packets, as well as the speed of energy and signal transfer, should be generally considered. For simplicity, it is assumed that the indicated segment lies in the range $0 \le z \le d$. Beyond this range, a particle moves as free and an electromagnetic wave moves in vacuum. For the Schrödinger equation, we introduce a potential $V(z, t)$ and the medium is characterized by the permittivity $\varepsilon(z, t)$. In the time-dependent case, the spectral permittivity $\varepsilon(z, \omega)$ can be obtained for the kernel of an integral operator. Such a formulation is topical not only for tunneling, but also for other applications. In the general case, this is the one-dimensional problem of scattering of wave packets.

Quantum tunneling is one of the most important quantum effects and is widely used in various fields such as the creation of electron sources in field-emission electron guns for vacuum electronics, in flat screen panels, hopping conductivity devices, tunnel-effect diodes and transistors, and Josephson-effect devices [3–10]. The time of flight or tunneling time through a potential barrier is important for all these devices. This problem also concerns electromagnetic waves [11–13]. It appeared immediately after the works by G.A. Gamow, R.W. Gurney, and E.U. Condon (1928), and, later, by F.T. Smith [14], T.E. Hartman [15], J.R. Fletcher [16], M. Buttiker and R. Landauer [17], etc. (see, e.g., reviews in [18–36]).

The paradoxical Hartman effect [15] is the saturation of the Bohm–Wigner tunneling time τ_{BW} with increasing barrier width, which can satisfy the inequality $\tau_{BW} < \tau_c = d/c$; i.e., superluminal tunneling is possible. In a few decades after work [15], several different times, in particular, complex were introduced (see [17–22, 26, 29, 31–33]) to describe the processes of tunneling but no one of them has become commonly accepted [33]. Among these times are the Buttiker–Landauer, Pollak–Miller, and Larmor times, as well as various dwell, interaction, reflection, and transmission times. It is stated that the Hartman effect was also detected for the tunneling of electromagnetic waves, and it was regularly reported in the last decades that light propagates "faster than light" (see, e.g., [10–13, 20–26, 29]). There are reports on zero tunneling time [37] and even on negative delay time [38–43]. Believing in a negative group velocity, some authors state that a signal at the output appears before it reaches the input; i.e., superluminal information transfer occurs. It was shown in [44] that such conclusions are absurd. There are several hundreds of works considering superluminal tunneling, and more than a hundred of them were published in high-impact journals.

The aims of this work are to derive equations for time-dependent tunneling and to prove that superluminal motion of wave packets is impossible. It will be shown that these equations are integral or integrodifferential equations. The difficulty of time-dependent tunneling is nonlocality [19, 45–47], which means that a wave packet is present in the entire space, is dispersed, and is split owing to diffraction. It is often stated that single-photon tunneling occurs in experiments, where a parametric down-conversion source is used: an initial parent photon, interacting with a nonlinear medium, is split into two photons with halved energies, and a reference photon is used for "photon" tunneling through the barrier to be detected in a Hong–Ou–Mandel interferometer (see, e.g., [11, 12]). It is remarkable that the "photon" having passed through a multilayer photonic crystal mirror or a tunnel is not the initial single photon. It is a polariton or quasiphoton, i.e., a quasiparticle undergoing multiple scattering events (absorption and emission) on atoms of a material. The "photon" having tunneled through the many-period photonic crystal is collected by means of the interference of scattered waves or quasiphotons, and it is incorrect to use an interferometer to detect it.

Steady-state tunneling is characterized not by time but by the reflection, $|R|^2$, and transmission, $|T|^2$, probabilities, i.e., probabilities of finding particles on the right and left, respectively. In the quantum case, the reflection and transmission coefficients satisfy the condition $|R|^2 + |T|^2 = 1$. For the tunneling of electromagnetic waves, where absorption is possible, $|R|^2 + |T|^2 < 1$. Steady-state tunneling described by the single-particle Schrödinger equation is tunneling of beams of noninteracting particles (when the interaction between electrons in a rarefied beam can be neglected); for this reason, it is necessary to specify the density of particles in the beam $|A\psi(z)|^2$ at the point z. If the time-independent Schrödinger equation is used for a single particle, the amplitude A of the wavefunction should be zero because the integral of $|A\psi(z)|^2$ over the infinite region is unity; for this reason, the wavefunction is normalized to a Dirac delta function. In this formulation, only the probabilities $|R|^2$ and $|T|^2$ of finding a particle at $z = -\infty$ and 0, respectively, are meaningful only at infinite time $t = \infty$; i.e., tunneling time is meaningless in this formulation. In reality (e.g., at field emission), there is usually a beam of particles often with different velocities [8]. At tunneling, beams in front of the barrier and inside it are always bidirectional. The normalization of the wavefunction to the delta function means that $|A\psi(z)|^2 = 1$, i.e., is equivalent to the normalization to the unit density of particles in the incident beam. The velocity of particles can be defined in the monochromatic beam. Such a definition for a polychromatic beam is more difficult and it is necessary to use the wavefunction with a spectrum, i.e., a wave packet. The wave packet is usually also polychromatic, except for the electromagnetic wave packet incident from vacuum at the speed of light c. However, such a wave packet becomes bi-velocity already at reflection and consists of two counterpropagating beams. The electromagnetic wave packet in a medium with dispersion is also polychromatic.

Steady-state electron tunneling inside a rectangular potential barrier is described by the wavefunction $\psi(z) = A^+\exp(-k''z) + A^-\exp(k''z)$, where $k'' = \sqrt{\mu_e(V-\mathcal{E})/\hbar^2}$, $A^{\pm} = T(1 \mp i\kappa)\exp(\pm k''d)/2$, $\kappa = \sqrt{\mathcal{E}/(V-\mathcal{E})}$, $\mu_e = 2m_e$, and m_e is the mass of the electron. This wavefunction gives a constant probability flux density

$$j(z) = i\hbar\mu_e^{-1}[\partial_z\psi^*(z)\psi(z) - \psi^*(z)\partial_z\psi(z)] = v_z|T|^2,$$

where $v_z = \sqrt{2\mathcal{E}/m_e}$ is the velocity of incident particles. This velocity is continuous; consequently, the velocity of particles is v_z in direct beams and $-v_z$ in opposite beams (taking into account that the density in the reflected beam is $|R|^2$). This result can be obtained defining the velocity as $v_{\mathcal{E}}(z) = j(z)/|\psi(z)|^2$. In the left region, $\psi(z) = \exp(ikz) + R\exp(-ikz)$, where $k = \sqrt{\mu_e\mathcal{E}/\hbar^2}$, and the coordinate-dependent velocity should be averaged over several de Broglie wavelengths, which gives the expression $v_z = \sqrt{\mu_e\mathcal{E}}$. The parts of the wavefunction corresponding to the incident and reflected waves give v_z and $-v_z$, respectively. Inside the barrier,

$$v_{\mathcal{E}}(z) = \frac{v_z}{\cosh^2(k''(z-d)) + \kappa^2\sinh^2(k''(z-d))}.$$

This velocity of motion of the particle density (probability density) is also below the speed of light and $v_{\mathcal{E}}(d) = v_z$; i.e., the particle leaving the barrier holds its velocity and energy. If the energy of the particle is $\mathcal{E} = V$, then

$$v_{\mathscr{E}}(z) = \frac{v_z}{1 + \mathscr{E}\mu_e(z-d)^2/\hbar^2}.$$

In electrodynamics, this corresponds to the so-called epsilon near zero (ENZ) region [48] characterized by the superluminal (and even infinite in the absence of dissipation) phase velocity. The time of passage of a particle in the beam through the barrier calculated by integrating $v_{\mathscr{E}}^{-1}(z)$ over the coordinate is significantly smaller than the luminal time $\tau_c = d/c$ and exponentially tends to infinity with increasing thickness. The steady-state tunneling of electromagnetic waves is described by the Helmholtz equation, which coincides in form with the time-independent Schrödinger equation; hence, it is not allow superluminal velocities. Speeds and times of time-dependent tunneling are of interest.

2. TIME-DEPENDENT ELECTROMAGNETIC TUNNELING

We solve one-dimensional Maxwell's equations in the form

$$\begin{aligned} -\partial_z H_y(z,t) &= \partial_t D_x(z,t) + J_x(z,t), \\ \partial_z E_x(z,t) &= -\mu_0 \partial_t H_y(z,t). \end{aligned} \tag{1}$$

The substitution of the time derivative of the first equation into the coordinate derivative of the second equation gives the wave equation

$$\partial_z^2 E_x(z,t) - \mu_0 \partial_t^2 D_x(z,t) = \mu_0 \partial_t J_x(z,t). \tag{2}$$

Its form depends on the relation between the induction and field. Neglecting time (frequency) dispersion, we obtain the instantaneous (local in time) relation $D_x(z, t) = \varepsilon_0\varepsilon(z, t)E_x(z, t)$. This relation is usually valid for slow processes where characteristic times of variation are much larger than inverse resonant frequencies of a material. The permittivity $\varepsilon(z, t)$ can often be considered as a slow function of time or as independent of time altogether. Below, $E_x = E$ and $H_y = H$. Then,

$$\partial_z^2 E(z,t) - c^{-2}\partial_t^2[\varepsilon(z,t)E(z,t)] = \mu_0 \partial_t J_x(z,t). \tag{3}$$

The time dependence of the permittivity means parametric excitation. A particular example of such excitation is a change in the carrier density in a semiconductor induced by laser pumping or injection. The amplitude of such a pumping should be a slowly varying function of the time [2]. Dispersion can be neglected at sufficiently low frequencies (for almost static slowly varying fields). In the general case,

$$D_x(z,t) = \varepsilon_0 \int_{-\infty}^{t} \tilde{\varepsilon}(z,t-t')E(z,t')dt'. \tag{4}$$

The absence of dispersion is described by the kernel $\tilde{\varepsilon}(z, t) = \varepsilon(z, t)\delta(t)$ of the integral operator given by Eq. (4). This kernel is related to the spectral permittivity through the Fourier transform:

$$\tilde{\varepsilon}(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon(z,\omega)\exp(j\omega t)d\omega, \tag{5}$$

$$\varepsilon(z,\omega) = \int_{-\infty}^{\infty} \tilde{\varepsilon}(z,t)\exp(-j\omega t)dt. \tag{6}$$

