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Quantum Hydrodynamics, Rotating Superfluid and Gravitational Anomaly

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Abstract—We present a consistent scheme of quantization of chiral flows (flows with extensive vorticity) in ideal hydrodynamics in two dimensions. Chiral flows occur in rotating superfluid, rotating turbulence, and also in electronic systems in magnetic field in the regime of a fractional Hall effect. The quantization is based on a geometric relation of chiral flows to two-dimensional quantum gravity and is implemented by the gravitational anomaly. The effect of the gravitational anomaly changes the major property of classical hydrodynamics, the Helmholtz law: vortices are no longer frozen into the flow. Effects of quantization could be cast in the form of quantum stress. We show that the quantum stress is a generator of Virasoro algebra, the centrally extended algebra of holomorphic diffeomorphisms.

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1. INTRODUCTION

The problem of quantization of hydrodynamics beyond linear approximation is commonly considered to be intractable. Nevertheless, nature confronts us with beautiful quantum ideal fluids with experimentally accessible and precise quantization. Among them, two quantum fluids stand out: superfluid helium and electronic fluid in the fractional quantum Hall state. In both cases the precise quantization of vortex circulation in superfluid helium and transport in FQHE leave no doubts of the quantum nature of their flows.

The fundamental aspects of quantization of fluid dynamics is seen in ideal flows: homogeneous, incompressible and inviscid. Dynamics of ideal two-dimensional flows is obtained by actions of area-preserving preserving diffeomorphism **SDiff**, and should be studied from a geometric standpoint, see, e.g., [1]. Then, the problem of quantization of incompressible flows is equivalent to a geometric quantization of **SDiff**. The problem is specialized in fast-rotating fluids where the quantum states are holomorphic. We refer to such flows as *chiral flows* and study them in this paper.

The two most perfect quantum fluids, fast-rotating superfluid helium [2] and FQHE [3, 4] fall to the class of chiral flows. The superfluid helium is a compressible fluid, however, a fast rotating helium could be considered as incompressible. States of electronic fluid in FQH regime, where electrons fractionally occupy the lowest Landau level are holomorphic; hence, their flows are incompressible.

In this paper we show how to quantize chiral flows and describe some not immediately obvious consequences of quantization. The guidance for the quantization comes from the intersection between FQHE, superfluid theory and quantum 2D gravity which we describe below.

Hydrodynamics of ideal flows is a Hamiltonian system. Its Hamiltonian is the kinetic energy of the flow

$$H = \frac{\rho_A}{2}\int \mathbf{u}^2 dV, \quad \nabla\cdot\mathbf{u} = 0,$$

and the Poisson structure

$$\{\omega(r), \omega(r')\} = \rho_A^{-1}(\nabla_r \times \nabla_{r'})\omega(r)\delta(r-r') \qquad (1)$$

is the Lie-Poisson algebra **SDiff**. Here ρ_A is a constant density of the fluid, and $\omega(r) = \nabla\times\mathbf{u}$ is the vorticity. Formally the quantization amounts the replacement of the Poisson brackets by the commutator $\{,\} \to \frac{1}{i\hbar}[,]$ [5], and identification of the Hilbert space with a representation space of **SDiff**. The latter, however, is not known. A difficulty is in ambiguous short-distance regularization. Canonical quantization of non-linear dynamics often yields divergencies which require a regularization. A regularization must be consistent with fundamental symmetries. The fundamental principle of fluid dynamics is the relabeling symmetry, or equivalently the invariance with respect to reparametrization of the fluid. Relabeling are diffeomorphisms in the manifold of Lagrangian coordinates. In this paper, we describe the diffeomorphism invariant regularization.

Invariance with respect to diffeomorphisms is also a guiding principle of quantum gravity. The analog between fluids and gravity we describe below suggests a proper and unique regularization. The result of it is the quantum correction to the Euler equation expressed in terms of the gravitational anomaly. The correction is small in superfluid helium, and may or may not be negligible in cooled atomic gases. But it is certainly large and important in FQH electronic fluids. But regardless of their size quantum corrections are fundamentally important for the consistency of the theory.

For the purpose of this paper we choose the quantization within the Eulerian specification. The quantization within the Lagrangian specification is briefly reviewed in the beginning of the paper. Other methods of quantization, such as, path integral and stochastic quantization yield the same result and will be published separately.

2. QUANTIZATION IN THE LAGRANGIAN SPECIFICATION

In this paper we consider only bulk flows, ignoring complications caused by boundaries. The bulk incompressible flows solely described by vorticity. The quantization scheme we propose consists of two almost independent steps. We demonstrate them within the Lagrangian specification.