In electromagnetic problems, imaginary unit $j = -i$ is used. Since $\varepsilon(z, \omega)$ is analytic in the lower half-plane of the complex ω plane, integral (5) is zero at $t < 0$ [1, 2], which expresses the principle of causality in Eq. (4): response in the form of polarization occurs only to preceding actions of the field. In the general case, the permittivity $\varepsilon(z, \omega) = \varepsilon'(z, \omega) - j\varepsilon''(z, \omega)$ is complex and its real and imaginary parts are even and odd functions of the frequency, respectively, and satisfy the Kramers–Kronig relation, which also expresses the principle of causality [49] (the time dependence for a monochromatic plane electromagnetic wave and the wavefunction of a quantum particle is taken in the form $\exp(j\omega t)$ and $\exp(-i\mathscr{E}t/\hbar)$, respectively. Relation (4) can also be represented in the form $D_x(z, t) = \varepsilon_0 E(z, t) + P(z, t)$ in terms of the polarization. Consequently, as in the case of vacuum, Eq. (3) can be written with the right-hand side $J(z, t) = \mu_0\partial_t(J_x(z, t) + \partial_t P(z, t))$. Here, the first and second terms in the parentheses correspond to the density of bias current (excitation current) and the polarization current density, respectively. Excitation is very convenient for determining the speed of pulse propagation. If excitation occurs in a finite region and begins at a certain time, the field is absent before this time; therefore, it is very simply to determine the velocity of wave packet propagation. All equations of electrodynamics show that the maximum velocity is c, and excitation appearing at the point z_0 cannot reach the point z in a time smaller than $|z_0 - z|/c$. This fact well known for a long time is already sufficient to avoid the consideration of tunneling of light at a speed higher than the speed of light. However, more than hundred such works were published and continue to be published in high-impact journals, which require the further consideration of the problem. The simplest case is a point source $J_x(z, t) = \delta(z - z_0)\delta(t)$. In the three-dimensional case, the point source corresponds to the scalar Green's function $g(\mathbf{r}, t) = (4\pi|\mathbf{r}|)^{-1}\delta(t - |\mathbf{r}|/c)$, which describes the propagation of the vector potential at the speed of light [50]. This Green's function is the inverse Fourier transform of the spectral scalar Green's function $G(\mathbf{r}, \omega) = (4\pi|\mathbf{r}|)^{-1}\exp(-j\omega|\mathbf{r}|/c)$. In our one-dimensional case, the spectral Green's function has the form $G(z, \omega) = -jc\exp(-j\omega|z|/c)/(2\omega)$ and the corresponding space–time Green's function is expressed in terms of the Heaviside step function:

$$g(z,t) = \frac{c\chi(t-|z|/c)}{2} = \frac{-jc}{4\pi}\int_{-\infty}^{\infty}\omega^{-1}\exp(j\omega(t-|z|/c))d\omega. \quad (7)$$

Here, the integral is closed by the lower semicircle in the complex ω plane. It is easy to verify that these functions satisfy the wave equations

$$(\nabla^2 - c^{-2}\partial_t^2)g(\mathbf{r},t) = -\delta(\mathbf{r})\delta(t),$$

$$(\partial_z^2 - c^{-2}\partial_t^2)g(z,t) = -\delta(z)\delta(t).$$

Indeed, substituting the Green's function in the form of the expansion

$$g(z,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\omega,\alpha)\exp(j\omega t - j\alpha z)d\omega d\alpha \quad (8)$$

into the last equation, we obtain $G(\omega, \alpha) = [-\alpha^2 + \omega^2/c^2]^{-1}$. The substitution of this expression into Eq. (8) and the calculation of the integral with respect to α by the method of residues give

$$g(z,t) = \frac{-j}{4\pi}\int_{-\infty}^{\infty}\frac{\exp(j\omega t - j\omega|z|)d\omega}{\omega/c},$$

which coincides with Eq. (7). The contour at different signs of z is closed by either the lower or upper semicircle and it is assumed that k_0 has an infinitely small negative imaginary (dissipative) addition; i.e., the Jordan lemma is valid on the semicircles. Sources in the range $-z_1 < z < -z_2$ generate the vector potential

$$\mathbf{A}(z,t) = \mathbf{x}_0 A(z,t) = \mathbf{x}_0\int_{-\infty}^{t}\int_{-z_1}^{-z_2}g(z-z',t-t')J_x(z',t')dz'dt'. \quad (9)$$

In this range, $P = 0$. Since $\mathrm{div}\mathbf{A}(z, t) = 0$, we have $E(z, t) = -\mu_0\partial_t A(z, t)$ or

$$E(z,t) = \frac{c\mu_0}{2}\int_{-z_1}^{-z_2}J_x(z',t-|z-z'|/c)dz'. \quad (10)$$

The differentiation of Eq. (9) with respect to the upper limit gives zero, whereas the differentiation of the Heaviside step function provides the delta function. Relation (10) follows from the integration by parts of the representation of solution (3) in terms of the Green's function. According to Eq. (10), sources appearing at the time t_0 in the indicated region cannot reach a certain point z earlier than after the time $|z - z'|/c$. Let the source $J_x(z, t) = I(t)\delta(z + z_1)$ be localized at the point $-z_1$. In the one-dimensional problem, it is a plane of emitting dipoles with the surface current density $I(t)$. In this case, $E(z, t) = c\mu_0 I(t - |z + z_1|/c)/2$. The argument of the function I should be larger than t_0; consequently, the field exists only in a finite region if the source emits for a finite time, say, stops to emit at the time t_1. In this case, the spectrum $I(\omega)$ of the function $I(t)$, which determines the spectrum of the field $E(z_1, \omega) = c\mu_0 I(\omega)/2$, can be defined at the source point. This spectrum is infinite (unlimited), containing infinite frequencies, but the wave packet with such a spectrum has a finite energy $\mathcal{E}$. Infinite frequencies cannot be attributed to field quanta because some frequency can satisfy the inequality $\mathcal{E} < \hbar\omega$. A wave packet with a finite spectrum is often considered. Such a wave packet is always delocalized in space and time. This also concerns a classical photon: it can be represented as a monochromatic wave. Its amplitude should be infinitely small for the energy of such a wave to be finite $\mathcal{E} = \hbar\omega$. A monochromatic plane wave with a finite amplitude involves a beam of photons and their density per unit length and area, i.e., per unit volume. Such a photon can be absorbed by matter with a $\mathcal{E}_1 \to \mathcal{E}_2$ transition between levels (infinitely narrow spectral lines) with the energy difference $\mathcal{E} = \mathcal{E}_2 - \mathcal{E}_1 = \hbar\omega$ in an infinite time. This situation is certainly abstract and all real processes are quasistationary, i.e., weakly nonstationary and occur in a finite time. An external background field is significant in this case. An atom that is described by the time-dependent Schrödinger equation and is irradiated by a wave packet can transit with a certain probability from one quasistationary state (before the action of the wave packet) to another one (after such action). The wave packet describing the photon with the energy $\hbar\omega$ has a limited spectrum near the frequency ω. Such a wave packet is infinite in space and time, but it is strongly damped at infinities because of the Paley–Wiener theorem [45]. The theory of functions with a limited spectrum and the analytic properties of a wave packet with a limited spectrum that lead to entire nonlocal wavefunctions because of the Paley–Wiener theorem become of great significance [19, 45–47]. The analytic properties of the S-matrix and Green's function are also important for scattering [47]. The excitation of a photon with the frequency ω_0 can be associated with the source function $I(t) = I_0\chi(t - t_0)\sin(\omega_0(t - t_0))$ and the field $E(z, t) = c\mu_0 I_0\chi(t - t_0 - |z + z_1|/c)/2$. Such a source emits to the both sides. The quantity $\varepsilon_0 E^2(z, t)$ is the energy density (taking into account the magnetic field). The right part of the wave packet at the time t is at the point $z_2 = -z_1 + c(t - t_0)$. It can be supposed that the domain with the unit area $S = 1$ of the source emits a photon to the right if

$$\hbar\omega_0 = \varepsilon_0 S\int_{-z_1}^{-z_1+c(t-t_0)}E^2(z,t)dz.$$

For this reason, this problem should be characterized by the photon density per unit area and length. If the source ends to emit at the time $t_1 = t_0 + \tau$, the factor in front of the sine should be taken in the form of two Heaviside step functions $\chi(t - t_0) - \chi(t_1 - t)$ (rectangular form factor). The spectrum of the wave packet is easily calculated and becomes very narrow (almost monochromatic) at a large duration.

The problem of the propagation of the wave packet in a homogeneous medium is solved using the Green's function [1, 2]. In vacuum, the wave packet propagates as a whole at the speed of light and such a problem does not arise. The propagation law for a homogeneous dispersive medium has the form

$$E(z,t) = \int_{-\infty}^{\infty} K_0(z - z_1, t - t')u(z_1, t')dt' = \int_{-\infty}^{t} K_0(z - z_1, t - t')u(z_1, t')dt'. \quad (11)$$

Here, $K_0(z, t) = \delta(t - z/c)$ for vacuum, $K_0(z, t) = \delta(t - z/(c\sqrt{\varepsilon}))$ for the ideal medium without dispersion with the permittivity $\tilde{\varepsilon}(t) = \varepsilon\delta(t)$, and quantity $K_0(z, t)$ is defined by the integral

$$K_0(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(j(\omega t - k(\omega)z))d\omega \quad (12)$$

for the homogeneous medium with the permittivity $\varepsilon(\omega)$ and the propagation constant $k(\omega) = \omega\sqrt{\varepsilon(\omega)}/c$ [1, 2]. The Green's function given by Eq. (12) is convenient if the wave packet with a sharp front is already given, moves in a medium with dispersion, and is then tunneled. In particular, the pulse tunneling in a plasma layer in a waveguide propagates before the interaction with the layer according to law (11), for which the Green's function (12) is known and is expressed in terms of the Bessel functions [1]. According to its analytic properties, the wave packet cannot propagate in a dispersive dissipative medium at a velocity exceeding the speed of light [1]. Tunneling occurs in inhomogeneous structures and, thereby, requires other methods of analysis. In this case, it is convenient to immediately specify the wave packet with a limited spectrum and a sharp front edge. A smooth front edge rapidly decreasing ahead of the main part of the wave packet cannot be detected and its energy is infinitely low. The authors of [2] discussed the detection of the envelope of an analytic signal. It has a superluminal precursor rapidly decreasing in the forward direction, which also cannot be detected. A detector records a sharp drop of the front edge with a nonideality-induced delay. For this reason, we consider the wave packet with a sharp edge in order to calculate the spectrum. The contribution to the spectrum from the precursor is very small and is usually neglected. Let the incident wave packet with the sharp edge at the time $t_0 = 0$ approach the beginning of the barrier (layer) $z = 0$. The wave is described by the function

$$E_0(z,t) = A_0\chi(t - z/c)\sin(\omega_0(t - z/c)).$$

If this wave packet is semi-infinite, the wave before the time of arrival is located in the range $(-\infty, 0)$. The front edge is sharp but begins with zero. Such a wave packet has infinite energy, i.e., is fundamentally multiphoton. A limited pulse with the duration $\tau_0 = z_0/c$ is described by the function

$$E(z,t) = A_0[\chi(t - z/c) - \chi(t + \tau_0 - z/c)] \times\sin(\omega_0(t - z/c)).$$

The energy of the limited pulse is finite:

$$\mathcal{E} = \varepsilon_0 A_0^2 \int_{-c\tau_0}^{0} \sin^2\left(\omega_0 \frac{z}{c}\right) dz = \frac{c\varepsilon_0\tau_0 A_0^2}{2}\left(1 + \frac{\sin(2\omega_0\tau_0)}{2\omega_0\tau_0}\right), \quad (13)$$

but its mathematical spectrum is infinite and is completely determined by the signal at the point $-z_1$. The spectrum naturally cannot include photons with the energies $\omega\hbar > \mathcal{E}$. For this reason, it is impossible to use the spatially limited wave packet for quantum tunneling of the spectrally limited pulse. The energy of a semi-infinite wave packet is infinite and the mentioned difficulties do not appear. However, such a wave packet describes a photon beam. An infinite wave packet exists everywhere at all times; therefore, times of passage through certain regions can hardly be determined for it. This is also valid for quantum tunneling described by the Schrödinger equation. A monochromatic wave describes a beam of photons with the frequency ω_0. The photon density should be introduced for such a wave with a finite amplitude A_0. The spectral energy density of photons is described by the spectral intensity:

$$W(\omega) = \frac{\hbar\omega_0}{\Delta\omega} = \frac{\varepsilon_0 cS|E(\omega)|^2}{\pi} \approx 2\pi A_0^2\varepsilon_0\delta(\omega - \omega_0).$$

Here, $\Delta\omega$ is the effective spectral width. The integral of $W(\omega)$ over positive frequencies gives the energy $\mathcal{E} = \hbar\omega_0$. Such a photon can be emitted or absorbed by an atom with the Lorentzian spectral linewidth $\Delta\omega$ in the time $\Delta t \sim 1/\Delta\omega$. These processes are also multifrequency and multiparticle because the atom is not isolated from the Universe and the lifetime of its excited state depends on, e.g., the external background field, neighboring particles, and temperature. Consequently, the system is not closed and the energy is not defined exactly. At $\Delta\omega \to 0$, we have $W(\omega) = \hbar\omega_0\delta(\omega - \omega_0)$. Such a "pure" photon is a monochromatic plane electromagnetic wave defined in the infinite spacetime, which is a convenient abstraction.

High-energy photons are not dispersed and pass through a plate or a layer at the speed of light in the time $\tau = d/c$. We derive a time-dependent integral equation. The layer affects diffraction only as a region with the polarization current $J_p(z, t) = \partial_t P = \partial_t(D(z, t) - \varepsilon_0 E(z, t))$. The vector potential of the diffraction field is determined by such a polarization current and has the form

$$A_x(z,t) = \int_0^d \int_0^t g(z - z', t - t') J_p(z', t') dz' dt' = \frac{c}{2} \int_0^d \int_0^{t - |z - z'|/c} J_p(z', t') dt' dz'. \quad (14)$$

Now, the diffraction field and total field are determined as $E_d(z, t) = -\mu_0 \partial_t A_x(z, t)$ and $E(z, t) = E_0(z, t) + E_d(z, t)$, respectively. For the plasma, it is necessary to find the kernel of the integral permittivity operator in Eqs. (4) and (6). It is given by the inverse Fourier transform of the permittivity of the plasma:

$$\varepsilon(t) = \delta(t) - \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} (\omega^2 - j\omega\omega_c)^{-1} \exp(j\omega t) d\omega = \delta(t) - \frac{\omega_p^2}{\omega_c} \chi(t)[1 - \exp(-\omega_c t)]. \quad (15)$$

Therefore, we obtain the expressions

$$D(z,t) = \varepsilon_0 E(z,t) - \frac{\varepsilon_0 \omega_p^2}{\omega_c} \times \int_0^t \exp(-\omega_c(t - t')) E(z, t') dt',$$

$$\partial_t D(z,t) = \varepsilon_0 \partial_t E(z,t) - \frac{\varepsilon_0 \omega_p^2}{\omega_c} \times \left[E(z,t) - \omega_c \int_0^t \exp(-\omega_c(t - t')) E(z, t') dt' \right],$$

$$J_p(z,t) = -\frac{\varepsilon_0 \omega_p^2}{\omega_c} \times \left[E(z,t) - \omega_c \int_0^t \exp(-\omega_c(t - t')) E(z, t') dt' \right],$$

$$E_d(z,t) = \frac{\omega_p^2}{c^2} \int_0^d \int_0^{t - |z - z'|/c} \exp(-\omega_c(t - t' - |z - z'|/c)) E(z, t') dt' dz'.$$

The integral equation for the homogeneous plasma layer has the form

$$E(z,t) = \chi(t - z/c) \left\{ \sin\left(\omega_0 \left(t - \frac{z}{c}\right)\right) + \frac{\omega_p^2}{c^2} \int_0^d \int_0^{t - |z - z'|/c} \exp\left(-\omega_c \left(t - t' - \frac{|z - z'|}{c}\right)\right) E(z, t') dt' dz' \right\}. \quad (16)$$

According to its form, this equation gives the maximum tunneling speed c. This integral equation should be solved numerically. It is most interesting to consider the wave packet at the point d. The time $t = \tau_c = d/c$ corresponds to the appearance of the edge of the precursor beginning with zero: $E(d, \tau_c) = 0$. Then, excitation at the nearest points of the layer contributes to this value. The field at this point is interesting at large times when all high-energy photons of the edge have propagated ahead and only photons with the frequency ω_0 remain (a monochromatic process is established) [2]. The energy transfer speed in the steady-state wave in the plate $v_{\mathscr{E}}(z, \omega_0) < c$ will be obtained below. The tunneling time τ is determined by integrating $v_{\mathscr{E}}^{-1}(z, \omega_0)$ over the plate and is always larger than τ_c. An ideal zero-delay detector determining the time of arrival of the front edge at this point [2] would determine this time as $\tau + \pi/(2\omega_0)$. The coordinate-dependent plasma frequency $\omega_p(z)$ should be used for tunneling through an inhomogeneous layer. For tunneling, the spectrum of the wave packet should be limited by this frequency: $\omega < \omega_p(z)$. If this condition is not satisfied, tunneling occurs for the low-frequency part of the wave packet, whereas its high-frequency part propagates. Dispersion of real rarefied media is often simulated by several terms of the Lorentz dispersion:

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\omega_c}.$$

In this case, the operator $\varepsilon(t)$ is simply calculated by the method of residues and has the form

$$\varepsilon(t) = \delta(t) + \chi(t) \frac{\omega_p^2 \exp(-\omega_c t/2)}{\sqrt{\omega_0^2 - \omega_c^2/4}} \sin(t\sqrt{\omega_0^2 - \omega_c^2/4}).$$

The quantities ω_p^2 are determined in terms of the oscillator strengths of the corresponding quantum transitions with the frequencies ω_0, and the quantities ω_c and $\tau_c = 1/\omega_c$ correspond to the widths of the corresponding spectral lines and lifetimes of excited states, respectively. Assuming that $\omega_0 = 0$ (uncoupled charges of the dipole), we obtain $\varepsilon(t) = \delta(t) + \omega_p^2 \omega_c^{-1}[1 - \exp(-\omega_c t)]$, which coincides with Eq. (15). Lorentz dispersion allows the condition $\varepsilon'(\omega) = \mathrm{Re}(\varepsilon(\omega)) < 0$ under which steady-state tunneling occurs in the corresponding, usually narrow range. In terms of dimensionless frequencies Ω measured in units of the plasma frequency, this range at an extremely low collision frequency is $\Omega_0 < \Omega < \sqrt{1+\Omega_0^2}$, and this frequency at a finite lifetime is given by the condition

$$\left|\Omega^2 - \Omega_0^2 - \frac{1-\Omega_c^2}{2}\right| \le \sqrt{\frac{(1-\Omega_c^2)^2}{4} - \Omega_0^2\Omega_c^2}.$$

In the case of large dissipation (low oscillator strength), when $\Omega_c > |1 - \Omega_c^2|/(2\Omega_0)$, the permittivity is a non-negative definite function.

Let U be the energy density per unit area of the cross section of a finite wave packet with the length l approaching a layer at the time $t_0 = 0$. If the area of the cross section is S, the wave packet has the energy $\mathcal{E} = SU$, the energy density $W = U/l$, and duration $\tau_0 = l/c$. The wave packet reaches the point $z_1 = -l$ at the time $t = -l/c$. At this point, the field $E(-l, t)$ over the entire duration of the signal $-l/c \le t \le 0$, energy U, spectrum $E(\omega)$, energy density $\varepsilon_0 E$, and spectral energy density W are known:

$$\begin{aligned} E(\omega) &= \int_{-l/c}^{0} E(-l,t)\exp(-j\omega t)dt \\ &= A\int_{0}^{\tau_0} \sin(\omega_0 t)\exp(-j\omega t)dt, \end{aligned} \tag{17}$$

$$\begin{aligned} U &= \varepsilon_0\int_{-l}^{0} E^2(z,0)dz = \varepsilon_0 c\int_{-l/c}^{0} E^2(-l,t)dt \\ &= \frac{\varepsilon_0 c}{\pi}\int_{0}^{\infty}|E(\omega)|^2 d\omega, \end{aligned} \tag{18}$$

$$W(\omega) = \varepsilon_0 cS|E(\omega)|^2/\pi. \tag{19}$$

The longer the wave packet, the better the description of the ideal photon with the frequency ω_0. Because of the Wiener–Khinchin theorem, the energy of the signal given by Eq. (18) can be expressed in terms of its spectrum specified by Eq. (17). The spectrum is given by the expression

$$E(\omega) = A\frac{\omega_0 - \exp(-j\omega\tau_0)[\omega_0\cos(\omega_0\tau_0) + j\omega\sin(\omega_0\tau_0)]}{\omega_0^2 - \omega^2},$$

and the energy has the form

$$\mathcal{E} = A^2\varepsilon_0 c\tau_0[1 - \sin(2\omega_0\tau_0)/(2\omega_0\tau_0)]/2.$$