2.1. Semiclassical Quantization

The first step is the quantization of vortex circulation. In quantum ideal fluids vorticity is not a smooth function. It consists of a discrete array of point-vortices with quantized circulation. If $\mathbf{u}$ is the velocity of incompressible flow, $\nabla \cdot \mathbf{u} = 0$, then vorticity is

$$\omega(r) \equiv \nabla \times \mathbf{u} = \sum_{i=1}^{N_v} \Gamma_i \delta(r - r_i), \quad (2)$$

where Γ_i are circulations of vortices quantized in units of h/m_A (mass of fluid atoms).

Hence, quantum flows confined in a finite volume are always finite-dimensional. If one stops here we face a problem of classical vortex dynamics. We can formulate the dynamics in the form of Kirchhoff.

2.2. Kirchhoff Equations

We recall the Helmholtz form of the hydrodynamics of ideal 2D flows. The Helmholtz law is the curl of Euler equations. It stays that vorticity is frozen into the flow: the *material derivative of vorticity vanishes*

$$D_t\omega = 0, \quad D_t = \partial_t + \mathbf{u} \cdot \nabla. \quad (3)$$

On other hands the flow itself is determined by positions of vortices. In complex coordinates z, $\bar{z}$ the complex fluid velocity $\mathbf{u} = u_x - \mathrm{i}u_y$ generated by an array of N_v vortices (2) is a meromorphic function

$$\mathbf{u}(z) = \frac{\mathrm{i}}{2\pi}\sum_{i=1}^{N_v} \frac{\Gamma_i}{z - z_i}. \quad (4)$$

Then the Helmholtz law and Kelvin circulation theorems say that the flow initially chosen as (4) retains its form with positions of vortices $z_i(t)$ moving accordingly the Kirchhoff equation (see, e.g. [6])

$$\dot{\bar{z}}_i = \frac{\mathrm{i}}{2\pi}\sum_{j \neq i} \frac{\Gamma_j}{z_i - z_j}. \quad (5)$$

Kirchhoff equations are Hamiltonian. The Hamiltonian and the Poisson brackets are

$$H = -\frac{\rho_A}{2\pi}\sum_{i>j} \Gamma_i \Gamma_j \log|z_i - z_j|, \quad \{\bar{z}_i, z_j\} = -\mathrm{i}(2/\rho_A \Gamma_i)\delta_{ij} \quad (6)$$

Kirchhoff equations, being exact, for flows with a finite number of vortices "approximate" arbitrary ideal flows by a finite dimensional dynamical system. However, in the quantum case, they are not an approximation. They are the starting point of quantization. Because the Kirchhoff system is Hamiltonian, the vortex dynamics could be quantized canonically. This is the second step of quantization.

When the vortex array is dense, vortices themselves form a fluid. In this approach trajectories of vortices z_i, rather than fluid atoms, are pathlines of the (vortex) fluid. Then enumeration of vortices, the label i is the Lagrangian coordinate of the vortex fluid.

2.3. Canonical Quantization

Within canonical quantization, we identify the Hilbert space with the space of holomorphic functions of coordinates of clockwise and anti-clockwise vortices, map the coordinates to operators acting in the space of vortices and replace the Poisson brackets by the commutator $\{\bar{z}_i, z_i\} \to \frac{1}{\mathrm{i}\hbar}[\bar{z}_i, z_i]$. Since the vortex circulation is quantized in units of $\Gamma = h/m_A$, and $\rho_A = m_A n_A$, where n_A is the density of fluid atoms, the commutation relations read

$$[\bar{z}_i, z_j] = \mathrm{sign}(\Gamma_i)(\pi n_A)^{-1}\delta_{ij}.$$

This scheme is a guidance for the quantization of hydrodynamics. It differs from earlier quantization attempts where Lagrangian coordinates were chosen to enumerate fluid atoms.

If the number of vortices is finite the quantization presents no fundamental difficulties. The difficulties arise when we attempt the large N_v limit, when an array of vortices approximate realistic flows of interest.

2.4. Chiral Flows

The 2D chiral flows are flows with an extensive net vorticity (vorticity is proportional to the volume of the fluid). Such flows consist of a dense liquid array of vortices. We assume that circulations of all vortices are the same $\Gamma_i = \Gamma > 0$ (anticlockwise). Then vorticity is proportional to the vortex density

$$n(r) = \Gamma^{-1}\omega(r) = \sum_{i=1}^{N_v}\delta(r - r_i). \tag{7}$$

We denote the mean density of vortices by $n_v = N_v/V$, the density of fluid atoms $n_A = \rho_A/m_A$ and the fraction $\nu = N_v/N_A$ of vortices per fluid atom.