Let $2\omega_0\tau_0 = n\pi$. The spectral density W has a maximum at the frequency of the "photon" and tends to $\hbar\omega_0\delta(\omega - \omega_0)$ in the limit $n \to \infty$. The spectral density for even n values is given by the simplest expression $W(\omega_0) = \tau_0/c\pi$. It is noteworthy that the spectrum at an odd number of half-waves includes photons at zero frequency or energy. Further, we assume that the spectral problem is solved; i.e., the functions $R(\omega)$ and $T(\omega)$ are determined. The reflected wave packet is now given by the spectral integral:

$$E_R(z,t) = \chi(t)\int_{-\infty}^{\infty} R(\omega)E(\omega)\exp(j\omega(T + z/c))d\omega. \tag{20}$$

If the layer is dispersive, i.e., $\varepsilon(t) \ne \varepsilon\delta(t)$, the duration of the wave packet specified by Eq. (20) can increase because of the tail of the pulse. Similarly, the transmitted wave packet has the form

$$\begin{aligned} E_T(z,t) &= \chi(t - d/c) \\ &\times\int_{-\infty}^{\infty} T(\omega)E(\omega)\exp(j\omega(t - (z-d)/c))d\omega. \end{aligned} \tag{21}$$

It contains the precursor, front edge, and back edge (tail). Such a single "photon" does not exist. The photon should either pass through the "barrier" at the speed of light or be scattered. Multiple scattering results in the formation of quasiphotons described by Eqs. (20) and (21). In particular, dispersion in the plasma layer and delay in a double prism with a gap and frustrated total internal reflection are due to these multiphoton effects; i.e., they cannot occur for a single photon. The time of arrival of the multiphoton wave packet should be identified as the time of establishing oscillations with the frequency ω_0 at the point $z = d$. The spectrum of the transmitted wave packet can change because of reflection and possible dissipation. Nonlinear effects and appearance of combination frequencies are possible in a high field. Integral (21) can obviously be calculated and the mentioned time can be determined but always with some uncertainty. It is obvious that $\tau > d/c$; i.e., superluminal tunneling is impossible. This is due to poles of the function $T(\omega)$ in the upper half-plane. The considered formulation is convenient because it allows modeling an almost monochromatic "photon." An emitting atom in a single-photon experiment should have a very narrow spectral line (long lifetime) and radiation directed toward the barrier should be generated by only one atom. The existence of such a "photon" requires the condition $\tau_0 = 2\hbar\omega_0/(\varepsilon_0 cA^2S)$, which determines the duration of the wave packet. Here, S can be treated as the area of the target. Such a wave packet should be much longer than the barrier. Scattered photons can be detected in front of the target and behind it. However, the "photon" passed with multiple reflection cannot be treated as the initial photon because it "have interacted" with matter: a part of the wave packet is reflected and another part is absorbed; as a result, the photon has a different spectrum. The noninteracting photon is a high-energy gamma-ray photon. Any barrier is transparent for it, its velocity is equal to the speed of light, and the probabilities of its elastic and inelastic scattering are negligibly small.

Using the matching method or transfer matrix method, it is convenient to determine now the reflection, $R(\omega)$, and transmission, $T(\omega)$, coefficients for a homogeneous plate and the speed of energy transfer by a monochromatic wave in it. The results for these coefficients have the form

$$T = [\cos(kd) + j\sin(kd)(\tilde{\rho} + \tilde{\rho}^{-1})/2]^{-1}, \qquad (22)$$

$$R = R_0 \frac{1 - \exp(-2jkd)}{1 - \exp(-2jkd)R_0^2}, \qquad (23)$$

where $R_0 = (\tilde{\rho} - 1)/(\tilde{\rho} + 1)$ is the reflection coefficient from the semi-infinite layer. Here, $\tilde{\rho} = \rho/\rho_0$, where ρ_0 and ρ are the normalized impedances for incidence of a wave at the angle $\theta = \arctan(k_x/k_{0z})$, where k_x is the component of the wavevector, $k_{0z} = \sqrt{k_0^2 - k_x^2}$ in vacuum, and $k_z = \sqrt{k_0^2\varepsilon - k_x^2}$ in the plate. The normalized impedances where ρ_0 and ρ are given by the expressions $\rho_0 = k_{0z}/k_0$ and $\rho = k_z/(k_0\varepsilon)$ for E-modes (p-polarization) and $\rho_0 = k_0/k_{0z}$, $\rho = k_0/k_z$ for H-modes (s-polarization). For normal incidence, $\tilde{\rho} = \rho = 1/\sqrt{\varepsilon}$. Inside the plate, $E(z) = A^+\exp(-jkz) + A^-\exp(jkz)$, where $k = k' - jk''$ and

$$k' = k_0\sqrt{\frac{\sqrt{\varepsilon'^2 + \varepsilon''^2} + \varepsilon'}{2}}, \quad k'' = k_0\sqrt{\frac{\sqrt{\varepsilon'^2 + \varepsilon''^2} - \varepsilon'}{2}}.$$

In the case of low dissipation, when $\varepsilon''^2 \ll \varepsilon'^2$, we have

$$k' = k_0\sqrt{\frac{|\varepsilon'| + \varepsilon''^2/2|\varepsilon'| + \varepsilon'}{2}}$$

$$k'' = k_0\sqrt{\frac{|\varepsilon'| + \varepsilon''^2/2|\varepsilon'| - \varepsilon'}{2}}.$$

Correspondingly, $k' = k_0\sqrt{\varepsilon'}(1 + \varepsilon''^2/(8\varepsilon'^2))$ and $k'' = k_0\varepsilon''/(2\sqrt{\varepsilon'})$ in the propagation regime ($\varepsilon' > 1$), and $k' = k_0\varepsilon''/(2\sqrt{\varepsilon'})$ and $k'' = k_0\sqrt{\varepsilon'}(1 + \varepsilon''^2/(8\varepsilon'^2))$ in the tunneling regime ($\varepsilon' < 0$), i.e., the propagation and damping constants are mutually exchanged. The amplitudes of waves propagating in opposite directions have the form $A^{\pm} = \exp(\pm jkd)T(1 \pm \tilde{\rho})/2$, which allows the calculation of the period-averaged component of the Poynting vector $S_z = \mathrm{Re}(EH^*)/2$ and the energy density W. If the layer is nondissipative, then $|R|^2 + |T|^2 = 1$. If the layer is dissipative, i.e., $\varepsilon = \varepsilon' - j\varepsilon''$, then $|R|^2 + |T|^2 < 1$. The relations obtained are convenient for the solution of the problem of tunneling from a medium at inclined incidence on the air gap with frustrated total internal reflection, when the angle of incidence is specified by the wavevector component k_x. In the infinite homogeneous medium, $H_y/E_x = \sqrt{\varepsilon}/Z_0$, where $Z_0 = (\varepsilon_0 c)^{-1}$; hence, $S_z = \mathrm{Re}(E_x H_y^*)/2 = c\varepsilon_0|E_x|^2 k'/(2k_0)$. If the accumulated potential energy and the kinetic energy of vibrations are absent in the dissipative medium (e.g., in distilled water described by the Debye formula), when $W = \varepsilon_0(\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2})|E_x|^2/4$ [51–54]; i.e., $v_{\mathscr{E}} = c/\sqrt{(\sqrt{\varepsilon'^2 + \varepsilon''^2} + \varepsilon')/2}$. Without dissipation, $v_{\mathscr{E}} = c/\sqrt{\varepsilon'}$. Dissipation reduces $|T|$ and increases retardation. For the collisional plasma,

$$W = \frac{\varepsilon_0|E|^2}{4}\left[\left(1 + \frac{\omega_p^2}{\omega^2 + \omega_c^2}\right) + \sqrt{\left(1 - \frac{\omega_p^2}{\omega^2 + \omega_c^2}\right)^2 + \frac{\omega_p^4\omega_c^2}{(\omega^2 + \omega_c^2)^2\omega^2}}\right], \qquad (24)$$

$$S_z = \frac{c\varepsilon_0|E|^2}{2^{3/2}}\sqrt{1 - \frac{\omega_p^2}{\omega^2 + \omega_c^2} + \sqrt{\left(1 - \frac{\omega_p^2}{\omega^2 + \omega_c^2}\right)^2 + \frac{\omega_p^4\omega_c^2}{(\omega^2 + \omega_c^2)^2\omega^2}}}. \qquad (25)$$

Consequently, the energy transfer speed is given by the expression [51–54]

$$v_{\mathscr{E}} = 2^{1/2}c\frac{\sqrt{1 - \frac{\omega_p^2}{\omega^2 + \omega_c^2} + \sqrt{\left(1 - \frac{\omega_p^2}{\omega^2 + \omega_c^2}\right)^2 + \frac{\omega_p^4\omega_c^2}{(\omega^2 + \omega_c^2)^2\omega^2}}}}{1 + \frac{\omega_p^2}{\omega^2 + \omega_c^2} + \sqrt{\left(1 - \frac{\omega_p^2}{\omega^2 + \omega_c^2}\right)^2 + \frac{\omega_p^4\omega_c^2}{(\omega^2 + \omega_c^2)^2\omega^2}}}. \qquad (26)$$

These formulas can be represented in the compact form

$$W = \frac{\varepsilon_0|E|^2}{4}[2 - \varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}],$$

$$S_z = \frac{c\varepsilon_0|E|^2\sqrt{\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}}}{2^{3/2}},$$

$$v_{\mathscr{E}} = \frac{2^{1/2}c\sqrt{\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}}}{2 - \varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}}.$$

For a weakly dissipative plasma, $v_{\mathscr{E}} = c\varepsilon''/(2|\varepsilon'| + 2|\varepsilon'|^2 + \varepsilon''^2/2)$. For tunneling in the absence of dissipation,

$$v_{\mathscr{E}} = c\frac{\sqrt{|\varepsilon'|}}{1 + |\varepsilon'|} = \frac{c\omega\sqrt{1 - \omega^2/\omega_p^2}}{2\omega_p} < c.$$

Weak dissipation is possible only in the frequency range $\omega \gg \omega_c$. Near the plasma frequency, $\varepsilon' \approx 0$, the expansion is inapplicable, and it follows directly from Eq. (26) that $v_{\mathscr{E}} = \sqrt{\varepsilon''/2}/(1 + \varepsilon''/2) \ll c$. The plasma at low frequencies $\omega \ll \omega_c$ can be considered as a medium without the accumulation of the energy of vibrations [53]. Then, $\varepsilon(\omega) = -j\omega_p^2/(\omega\omega_c) = -j\sigma\varepsilon_0/\omega$, and it follows from Eq. (26) that $v_{\mathscr{E}} \approx 2c\sqrt{\omega\omega_c}/\omega_p \ll c$. Dispersion in such a medium is due to the conductivity σ. An expression for $v_{\mathscr{E}}$ similar to Eq. (26) can be obtained in the case of Lorentz dispersion, for which also $v_{\mathscr{E}} \le c$. The corresponding lengthy expression is not presented here. The inequality $\varepsilon' < 0$ and tunneling are possible for such a medium in the region of anomalous negative dispersion at small dissipation. It is noteworthy that Lorentz dispersion at zero resonant frequency $\omega_0 = 0$ (free oscillators) gives the dispersion of the plasma, and the passage to the limit $\omega_0 \to \infty$, $\omega_c \to \infty$, $\omega_p \to \infty$ (infinitely rigid dipoles) under the condition $\omega_p^2/\omega_0^2 = \kappa$, $\omega_c^2/\omega_0^4/\tau^2$ yields the Debye formula $\varepsilon' = 1 + \kappa/(1 + \omega^2\tau^2)$, $\varepsilon'' = (\varepsilon' - 1)\omega\tau$. In the collisionless plasma, $v_{\mathscr{E}} = c\sqrt{1 - \omega_p^2/\omega^2}$, which coincides with the group velocity. The group velocity below the plasma frequency is imaginary; i.e., propagation is impossible. However, tunneling through the collisionless plasma layer occurs at the speed below the speed of light both below and above the plasma frequency taking into account that the power flux density is proportional to $|T|^2$. This speed is below c in both the presence and absence of dissipation and depends on the coordinate. The group velocity corresponds to the energy transfer speed, according to the definition by W. Hamilton, only for a monochromatic wave and only in absolutely nondissipative (conservative or Hamiltonian) systems and media under the conditions of the Leontovich–Lighthill–Rytov theorem [55]. Only in these cases, the group velocity is a real quantity transformed as a polar vector, i.e., as the velocity of a material point. In stopbands, in particular, inside barriers, the group velocity is a kinematic quantity determining the velocity of motion of two waves infinitely close in frequency (according to the definition by G.G. Stokes) and can have any magnitude: exceeding c, infinite, and even negative (directed against the direction of energy transfer) [56]. The same is true for the group delay time or Bohm–Wigner time, which can be zero and negative [38–44]. For this reason, the velocity of motion of the wave packet, as well as the energy transfer speed by the wave packet, particularly at a sufficiently wide spectrum, should not be treated as the group velocity.

According to Eqs. (22) and (23) obtained by matching the transverse components, $S_z(z)$ in the absence of dissipation is continuous at all points including the points 0 and d, and $2Z_0S_z(d) = |T|^2$, $2Z_0S_z(0) = (1 - |R|^2)$. In the notation $\tilde{\rho} = \tilde{\rho}' + j\tilde{\rho}''$, we obtain

$$S_z(z) = \frac{c\varepsilon_0|T|^2(\rho'\alpha(z) - \rho''\beta(z))}{2},$$

where

$$\alpha(z) = \frac{1}{4|\tilde{\rho}|^2}[|1 + \tilde{\rho}|^2\exp(-k''(z - d)) - |1 - \tilde{\rho}|^2\exp(k''(z - d))],$$

$$\beta(z) = \frac{1}{|\tilde{\rho}|^2}[\sin(2k'(z - d))(1 - |\tilde{\rho}|^2)/2 - \tilde{\rho}''\cos(2k'(z - d))].$$

For the plasma layer instead of the plate, we have

$$W(z) = \varepsilon_0|T|^2\left(1 + \frac{\sqrt{\varepsilon'^2 + \varepsilon''^2} - \varepsilon'}{2}\right)\frac{a^2(z) + b^2(z)}{4},$$

where

$$a(z) = \cos(k'(z - d))[\cosh(k''(z - d)) - \tilde{\rho}'\sinh(k''(z - d))] + \tilde{\rho}''\sin(k'(z - d))\cosh(k''(z - d)),$$

$$b(z) = \tilde{\rho}''\cos(k'(z - d))\sinh(k''(z - d)) + \sin(k'(z - d))[\tilde{\rho}'\cosh(k''(z - d)) - \sinh(k''(z - d))].$$

Therefore, the energy transfer speed is given by the formula

$$v_{\mathscr{E}}(z) = c\frac{\tilde{\rho}'\alpha(z) - \tilde{\rho}''\beta(z)}{[1 + (\sqrt{\varepsilon'^2 + \varepsilon''^2} - \varepsilon')/2](a^2(z) + b^2(z))}. \quad (27)$$

This speed is always lower than the speed of light. For the ideally transparent plate, the tunneling speed is

$$v_{\mathscr{E}}(z) = \frac{2c}{\varepsilon + 1 - (\varepsilon - 1)\cos(2k_0\sqrt{\varepsilon}(z - d))} \le c$$

and the tunneling time is

$$\tau = \frac{d[\varepsilon + 1 - (\varepsilon - 1)\operatorname{sinc}(2k_0d\sqrt{\varepsilon})]}{2c} \ge \frac{d}{c}.$$

In the case of tunneling through a nondissipative layer with a negative permittivity,

$$E = A^+\exp(-k''z) + A^-\exp(k''z),$$

where

$$H = -jc\varepsilon_0\sqrt{|\varepsilon|}(A^+\exp(-k''z) - A^-\exp(k''z)),$$

$$k'' = k_0\sqrt{|\varepsilon|}, \quad A^\pm = \frac{\exp(\pm k''d)T}{2}\left(1 \pm \frac{j}{\sqrt{|\varepsilon|}}\right).$$

Consequently,

$$S_z(z) = \frac{c\varepsilon_0|T|^2}{2}, \quad W = \frac{\varepsilon_0|E|^2(2-\varepsilon+|\varepsilon|)}{4},$$

$$|E|^2 = |T|^2\left[\cosh^2(k''(z-d)) + \frac{\sinh^2(k''(z-d))}{|\varepsilon|}\right],$$

and the energy transfer speed at a frequency below the plasma frequency is

$$v_{\mathscr{E}}(z,\omega) = \frac{c(\omega_p^2/\omega^2-1)}{(\omega_p^2/\omega^2)[\omega_p^2/\omega^2\cosh^2(k''(z-d))-1]} < c. \quad (28)$$

Relation (28) is approximate since for simplicity we took the energy density $W = W_E + W_H$ as for a wave of one direction, i.e., $W_E = W_H$ neglecting the small reflected wave, which is justified. According to Eq. (28), $v_{\mathscr{E}}(z, \omega_p) = 0$, $v_{\mathscr{E}}(d, \omega) = c(\omega/\omega_p)^2 \le c$, and $v_{\mathscr{E}}(\omega)/c \approx \omega^2/[\omega_p^2\cosh^2(k''(z-d))]$ at $\omega \ll \omega_p$. This speed is very low, particularly at the beginning of the wide barrier.

We now consider another method based on the inverse Fourier transform of the quantity $E(z, \omega)T(\omega)$ that determines the transmitted field. Here, $T(\omega)$ is the transmission coefficient given by Eq. (22) with respect to the point $z = d$, and $E(0, \omega)$ is the spectrum of the incident pulse in front of the barrier. Therefore, it is necessary to add the factor $\exp(-jk_0z)$ and to consider the integral

$$E_T(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(0,\omega)T(\omega)\exp(j\omega(t-z/c))d\omega. \quad (29)$$

As seen, this integral is nonzero only if $t > z/c$, i.e., at the point $z = d$ and time $t = d/c$. Indeed, the function $E(0, \omega)$ has poles on the real frequency axis, whereas all poles of the function $T(\omega)$ are located in the upper half-plane [1, 2]. Shifting the contour of integration at $t < z/c$ to the lower half-plane, we find that integral (29) is zero.

We now consider the time of time-dependent tunneling. Knowing the field $E(z, t)$ and $H(z, t)$, one can calculate $S_z(z, t) = E(z, t)H(z, t)$. However, the energy density $W(z, t)$ cannot be obtained in the general form. Only the quantity $\partial_t W(z, t)$ is known from the Poynting theorem. Therefore, the quantity

$$W(z,t) = \int_{-\infty}^{t}\partial_{t'}W(z,t')dt'$$

can be calculated at each point, knowing the prehistory of the generation of the field by sources or the prehistory of the arrival of the pulse with the sharp front at the point z. Correspondingly, the velocity can be determined as

$$v_{\mathscr{E}}(z,t) = \frac{S_z(z,t)}{\int_{-\infty}^{t}\partial_{t'}W(z,t')dt'}.$$

Unfortunately, a general proof that this velocity is always lower than c is absent. It can only be stated that the velocity of motion of sharp fronts for the Poynting vector does not exceed c. The corresponding time interval is given by the formula

$$\tau(t) = \int_0^d\int_{-\infty}^{t}\frac{\partial_{t'}W(z,t')}{S_z(z,t)}dt'dz$$

$$= \int_0^d\int_{-\infty}^{t}\frac{E(z,t')\partial_{t'}D_x(z,t') + \mu_0 H(z,t')\partial_{t'}H(z,t')}{E(z,t)H(z,t)}dt'dz.$$

This time interval depends on the current time, which is reasonable for the time-dependent theory.