The well known examples of quantum chiral flows are superfluid helium rotating with the angular frequency $\Omega = n_v\Gamma/2$, and also FQHE, where N_v electrons confined in a 2D layer are placed in a uniform magnetic field $B = (h/\nu e)n_v$, whose total number of flux quanta $N_\Phi = eBV/h$ equals the number of fluid atoms N_A. In this case electrons occupy a fraction ν of the lowest Landau level.

The chiral flow sets the scale of length, the mean distance between vortices $\ell_v = (2\pi n_v)^{-1/2}$, and $\hbar\Omega$ sets the energy scale. We collect all these units below

$$n_A = \rho_A/m_A = (eB)/h, \quad \Gamma = h/m_A,$$
$$n_v = 2\Omega/\Gamma, \quad \nu = n_v/n_A, \quad \ell_v = (2\pi n_v)^{-1/2}.$$

In these units the Kirchhoff Hamiltonian and the commutation relations read

$$H = -\frac{\hbar\Omega}{\nu}\sum_{i\neq j}\log|z_i - z_j|,$$
$$[z_i, \bar{z}_j] = (\pi n_A)^{-1}\delta_{ij}. \tag{8}$$

Then the Kirchhoff Eqs. (5) become Heisenberg equations for vortices

$$i\hbar\dot{\bar{z}}_i = (\pi n_A)^{-1}\partial_{z_i}H = -\frac{\hbar}{2\pi}\sum_{j\neq i}\frac{\Gamma}{z_i - z_j}.$$

The fraction ν is de facto a semiclassical parameter. It is small in helium, but is of the order one, say $\nu = 1/3$ in FQHE.

The Hilbert space of the chiral flow, where vortices are the same sense, is the Bergmann space [7, 8]. It is a space of holomorphic polynomials of z and ∂_z with the inner product

$$\langle g|f\rangle = \int g(\bar{z})f(z)d\mu, \quad d\mu = e^{-\pi n_A|z|^2}dzd\bar{z}.$$

Then the anti-holomorphic coordinates operators $\bar{z}_i$ are Hermitian conjugations of holomorphic coordinates z_i realized as

$$\bar{z}_i = z_i^\dagger, \quad z_i^\dagger = (\pi n_A)^{-1}\partial_{z_i}^T.$$

All holomorphic states are eliminated by $z_i^\dagger$

$$\langle\text{holomorphic states}|z_i^\dagger = 0.$$

We can easily determine the ground state wave function of the chiral flow, the stationary flow. It corresponds to a solid rotation of the vortex system $z_i(t)|0\rangle = e^{-i\Omega t}z_i(0)|0\rangle$. In this case the velocity $\dot{\bar{z}}_i|0\rangle = -i\Omega\bar{z}_i|0\rangle = \nu(\Gamma/2\pi i)\partial_{z_i}|0\rangle$. Hence the ground state is killed by the operator

$$(\hbar\Omega\partial_{z_i} + \partial_{z_i}H)|0\rangle = 0, \quad \text{or}$$

$$\left(\nu\partial_{z_i} - \sum_{j\neq i}\frac{1}{z_i - z_j}\right)|0\rangle = 0.$$

This equation has single-valued solutions only if $1/\nu \in \mathbb{Z}$ is an integer: each vortex is surrounded by the integer number of atoms. This is well known quantization of the inverse fraction in FQHE. Then the solution is the holomorphic polynomial

$$|0\rangle_B = \prod_{i>j}(z_i - z_j)^{1/\nu}. \tag{9}$$

This is, of course, the Laughlin wave function for FQHE in the Bargmann space.

If we choose the standard L_2 scalar product, the factor $e^{-\frac{1}{2}\pi\sum_i n_A|z_i|^2}$ must be added to the holomorphic polynomial (9). In this case the Laughlin wave function of a stationary chiral flow appears in a more familiar form [9]

$$|0\rangle_{L_2} = e^{-\frac{1}{2}\pi\sum_i n_A|z_i|^2}\prod_{i>j}(z_i - z_j)^{1/\nu}. \tag{10}$$

Hence, the probability distribution of vortices in the ground state (10) could be expressed through the Kirchhoff free energy $H - \Omega L$, where

$$L = m_v\sum_i \mathbf{r}_i \times \mathbf{v}_i \tag{11}$$

is the angular momentum of the vortex matter, also called angular impulse [6], $m_v = m_A/\nu$, is a "vortex mass," a mass of atoms trapped by each vortex, and $\mathbf{v}_i = \dot{\mathbf{r}}_i - \Omega \times \mathbf{r}_i$, the velocity of a vortex in rotating frame.

We have

$$d\mathcal{P}/dV = \langle 0|0\rangle = e^{-\frac{1}{T_*}(H-\Omega L)}. \tag{12}$$

This formula looks like a canonical Gibbs distribution with the temperature $T_* = \hbar\Omega$. This ensemble appears in various independent contexts and had been extensively studied (see, e.g., [10]). Among them are one-component plasma, Dyson diffusion, and also

Onsager ensemble of vortices. It is also the equilibrium distribution of a non-determinantal stochastic point process, called β-ensemble, where β labels $1/\nu$. At large N_v it describes a distribution with an equilibrium density of vortices equal $n_v = 2\Omega/\Gamma$ [11].

3. LIE ALGEBRA OF AREA PRESERVING DIFFEOMORPHISMS

It is a straightforward check that the brackets (6) with the relations (2), (4) yield the canonical Poisson structure for the ideal fluid (1). It is also well known that the Poisson structure of ideal flows is the Lie-Poisson structure. The Lie group is the group of area-preserving diffeomorphisms **SDiff** (see, e.g., [1]). Hence, **SDiff** could be realized by operators acting in the space of vortices. We use this realization for the quantization.

Consider Fourier mode expansion of the vortex occupation Eq. (7) $n_k = \int e^{-i\mathbf{k}\cdot\mathbf{r}} n(r) dV = \sum_i e^{-\frac{i}{2}(k\bar{z}_i + \bar{k}z_i)}$. In the Bargmann space, the occupation number is realized by the normally ordered operator with respect to holomorphic states

$$:n_k: = \sum_{i \le N} e^{-\frac{i}{2}kz_i^\dagger} e^{-\frac{i}{2}\bar{k}z_i},$$

where $k = k_x + ik_y$ is a complex wave vector. It gives the quantum meaning of vorticity $:\omega(r): = -(\Gamma/V)\sum_k e^{i\mathbf{k}\cdot\mathbf{r}} :n_k:$. For certain physical problems, a not-ordered operator is also considered $n_k = \sum_i e^{-\frac{i}{2}(k\bar{z}_i + \bar{k}z_i)} = e^{\frac{1}{4}\frac{k^2}{n_A}} :n_k:$. When the difference between two operators is not important we may drop the normal ordering symbol.

The occupation number operator is chiral $n_k^\dagger = n_{-k}$, and eliminates the ground state

$$\langle 0| :n_k: = \langle 0| N_v \delta_{k0}. \tag{13}$$

With the help of (8) we obtain their algebra

$$[:n_k:, :n_{k'}:] = ie^{\frac{(k\cdot k')}{4\pi n_A}} e_{kk'} :n_{k+k'}: \tag{14}$$
$$[n_k, n_{k'}] = ie_{kk'} n_{k+k'}$$

with the structure constants

$$e_{kk'} = 2\sin\left(\frac{\mathbf{k}\times\mathbf{k}'}{4\pi n_A}\right). \tag{15}$$

Since N_A is an integer (a large integer), the algebra (14), also known as sine-algebra [12], is finite dimensional. It is a quantum deformation or an "approximation" of **SDiff**. In the classical limit, as well as in the long-wave limit when we neglect the discreetness of atoms $n_A \to \infty$

$$e_{kk'} \approx (2\pi n_A)^{-1}(\mathbf{k}\times\mathbf{k}').$$

It brings us back to the Lie-Poisson structure of classical hydrodynamics (1).