3. TIME-DEPENDENT QUANTUM TUNNELING

Time-dependent quantum tunneling is described by the solution of the Schrödinger equation

$$i\hbar\partial_t\psi(z,t) = \hat{H}\psi(z,t) = \left[\frac{\hat{p}^2}{\mu_e} + V(z,t)\right]\psi(z,t), \quad (30)$$

where $\hat{p} = -i\hbar\partial_z$ is the z-component momentum operator and $\mu_e = 2m_e$. The problem is difficult because the wavefunction $\psi(z, t)$ is nonlocal and exists in the entire space. The probability density of finding the electron $|\psi(z, t)|^2$ is also a nonlocal function. Consequently, the position of the electron cannot be measured at the point $z = 0$ at the time t_0 in front of the barrier and, then, at the point $z = d$ and time $t_0 + \tau$ because each such measurement results in the collapse of the wavefunction and in the uncertainty in the momentum of the electron, i.e., again in its delocalization. If the wave packet is almost monochromatic, i.e., is almost infinitely extended and is much wider than the barrier, the problem of times of its passage is the more so open. Just in this case, the stationary phase approximation is often used for the transmission coefficient

$$T = \left[\cosh(k''d) - i\sinh(k''d)\frac{\tilde{\rho}'' - \tilde{\rho}''^{-1}}{2}\right]^{-1},$$

which leads to the Bohm–Wigner time $\tau_{BW} = \hbar\partial_{\mathscr{E}}\phi$ for the phase $\phi = \arg(T)$, i.e.,

$$\phi = \arctan\left[\tanh\left(\sqrt{\mu_e(V-\mathscr{E})}\frac{d}{\hbar}\right)\left(\frac{2-V/\mathscr{E}}{2\sqrt{V/\mathscr{E}-1}}\right)\right].$$

The saturation for a wide barrier gives $\tau_{BW} = \hbar/\sqrt{V\mathscr{E}-\mathscr{E}^2}$, which does not reflect the process of tunneling. Measurement should be treated as the introduction of an additional potential to Eq. (30) at the time of measurement. For this reason, the time-dependent tunneling time of the wave packet is meaningless. The wave packet is always infinite, polychromatic, and dispersive, i.e., varying in the process of motion even in the absence of the potential. Interference maxima appear in the wave packet incident on the barrier before the principal maximum approaches the barrier. The wave packet is split and inverse probability densities arise in it in front of the barrier and inside it. Hence, the energy velocity $v_{\mathscr{E}}(z, t) = j(z, t)/|\psi(z, t)|^2$ also does not determine the velocity of an individual particle. Here,

$$j(z,t) = i\hbar\mu_e^{-1}[\partial_z\psi^*(z,t)\psi(z,t) - \psi^*(z,t)\partial_z\psi(z,t)].$$

However, the velocity in a steady beam can be determined. The velocity in the monochromatic steady beam coincides with the velocity of incident particles even inside the barrier, where it should be imaginary for a negative energy, but depends on the coordinate taking into account the reflected beam. If the electron beam incident on the barrier from the interior (e.g., on the cathode–vacuum interface) is given and the anode voltage begins to vary at the time t_0, it is possible to consider the delay time of the variation of the anode current, i.e., the tunneling time through the barrier.

The problem of tunneling is usually considered in the infinite region. The relativistic quantum scattering theory deals with the incident and scattered amplitudes at $z = \pm\infty$ and times $t = \pm\infty$, whereas the interaction time is usually ignored. However, according to the Schrödinger equation, the wave packet at $t \to \infty$ is extended over the infinite region. The time-dependent Schrödinger equation is not relativistically covariant and, thereby, allows infinite velocities. In particular, the wavefunction in the form of a wave packet with a fixed momentum range is extended over the entire space [19]. The response of the Green's function to the appearance of the probability density exists in the entire infinite region [57]. Correspondingly, the Green's function (propagator) $K_0(z, t) = \sqrt{\mu_e/(4i\pi\hbar t)}\exp(iz^2\mu_e/(4\hbar|t|))$ of the free field ψ can be introduced [57]. Even the wave packet $\varphi(z)$ limited in the spatial region $a < z < b$ at the time $t_0 = 0$ becomes unlimited at $t > t_0$:

$$\psi(z,t) = \int_a^b K_0(z-z',t)\varphi(z')dz'.$$

However, the spatially limited wave packet cannot be represented as an integral over a finite momentum range; i.e.,

$$\varphi(z) = \int_{-\infty}^{\infty} A(k_0)\exp(-ik_0 z)dk_0. \quad (31)$$

According to the wavefunction given by Eq. (31), such a "particle" in the range $a \le z \le b$ can have infinite momentum and energy; at the localization of the particle ($a = b$), we have

$$|\varphi(z)|^2 = \delta(z-a) = \frac{1}{\pi}\int_0^{\infty}\cos(k_0(z-a))dk_0.$$

Such a particle cannot be considered as free and incident on the barrier. The localization of the free particle results in the collapse of the wavefunction and is impossible in the absence of external fields. For this reason, the free particle should be described as a wave packet that is specified in a finite momentum range and is infinite in space. The narrower the spectrum, the better the correspondence of the wave packet to a plane wave and the better the correspondence of tunneling to the steady state. For time-dependent tunneling of the particle, the Green's function should be defined in a region with a potential. This Green's function satisfies the integral equation [57]

$$G(z,t|z',t') = K_0(z,t|z',t') - i\hbar^{-1} \times \iint K_0(z,t|z'',t'')V(z'',t'')G(z'',t''|z',t')dz''dt'', \quad (32)$$

and the motion of the "particle" is described by the function

$$\psi(z,t) = \int_{-\infty}^{t}\int_{-\infty}^{\infty} G(z,t|z',t')\psi(z',t')dz'dt'. \quad (33)$$

In the presence of the potential, the energy is not conserved in time (the system is nonconservative). Let the wavefunction at the initial time $t_0 = 0$ be a wave packet $\varphi(z, t)$ and the potential before the time $t_0 = 0$ be absent. Such a wave packet at the point $-z_0$ can be taken Gaussian, e.g.,

$$\varphi(z,t) = \int_{k_0-\Delta k}^{k_0+\Delta k}\exp\left(ik(z+z_0) - \frac{i\hbar k^2}{\mu_e}t - \frac{(k-k_0)^2}{2\Delta k^2}\right)dk. \quad (34)$$

It satisfies the Schrödinger equation with $V \equiv 0$ and has a momentum spread $\pm\Delta p = \pm\hbar\Delta k$, its probability density has a maximum at $k = k_0$, and its spectral density $\Phi(k) = \exp(-(k-k_0)^2/(2\Delta k))$ begins to vary according to Eqs. (32) and (33) at the appearance of the potential. This spectral density for free motion in the absence of the potential is also dispersed because of polychromatic motion described by Eq. (34). This is clear at the decomposition

$$\omega(k) = \omega(k_0) + \partial_k\omega(k_0)\Delta k + \partial_k^2\omega(k_0)\frac{\Delta k^2}{2}$$

and substitution into Eq. (34). Taking into account that $\partial_k\omega(k_0) = v_g(k_0)$, $\partial_k^2\omega(k_0) = \partial_k v_g(k_0)$, it is seen that the wave packet is dispersed because of the dispersion of the group velocity. A correction is based on the Airy function [1]. Here, the limit $\Delta k \to 0$ corresponds to localization in the momentum space (since the Gaussian form factor tends to $\delta(k - k_0)$), whereas the wave packet at $\Delta k = \infty$ has infinite limits and a uniform momentum distribution density. A single group velocity is certainly insufficient to describe the wave packet wide in the momentum space (i.e., strongly localized in the coordinate space). A more accurate description is based on several group velocities $v_g(k_n) = \hbar k_n/m_e$ for several n values from the spectrum [1]. We consider now the wave packet in the form

$$\varphi(z,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} \exp\left(ik(z+z_0) - \frac{i\hbar k^2}{\mu_e}t - \frac{|z+z_0||k-k_0|}{2\Delta z\Delta k}\right)dk. \quad (35)$$

It is localized near the spatial point z_0 and near $\hbar k_0$ in the momentum space, but it does not satisfy the Schrödinger equation for the free particle. The action of the Schrödinger operator $\hat{S} = i\hbar\partial_t + \hbar^2\partial_z^2/\mu_e$ on the wave packet given by Eq. (35) gives not zero but the function $\varphi'(z, t) = V(z, t)\varphi(z, t)$, which can be attributed to the potential

$$V(z,t) = -\frac{\delta(z+z_0)\varphi_1(z,t) - \varphi_2(z,t)}{\mu_e\Delta z\Delta k\,\varphi(z,t)},$$

where

$$\varphi_1(t) = \int_{k_0-\Delta k}^{k_0+\Delta k} |k-k_0|\exp\left(-\frac{i\hbar k^2 t}{\mu_e}\right)dk,$$

$$\varphi_2(z,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} \frac{(k-k_0)^2}{4\Delta z\Delta k} \times\exp\left(ik(z+z_0) - \frac{i\hbar k^2}{\mu_e}t - \frac{|z+z_0||k-k_0|}{2\Delta z\Delta k}\right)dk.$$

The localization of the wave packet can be associated with the additional interaction potential (appearing, e.g., at the emission of an electron from an atom). The wave packet localized in the coordinate and momentum spaces can be written in the form

$$\varphi(z,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} \exp\left(ik(z+z_0) - \frac{i\hbar k^2}{\mu_e}t - \alpha\frac{|z+z_0||k-k_0|}{2\Delta z\Delta k}\right)dk. \quad (36)$$

The factor α is responsible for the localization of the envelope. At $\alpha = 0$, localization is only due to the stationary phase. The wave packet specified by Eq. (36) moves with dispersion in the both directions and satisfies the inhomogeneous Schrödinger equation $\hat{S}\varphi(z, t) = \delta(z + z_0)\varphi_1(z, t) + \varphi_2(z, t)$. The function $\varphi_2(z, t)$ differs from Eq. (36) in the factor $\hbar^2\alpha^2/(4\Delta z^2\Delta k^2\mu_e)$ in front of the integral and the factor $(k - k_0)^2$ in the integrand. At $\alpha = 0$, i.e., $\varphi_2(z, t) = 0$, the wavefunction given by Eq. (36) satisfies the Schrödinger equation with the potential $V(z, t) = \delta(z + z_0)\varphi_1(z, t)/\varphi(z, t)$, where

$$\varphi_1(z,t) = \frac{2\hbar^2}{\mu_e}\int_{k_0-\Delta k}^{k_0+\Delta k}\left(ik + \alpha\frac{|k-k_0|}{2\Delta z\Delta k}\right) \times\exp\left(-\frac{i\hbar k^2}{\mu_e}t\right)dk.$$

Here, the maximum (but incomplete) localization occurs at the time t_0 and the point $-z_0$, but this is possible only if the potential is proportional to the delta function. Finally, let the particle at the time t_0 be completely localized at the point $-z_0$. Its wavefunction should satisfy the initial condition $\varphi(z, t_0) = \delta(z - z_0)$ and the Schrödinger equation at $t > t_0$. Consequently, it coincides with the Green's function introduced above: $\varphi(z, t) = K_0(z - z_0, t - t_0)$. As seen, small deviation $t = t_0 + \Delta t$ will result in delocalization; i.e., the function is nonzero in the entire space at any Δt value. However, the difference at large distances is exponentially small. Delocalization increases with time: the wave packet is dispersed in all directions. Then, the propagation of such (and any other) a wave packet can be described by the propagator of the form

$$\varphi(z,t) = \int_{-\infty}^{\infty} K_0(z-z', t-t')\varphi(z',t')dz'.$$

Here, nonlocality is fundamental: motion from the point z' at the time t' requires integration over the entire infinite coordinate range. The representation of the wave packet with an arbitrary spectral amplitude $A(k)$ is often used. In this case, the first approximation of dispersion theory for the wave packet narrow in k gives the wavefunction $\psi(z, t) \approx \tilde{A}(z, t, k_0)\exp(ik_0(z + z_0) - \omega t)$ with the envelope $\tilde{A}(z, t, k_0) = 2A(k_0)\,\mathrm{sinc}((z + z_0) - tv_g(k_0))$. Here, $\mathrm{sinc}(x) = \sin(x)/x$. Such a wavefunction is not localized although it has the principal maximum at the point $-z_0$.