4. QUANTIZATION IN THE EULERIAN SPECIFICATION

Despite straightforward quantization of flows with a finite number of vortices described above, the quantization of general flows approximated by a large number of vortices meets essential difficulties. The source of difficulties is the same as in any quantum theory of continuous media—a passage from the finite dimensional to the infinite dimensional system. Formally the problem arises as follows. The advection term $\mathbf{u}\cdot\nabla\omega$ in the Helmholtz Eq. (3) possesses two operators sitting at the same point. It requires a regularization. A standard recipe of a regularization commonly adopted in the field theory is the point splitting $\mathbf{u}\left(r + \frac{\ell_v}{2}\right) \cdot \nabla\omega\left(r - \frac{\ell_v}{2}\right)$, when points are split by the shortest distance between atoms. However, in quantum hydrodynamics, this recipe leads to inconsistencies. It violates the relabeling symmetry of the fluid. The difficulty is that in hydrodynamics contrary to a field theory the point-splitting distance, the short distance cut-of, itself depends on the flow $\ell[\mathbf{u}]$. Similar difficulties also arise in the quantization of gravity, where this problem had been understood and resolved [13]. In the nutshell, the recepie suitable for the hydrodynamics is that the variable short-distance cutoff is the distance between vortices $\ell[\mathbf{u}] \sim 1/\sqrt{n}$. This regularization had been explored in the theory of rotating superfluid a long ago for regularization of vortex energy in rotating superfluid and explained in details in Khalatnikov's book [2], see also original papers [14–17]. Correction to the energy of the flow obtained in these papers has a classical nature and locally depends on vorticity. Nowadays, it is called odd, or anomalous viscosity [18], but because it is local, its contribution to the stress is divergent-free. It does not affect the Euler equations for the bulk flow. However, it is essential at the boundary [15], on the interface between the vortex array and a potential flow [19], and also, on curved surfaces [20].

Here we push this idea forward. We determine the quantum correction. The quantum correction depends on a gradient of vorticity, hence enters the equation for the bulk flow. We express it in terms of the quantum stress.

4.1. Quantum Corrections to the Helmholtz Equation and Quantum Stress

The quantization of the field equations, such as Helmholtz equation essentially means normal ordering of operators entered the equation with respect to a flow (quantum state) of interest. The problem of regularization problem comes when we attempt to order the advection term. The advection term of the Helmholtz equation reads

$$\mathbf{u}\cdot\nabla\omega = \epsilon^{ik}\partial_k\nabla^j(u_iu_j),$$

Hence, we have to understand the quantum meaning of u_iu_j by ordering the product of two velocity operators. We denote the Wick contraction $\overline{AB} =: AB: - :A::B:$ and compute $\overline{u_iu_j}$. This is the quantum correction received by the momentum flux tensor. It should be interpreted as a quantum analog of the Reynolds stress

$$:u_iu_j: = :u_i::u_j: - \rho_A^{-1}T_{ij}, \quad T_{ij} = -\rho_A\,\overline{u_iu_j}. \tag{16}$$

Since quantization is the sole origin of the stress we refer it as *quantum stress*. The quantum stress corrects the Helmholtz law

$$D_t\omega = \epsilon^{ik}\partial_k\nabla^j(\rho_A^{-1}T_{ij}). \tag{17}$$

We comment that only traceless part of the stress enters the Helmholtz equation.

In this equation, all entries are assumed to be normally ordered. The implication of quantum stress is that the Helmholtz law held for quantum operators does not hold for their matrix elements.

We will show that the stress is not divergence-free as it happens in the case of odd-viscous stress [18], and that vorticity is no longer frozen into the flow. Nevertheless the Kelvin theorem, the conservation of vorticity in a fluid parcel remains intact. Condition for the Kelvin theorem to hold is that the divergence of the stress has no circulation along the boundary of a parcel. We will see below that $\oint\nabla^jT_{ij}\mathrm{d}x^i = \frac{1}{2}\oint T_i^i\,\mathrm{d}n$, indeed, vanishes.

We will show that the quantum stress reads

$$T_{ij} = \frac{\hbar\Omega}{96\pi}\Big[(\nabla_i\nabla_j - \delta_{ij}\Delta)\log n - \Big(\nabla_i\log n\nabla_j\log n - \frac{1}{2}\delta_{ij}(\nabla\log n)^2\Big)\Big].$$

For references we write the stress in complex coordinates

$$T_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{4}[T_{zz}(\mathrm{d}z)^2 + 2T_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z} + T_{\bar{z}\bar{z}}(\mathrm{d}\bar{z})^2].$$

$$T_{zz} = \frac{\hbar\Omega}{12\pi}\Big(\partial_z^2\log n - \frac{1}{2}(\partial_z\log n)^2\Big), \tag{18}$$

$$T_{z\bar{z}} = -\frac{\hbar\Omega}{48\pi}\partial_z\partial_{\bar{z}}\log n.$$

The quantum stress is small. It consists of higher derivatives, but it is the only source which deviates vortices away from the flow.

Below we obtain these formulas.

5. GEOMETRIC INTERPRETATION OF THE CHIRAL FLOWS AS QUANTUM GRAVITY

We start from a general discussion of a parallel between 2D chiral flows and 2D gravity. The surface which hosts the fluid is a complex manifold equipped with a closed vorticity 2-form, $\omega_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^i$, $\omega_{ij} = \partial_iu_j - \partial_ju_i$. Because vorticity of the chiral flow does not change sign, in our convection, it is positive $\omega = \frac{1}{2}\epsilon^{ij}\omega_{ij} > 0$, it gives the host surface a Kähler structure with the Kahler form $\Gamma^{-1}\omega\mathrm{d}z\wedge\mathrm{d}\bar{z} = n\mathrm{d}z\wedge\mathrm{d}\bar{z}$. Hence, the differential $\mathrm{d}s^2 = 2n|\mathrm{d}z|^2$ can be treated as a Riemannian metric. The interval $\mathrm{d}s = |\mathrm{d}z|/\ell[n]$, where $\ell[n] = 1/\sqrt{n}$ is a distance between vortices in the parcel of the size $|\mathrm{d}z|$, and the volume element of the metric is $\mathrm{d}N_v = n\mathrm{d}V$, the number of vortices in the fluid parcel of the volume $\mathrm{d}V = dz\wedge\mathrm{d}\bar{z}$. Obviously a map of coordinates of vortices to points of an auxiliary surface whose metric is $\mathrm{d}s^2$ is the map to Lagrangian coordinates.

We can interpret flows of the fluid as a flow of the metric. Consider the stationary state, where vortices are distributed uniformly with the density n_v. The (background) metric which corresponds to this state is $\mathrm{d}s_0^2 = n_v|\mathrm{d}z|^2$. Then a general flow can be seen as a Weyl transformation of the background metric

$$\mathrm{d}s_0^2 = n_v|\mathrm{d}z|^2 \to \mathrm{d}s^2 = n|\mathrm{d}z|^2. \tag{19}$$

This map constitutes a flow. Adopting the language of quantum gravity we may identify the manifold of Lagrangian coordinates with a target space, and the host surface as a world sheet. The metric ds had been used in studies of a vortex lattice in [21, 22]. Here we utilize it for a liquid.

If the fluid resides on a flat surface, then, generally the metric obtained by the Weyl transformation is curved. Its scalar curvature is

$$\mathcal{R} = -(4/n)\partial_z\partial_{\bar{z}}\log n. \tag{20}$$

A reparametrization of the axillary surface is equivalent to relabeling vortices with no effect on the flow (a diffeomorphism invariance or relabeling symmetry). In hydrodynamics, the relabeling symmetry usually refers to fluid atoms. In our approach, it is a relabeling symmetry of vortices. We want to keep this major symmetry intact in quantization. This amounts to implement a uniform short distance cut-off on the target space, the space of labels. In the host plane, the world sheet, the cut-off is a function of the flow. This principle uniquely determines the regularization.

We illustrate this idea in terms of the path integral approach to quantization. In this approach we integrate over all pathlines of fluid parcels. We choose to integrate over pathlines of vortices, instead. The measure of the path integral must be invariant with respect to relabeling. We want to integrate over flows which consists of a large number of vortices when vorticity admits a coarse-grained limit and could be approximated by a smooth function. Since vorticity is a metric we effectively integrate over all metrics $ds^2 = n|dz|^2$. The measure in the space of metrics which is invariant with respect to diffeomorphisms is unique. It has been well established in quantum gravity [13]. That measure possesses a Jacobian of the map of the space of pathlines to the space of metrics. That Jacobian is a source of the quantum stress.

Below we present a somewhat heuristic, but economic approach to the quantization based on diffeomorphism (relabeling) invariance. In practice it requires that all observable quantities are expressed through invariant geometric objects of the metric, such as geodesic distance, curvature, etc.

6. GRAVITATIONAL ANOMALY

We now describe the major effect of quantization, the gravitational anomaly. In complex coordinates $u_z = u_x - iu_y$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ the advection term and the quantum stress read

$$:\mathbf{u}\nabla\omega: = \rho_A^{-1} i[\partial_{\bar z}^2 T_{zz} - \partial_z^2 T_{\bar z\bar z}], \quad T_{zz} = -\rho_A\, \overline{u_z u_z}. \quad (21)$$