In view of superposition, the wavefunction at $t > t_0$ should be sought as the sum of the solution of the Schrödinger equation with $V = 0$,

$$\psi_0(z,t) = \int_{-\infty}^{\infty} K_0(z - z',t)\varphi(z')dz'$$

and the solution due to the interaction,

$$\psi_V(z,t) = -i\hbar\int_0^t\int_{-\infty}^{\infty} G_0(z,z',t-t')V(z',t')\psi(z',t')dz'\,dt'.$$

As a result, we obtain the integral equation for the wavefunction

$$\psi(z,t) = \psi_0(z,t) + \psi_V(z,t). \quad (37)$$

The action of the Schrödinger operator $\hat{S}$ indicates that $\psi(z, t)$ satisfies the Schrödinger equation (30).

Thus, it is possible to ensure sufficient localization of the particle at a given time with some momentum spread near the mean value in the absence of interaction and to analyze how such a "particle" approaches the barrier. The moving wave packet is dispersed and extended to the entire region, as well as interferes with the barrier already before the arrival of the principal maximum. Consequently, it is necessary to specify a certain edge of the wave packet and to switch on of the barrier when it approaches. It is hardly convenient to use the maximum for this aim because this does not reflect the formulation of the problem. The barrier can be switched on both smoothly and suddenly. Then, it is necessary to use the Green's function G and Eq. (33). The solution requires the calculation of complex integrals and can be obtained only numerically. The diagram technique and perturbation theory are often used [57]. It is remarkable that Hartman [15] considered a static potential and believed that wave packets beyond the barrier satisfy the Schrödinger equation for a free particle, which is invalid. The maximum moves at the velocity $v_z = \hbar k_0/m_e$, but its approach to the barrier is accompanied by reflections and interference, and several maxima can appear. Nevertheless, the time of its arrival can be interpolated in terms of the velocity. Knowing the steady-state tunneling time, one can determine an approximate time of arrival of the maximum at the barrier. The Green's function allows the approximate determination of the maximum at the output after a certain time. This time should be larger than that necessary for the transmission of the barrier at the velocity v_z. The tunneling time can be calculated in terms of the position of the maximum and velocity v_z. However, tunneling distorts the spectral content of the outgoing wave packet: transmittance (transparency) is usually higher for high k values. The output spectrum is determined by the function $T(\omega)$. The definitions of the beam and spectrum and all related quantities are meaningful only in the region behind the barrier, where the beam is unidirectional. The presented scheme is likely most reasonable for the definition of the speed and time of time-dependent tunneling. The result will apparently close to the average time and speed in the steady-state problem.

We discuss the energy velocity $v_{\mathscr{E}}$. Solving the integral equation (19), we obtain

$$j(z,t) = \frac{1}{i\hbar\mu_e}[\partial_z\psi^*(z,t)\psi(z,t) - \psi^*(z,t)\partial_z\psi(z,t)],$$

$$v_{\mathscr{E}}(z,t) = \frac{j(z,t)}{|\psi(z,t)|^2}.$$

This velocity depends on the coordinate and time (as should be in the time-dependent case) and can be defined in any region, including the barrier. It can be averaged over the barrier; the result of this averaging $\bar{v}_{\mathscr{E}}(t)$ depends only on the time. It is reasonable to define the tunneling time as

$$\tau(t) = \int_0^d \frac{dz}{v_{\mathscr{E}}(z,t)}.$$

This time is a function of the time and determines the dwell time. It is necessary to find the time $t_{\min}$ when the tunneling time is minimal. It is possible to consider a certain average velocity $\langle v_{\mathscr{E}}\rangle$ and an average tunneling time $\bar{\tau} = d/\langle v_{\mathscr{E}}\rangle$ if $\bar{v}_{\mathscr{E}}(t)$ is averaged near this time:

$$\langle v_{\mathscr{E}}\rangle = \frac{1}{\tau_{\min}}\int_{t_{\min}-\tau_{\min/2}}^{t_{\min}+\tau_{\min}/2} \bar{v}_{\mathscr{E}}(t)dt.$$

Because of reflections from the boundaries, the wavefunction inside the barrier does not vanish sharply with time, and the infinite limits of integration can distort the tunneling time. When a finite wave packet of electromagnetic waves passes through the plate, infinite tails in the transmitted and reflected pulses appear because of infinite reflections and damping oscillations arise in the plate region.

4. QUASISTATIONARY APPROACH TO QUANTUM TUNNELING

Let a static rectangular barrier $V(z) = V_0 > E$ and an incident electron beam with the energy $\mathscr{E}$ and unit probability density $|\psi^+(z)|^2 = 1$ exist until the time $t_0 = 0$, the reflection, $R(\mathscr{E})$, and transmission, $T(\mathscr{E})$, coefficients, as well as the wavefunction $\psi(z) = A^+\exp(-k''z) + A^-\exp(k''z)$ in the range $0 \le z \le$ d, the wavefunction $\psi(z) = \exp(ikz) + R\exp(-ikz)$ on the left of the barrier, and the wavefunction $\psi(z) = T\exp(ik_0(z - d))$ on the right of the barrier be known. Here,

$$A^{\pm} = \frac{T(1 \mp i\kappa)\exp(\pm k''d)}{2}, \quad T = |T|\exp(i\phi),$$

$$k'' = \sqrt{\frac{\mu_e(V - \mathcal{E})}{\hbar^2}}, \quad \kappa = \rho'' = \sqrt{\frac{\mathcal{E}}{V - \mathcal{E}}},$$

$$k = \frac{\sqrt{\mu_e \mathcal{E}}}{\hbar}.$$

Let the time-dependent potential $U(z, t)$ appear at the time $t_0 = 0$ for the time interval τ. The problem cannot be solved separately, i.e., satisfying the Schrödinger equation separately in three regions and matching the solutions (this was one of the mistakes by Hartman). As can be shown, such a solution in the region of the barrier in the form

$$\psi(z,t) = \exp\left(-\frac{i\mathcal{E}t}{\hbar}\right)\left[\psi(z) + \sum_{n=0}^{\infty} a_n(t)\cos(n\pi z/d)\right]$$

allows obtaining differential equations for $a_n(t)$ and their solutions, but leads to a contradiction at the matching of the wavefunction. Let us seek the solution in the form of a superposition in the entire region: $\psi(z, t) = \exp(-i\mathcal{E}t/\hbar)\psi(z) + \Delta\psi(z, t)$. In view of linearity, we have

$$\psi(z,t) = \exp\left(-\frac{i\mathcal{E}t}{\hbar}\right)\psi(z) + \int_0^{\tau}\int_0^{d} K_0(z - z', t - t')U(z', t')\psi(z', t')dz'dt'. \quad (38)$$

This integral equation is simplified for a static rectangular potential jump $U(z, t) = U_0$. In the infinite region, it is convenient to use the perturbation method, which can be treated as multiple scattering by the potential U_0. In the case of single scattering at $t > \tau$,

$$\psi(z,t) = \exp\left(-\frac{i\mathcal{E}t}{\hbar}\right)\psi(z) - i\frac{U_0}{\hbar}\int_0^{\tau}\int_0^{d} K_0(z - z', t - t') \times\exp\left(-\frac{i\mathcal{E}(t - t')}{\hbar}\right)\psi(z - z')dz'\,dt'.$$

This integral can be calculated numerically and change in the quantity $|\psi(d, t)|^2$ compared to $|T|^2$ can be estimated. If the potential appears suddenly as the delta function $\delta(t)$,

$$\psi(z,t) = \exp\left(-\frac{i\mathcal{E}t}{\hbar}\right)\psi(z) - i\hbar U_0\exp\left(-\frac{i\mathcal{E}t}{\hbar}\right) \times\int_0^{d} K_0(z - z', t)\psi(z')dz'.$$

For a wide barrier,

$$\psi(z) \approx \frac{|T|}{2}\exp(i\phi)(1 - i\kappa)\exp(-k''(z - d)).$$

Now, it is necessary to calculate the integral. Its estimate by the mean-value theorem at the point $z' = d/2$ gives

$$\psi(d,t) = |T|\exp\left(-\frac{i\mathcal{E}t}{\hbar} + i\phi\right) \times\left[1 - \frac{id\hbar^{-1}U_0(1 - i\kappa)}{2}\exp\left(\frac{k''d}{2}\right) \times\sqrt{\frac{\mu_e}{4i\pi\hbar t}}\exp\left(i\frac{d^2\mu_e}{16\hbar t}\right)\right].$$

This result means an instantaneous change in the wavefunction. However, it does not correspond to the tunneling time. The potential instantaneously acts on the de Broglie wave inside the barrier and beyond it. It is seen that the wavefunction changes at large times because of the action of the potential changing the energy of the system. Finally, we consider a wave packet with an infinitely narrow spectrum $\Delta k \to 0$ in the space free of the potential: $\psi_0(z, t) = \mathrm{sinc}(\Delta k(z - tv))\exp(ik_0 z - i\mathcal{E}t/\hbar)$. Here, $v = \sqrt{\mu_e \mathcal{E}}$ is the velocity of the incident particle. The maximum $|\psi(z, t)|^2 = 1$ is reached at the time $t = 0$ and point $z = 0$. It can be asked where the maximum occurs, e.g., at the time $\tau = d/v$ if the rectangular potential barrier V_0 appears at the time $t = 0$ for the time interval τ:

$$\psi(z,\tau) = \psi_0(z,\tau) - i\hbar^{-1}V_0\sqrt{\frac{\mu_e}{4i\pi\hbar}} \times\int_0^{\tau}\int_0^{d}\frac{\sin(\Delta k[(z - z') - (\tau - t)v])}{\Delta k[(z - z') - (\tau - t)v} \times\exp(i\varphi(z, z', t, \tau))dz'dt, \quad (39)$$

where

$$\varphi(z, z', t, \tau) = \frac{(z - z')^2\mu_e}{4\hbar(\tau - t)} + k_0(z - z') - \frac{\mathcal{E}(\tau - t)}{\hbar}.$$