We express the quantum stress through the occupation number with the help of the formula $u_z = 2i\Gamma\partial_z V^{-1}\sum_k e^{i\mathbf{k}\cdot\mathbf{r}} k^{-2} n_k$

$$\overline{u_z u_z} = -\Gamma^2 \lim_{r\to r'}(4\partial_{z'}\partial_z)$$

$$\times\frac{1}{V^2}\sum_{k,k'} e^{i(\mathbf{k}\cdot\mathbf{r}+\mathbf{k}'\cdot\mathbf{r}')}(k'k)^{-2}\,\overline{n_k n_{k'}}.$$

Hence, we need to compute the contraction of the normally ordered two-modes operator. Its normal ordering reads

$$:n_k n_{k'}: = \sum_{i,j} e^{-\frac{i}{2}kz_i^\dagger} e^{-\frac{i}{2}k'z_j^\dagger} e^{-\frac{i}{2}\bar k z_i} e^{-\frac{i}{2}\bar k' z_j}. \quad (22)$$

With the help of the algebra (8) we find (we recall that $(2\pi n_A)^{-1} = \nu\ell_v^2$)

$$\overline{n_k n_{k'}} = \left(1 - e^{\frac{\nu}{2}\ell_v^2 \mathbf{k}\cdot\mathbf{k}'}\right):n_{k+k'}:. \quad (23)$$

Then

$$\overline{u_z(r)u_z(r')} = -\Gamma^2(4\partial_z\partial_{z'})$$

$$\times\frac{1}{V^2}\sum_{k,k'}\left(\frac{1-e^{\frac{\nu}{2}\ell_v^2(\mathbf{k}\cdot\mathbf{k}')}}{k^2k'^2}\right):n_{k+k'}:e^{i\mathbf{k}\mathbf{r}+i\mathbf{k}'\mathbf{r}'}.$$

Now we have to evaluate the expectation value $\langle n|u(r)u(r')|n\rangle$ on the flow with the density $n(r)$. But we know that flows belong to the orbit of the Lie algebra (14). Therefore, it is sufficient to start from any state. The easiest is the ground state (the stationary flow) $|0\rangle$. It is a uniform state, hence, we have (13). Hence,

$$\langle 0|\overline{u_z(r)u_z(r')}|0\rangle = -\Gamma^2(4\partial_z\partial_{z'})$$

$$\times\int e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\left(\frac{1-e^{-\frac{\nu}{2}\ell_v^2p^2}}{2\pi\ell_v^2p^4}\right)\frac{d^2p}{(2\pi)^2}.$$

In the limit, where points are well separated, p is small and the integrand behaves as $1/p^2$. At short distances p is large, the integral converges. The crossover between two regimes is assisted by the factor $e^{-\frac{\nu}{2}\ell_v^2p^2}$

$$\langle 0|\overline{u_z(r)u_z(r')}|0\rangle = -\frac{\nu}{2\pi}\Gamma^2(\partial_z\partial_{z'})$$

$$\times\begin{cases}-\frac{1}{2\pi}\log(|r-r'|/\sqrt{V}), & |r-r'|\gg\ell_v\\ -\frac{1}{2\pi}\log(\ell_v/\sqrt{V}), & r\to r'.\end{cases} \quad (24)$$

This is of course the Green function of the Laplace operator G_R regularized at short distances by the intervortex distance ℓ_v. We arrive to the expression for the quantum stress

$$T_{zz} = 2(\hbar\Omega)\lim_{r\to r'}\partial_z\partial_{z'}G_R(r,r'). \quad (25)$$

We now extend this result to a non-uniform flow. An economic way to do this is to invoke the geometric interpretation of the flow, where the vorticity is understood as a metric. Then, at large distance one expects to have the Green function of the Laplace-Beltrami operator $\Delta = -(4/n)\partial_z\partial_{\bar z}$ in the metric $ds^2 = n|dz|^2$. At short distances the Green function diverges as $r\to r'$, but the result must be finite as in (24). We obtain the crossover by a unique covariant regularized Green function consistent with the metric. The infinity at short distances is subtracted by adding to the Green function the logarithm of the geodesic distance between merging points

$$G_R(r,r') \underset{r\to r'}{\approx} G(r,r') + \frac{1}{2\pi}\log d(r,r'). \quad (26)$$

Formally, this is equivalent a modification of the factor $e^{-\frac{\nu}{2}\ell_v^2p^2}$ in the integrand of (26) by the operator $e^{\frac{\nu}{2}\ell_v^2\Delta}$ and the replacing the integral by the trace. This is equivalent the usual heat-kernel regularization. The result of the limit of merging points in (25) is known to be the Schwarzian of the metric

$$\lim_{r\to r'}\partial_z\partial_{z'}\left[G(r,r')+\frac{1}{2\pi}\log d(r,r')\right] = \frac{1}{24\pi}\left(\partial_z^2\log n-\frac{1}{2}(\partial_z\log n)^2\right).$$

Hence

$$T_{zz}=\frac{\hbar\Omega}{12\pi}\left(\partial_z^2\log n-\frac{1}{2}(\partial_z\log n)^2\right). \quad (27)$$

This effect is analogous to the gravitational anomaly in quantum gravity. Physically, it amounts a flow-dependent cut-off $\ell[n]=(2\pi n)^{-1/2}$, rather than a uniform cut-off ℓ_v.