Here, the first order of the perturbation theory was used. The integral is a rapidly oscillating function. According to the structure of the wavefunction given by Eq. (39), the maximum at a very small V_0 value will be determined by the term $\psi_0(z, \tau)$, i.e., by the point of the maximum $z_0 = d$. In this case, any relation between V_0 and $\mathcal{E}$ is possible. In the stationary phase approximation with $\partial_{z'}\varphi(z, z', t, \tau) = 0$, or $z_s'(t) = z + 2\hbar(\tau - t)k_0/\mu_e$, the double integral in Eq. (39) is reduced to the single integral

$$\frac{d}{2\Delta k v}\int_0^{\tau}\frac{\sin(2\Delta k t v)}{t}\exp\left(-\frac{2it\mathcal{E}}{\hbar}\right)dt = I_0 - I_1 + \dots .$$

Here, we use the expansion $\sin(2\Delta k t v) \approx 2\Delta k t v - 8(\Delta k t v)^3/3 + \dots$ in the small parameter Δk, but the integral can be calculated in terms of the exponential integral. As a result, $I_0 = d(1 - \exp(-2i\tau\mathcal{E}/\hbar))/(2i\mathcal{E}/\hbar)$. The integral I_1, as well as next corrections, is calculated easily, but we retain only I_0. As a result,

$$\psi(z,\tau) = \psi_0(z,\tau) - d(V_0/\mathscr{E}) \times \sqrt{\frac{\mu_e}{4i\pi\hbar}}\left[1 - \exp\left(-\frac{2i\mathscr{E}\tau}{\hbar}\right)\right].$$

Addition to the wavefunction is independent of z; therefore, the maximum remains at the point d, but becomes more smeared. The strict consideration requires the solution of the integral equation (38), but the velocity of motion of the maximum in this case should also have the same order.

5. CONCLUSIONS

The problems of speeds and times of tunneling or transmission through certain regions should be formulated in terms of correct definitions. Steady-state tunneling of photons is a multiphoton process involving an incident beam of photons with the same frequency. Quasiphotons (polaritons) propagate in the region of the barrier or layer at a different phase velocity, which can be higher than c, but the speed of energy transfer by them is always below c. The spectral parameters $\hat{\varepsilon}(\mathbf{r}, \omega)$ and $\hat{\mu}(\mathbf{r}, \omega)$ of the linear medium, which can generally be inhomogeneous and anisotropic (even bianisotropic), as well as correct expressions for the energy density, are important for the determination of velocities. The spectral properties ensure the principle of causality. The scattering parameters for the inhomogeneous barrier or layer are obtained by solving an integral equation or by integrating a Helmholtz-type wave differential equation [58–60]. For time-dependent tunneling, it is necessary to consider the motion of a nonlocal wave packet. Here, its spectrum is important. The wave packet for electromagnetic waves can have a sharp front with discontinuity moving at the speed of light, whereas other its part moves in matter at a velocity below c. The packet of electromagnetic waves with a finite energy is fundamentally delocalized. However, the motion of its main part in this case cannot be superluminal. A finite wave packet is not single-photon, and an almost single-photon wave packet after interaction is not already the initial (incident) "photon." For this reason, the superluminal detection of such a photon in some experiments, particularly with interferometers is doubtful. Reports such as [61] on the superluminal propagation of microwaves through narrowed segments of waveguide or light through a gap in double prisms with frustrated total internal reflection are also doubtful. The picture of propagation of evanescent waves in the waveguide is similar to the motion of waves in a channel with a liquid, but it is three-dimensional (see [62]). The narrowing of the channel leads to reflection and passage of a small part of fast waves at the same velocity, and the time of transmission of the main slow part of the pulse only increases. The integral equation and functionals for $R(\omega)$ and $T(\omega)$ can be derived for the spectral problem. Specifying approximately the field in the narrow section, one can easily obtain their explicit form. The Fourier transform gives the transmitted and reflected wave packets. The approximate solution does not change the poles of the transmission coefficient $T(\omega)$, which lie in the upper half-plane. This means that the signal cannot appear before the time of passage of a narrow segment at the speed of light. Dispersion cannot occur without dissipation, which (as well as reflections) changes the spectral composition of the wave packet and results in its dispersion. The detection of the wave packet requires the determination of its envelope based on an analytical signal. The time of arrival of the front edge should be defined as the time of the maximum of the envelope derivative (the maximum steepness of the front). Tunneling is the interference of processes of multiple reflections and transmissions of the layer or barrier at a subluminal velocity, which cannot give the resulting superluminal velocities.

The wavefunction of the electron or other particle described by the time-dependent Schrödinger equation is not localized in the coordinate and momentum spaces, as well as in time. It is impossible to strictly determine the velocity of an individual particle and its tunneling time. The description of the particle by the wave packet is approximate because neither the coordinate nor the momentum is specified exactly. The wave packet begins to strongly interfere, which results in the formation of several maxima already near the barrier, and is split into two wave packets propagating with dispersion in different directions after "tunneling." Tunneling occurs for the wave packet and probability density rather than for the particle, which means that the particle in a series of identical experiments can be found in the states $z = \pm\infty$ with the probabilities whose ratio is $|T/R|^2$. It is possible to determine the velocities of the maxima of the reflected and transmitted packets far from the barrier or the maximum of the derivative of their envelope at a certain point of the (z, t) plane. However, this cannot be made analytically and requires numerical simulation, in particular, based on the equations presented above. For an arbitrary barrier, it is not obvious that $|\psi(z, t)|^2$ has only one maximum in the range $0 < z < d$. Similar to the quasistationary monochromatic (long) wave packet, it is meaningless to define steady-state tunneling times in terms of the complex coefficient $T(\omega)$.

A longer wave packet more accurately corresponds to the particle, but when the wave packet is much longer than d, it is meaningless to consider any times. This is the manifestation of wave–particle duality. For simplicity, the phase of $T(\omega)$ is defined with respect to the end of the layer, i.e., with the accuracy to kd with respect to the beginning, which gives a factor of $\exp(-jkd)$. It makes the contribution to the delay $\tau = d\sqrt{\varepsilon}/c$ in a transparent medium. The propagation coefficient in an opaque medium is transformed to the damping coefficient; in this case, delay seems to be absent, but it should be introduced. However, such a formulation of the problem is incorrect. The speed of steady-state tunneling of the particle in the beam, which is not superluminal, is meaningful. It should be defined in the energy treatment in terms of the probability flux and probability flux density. Correspondingly, for this speed, one can introduce a time that cannot be attributed to tunneling of an individual particle. It is also meaningless to apply the stationary phase method to the output amplitude of the wave packet. First, it should be determined as a solution of the integral or integrodifferential equation. As far as we know, this was not done in any works concerning tunneling times. The mistake by Hartman is that he separate one wavefunction that is the solution of the time-dependent Schrödinger equation into three functions, as in the matching method when solving the time-independent Schrödinger equation. The second his mistake is the use of the stationary phase method. Second, it is necessary to define the desired velocity. The wave packet $\psi(z, t)$ is fundamentally dispersed and polychromatic. It is possible to seek the velocity of the maximum or, e.g., the averaged energy velocity $\bar{v}(t) = \langle\psi|v_{\mathscr{E}}(z, t)|\psi\rangle\langle\psi/\psi\rangle$. The velocity $v_{\mathscr{E}}(z, t)$ should naturally be defined in terms of the flux $j(z, t)$ and its density $|\psi(z, t)|^2$. It specifies the speed of probability density transfer. Since the flux on the right of the barrier propagates to the right, the region (d, ∞) to the right of the barrier should be used for averaging; otherwise, the part of the wavefunction reflected from the barrier and waves inside the barrier will affect the velocity. To determine the spectrum $\Phi(k, t)$ of the part of the wavefunction behind the barrier, it is possible to average the spectral energy velocity over the spectrum, i.e., to take weight-averaged energy velocity of the polychromatic beam. This spectrum is instantaneous, i.e., depends on the time. If the initial wavefunction has an infinite shallow ascending front, the main part of the pulse reaches the point $z = d$ after a certain time. The function $\psi(d, t)$, which should be decomposed into the instantaneous spectrum $\Phi(k, t)$, should be defined just at this point. Correspondingly, the velocity will be defined at this point. Using the Green's function, one can transfer the wavefunction to another point (z, t) and determine the velocity at it. In any case, steady-state theory should provide quantities depending on the coordinates and time. Consequently, the tunneling time of the wave packet as a whole is meaningless because this time at each point depends on the current time. The same can be done to define the velocity inside the barrier. It is convenient to average the spectral energy velocity of the electromagnetic wave over the spectrum $W(\omega)$ [27]. However, there is the simple formula $v_e(z, t) = Z_0E^2(z, t)/W(z, t)$. It is only necessary to solve the integrodifferential equation for $E(z, t)$ and to determine the energy density $W(z, t)$ (see [51–54]). This problem is difficult because only the quantity $\partial_t W(z, t)$ is known from the Poynting theorem, and the determination of $W(z, t)$ in the general case requires integration taking into account the entire prehistory of the process.

Microwave electronics involves traveling-wave tubes operating in a pulsed mode both in transparency bands and beyond them. In any case, the signal appears at the output after a certain time $\tau \sim l/v < l/c$, where l is the length of a device, with respect to the signal at the input. Operation beyond the transparency band is in essence the tunneling of the wave, which is enhanced by the electron beam. Only when tunneling damping becomes larger than enhancement, the device ceases to work. Recent studies of transient processes in resonant tunneling diodes do not reveal superluminal motions. Microwave technique includes waveguide filters in evanescent waveguide segments with dielectric resonators. Such resonators, which are evanescent waveguide segments filled with a dielectric, support propagation, whereas tunneling occurs in regions between them. Hyperbolic metamaterials, i.e., periodic structures consisting, in particular, of metallic and dielectric layers are known. Superluminal propagation was not observed experimentally in all these structures. Simulation with standard software packages for them also does not reveal such motions. It is not fundamentally difficult to perform very accurate experiments with iris-loaded waveguide sections (e.g., evanescent sections) fed by pulsed signals and disprove the results of works such as [61] on superluminal tunneling.

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