7. QUANTUM STRESS

The following arguments help to determine the trace of the quantum stress $T_{z\bar z}$. The stress tensor of the metric (19) is divergence-free. In complex coordinates this means

$$\partial_{\bar z}T_{zz}+n\partial_z(n^{-1}T_{z\bar z})=0. \quad (28)$$

With the help of (27) and (28) we obtain the trace anomaly: the trace of the quantum stress is proportional to the Ricci curvature

$$T_{z\bar z}=\frac{\hbar\Omega}{48\pi}n\mathcal{R}. \quad (29)$$

It follows that in the metric of the host surface, the tensor is not conserved (we raise indices with the metric of the host surface)

$$\nabla^i T_{ij}=T_{z\bar z}\nabla_j\log\sqrt{n}. \quad (30)$$

The right-hand side of this equation is the source of the quantum correction.

8. QUANTUM CORRECTIONS TO THE HELMHOLTZ LAW

Few equivalent forms of the quantum Helmholtz equation follows from (17), (29), and (30). One is

$$D_t\omega=\frac{\nu\Gamma}{192\pi^2}\epsilon^{ik}\partial_k\mathcal{R}\nabla_i\omega. \quad (31)$$

In this equation, all terms are normally ordered. It could be treated as a classical equation.

If vorticity waves are small, we may expend the quantum Helmholtz Eq. (31) about the net vorticity $\omega=2\Omega+V^{-1}\sum_{k\neq 0}e^{\mathbf{ik\cdot r}}\omega_k$. In harmonic approximation (the leading order in n_k) we obtain

$$n_\mathrm{v}D_t\omega_k=\frac{\nu}{192\pi^2}\frac{1}{N_\mathrm{v}}\sum_q q^2(\mathbf{q}\times\mathbf{k})\omega_q\omega_{k-q}. \quad (32)$$

For some physical applications the sum over modes could be truncated.

9. FLUID FLOW AS METRIC FLOW

Since vorticity of the chiral flow could be thought as a metric, the quantum Helmholtz equation describes a metric flow. We cast it in the form

$$m_\mathrm{v}D_t\log n=-\frac{\hbar}{48\pi}\varpi^i\partial_i\mathcal{R},$$

where $\varpi^i=\frac{1}{2}\epsilon^{ij}\partial_j\log n$ is the transversal part of the spin connection of the target surface. This form suggests that the chiral flow is merely a dilatation of the target space.

10. VIRASORO ALGEBRA

The gravitational anomaly is essentially equivalent to the Virasoro algebra. To shorten the formulas we set $\hbar\Omega=1$.

Since, a change of vorticity is interpreted as a Weyl transformation of the metric, the trace of the quantum stress is a generator of dilatations

$$:n\frac{\delta}{\delta n}\mathbb{O}:=-\overline{T_{z\bar z}\mathbb{O}},$$

where $\mathbb{O}$ is an arbitrary operator. With the help of the conservation law (28) and the $\bar\partial$—formula we write it as

$$\frac{1}{\pi}\int\frac{n(\xi)\partial_\xi(\delta/\delta n(\xi))}{z-\xi}\mathbb{O}d^2\xi=\overline{T_{zz}\mathbb{O}} \quad (33)$$

(we dropped the normal ordering symbol).

We now specify $\mathbb{O}$ to be $\mathbb{O}=T(z')\tilde{\mathbb{O}}$ and evaluate the relation (33) with the help of (27). The calculations are standard in conformal field theory (CFT) literature (e.g., [23, 24]), since (33) is equivalent to the CFT Ward identity. They yield

$$:T_{zz}(z)T_{zz}(z')\mathbb{O}:-T_{zz}(z):T_{zz}(z')\mathbb{O}: = \frac{c/2}{(z-z')^2}+\left(\frac{2}{(z-z')^2}+\frac{1}{z-z'}\partial_z\right):T_{zz}(z')\mathbb{O}:$$

It is well known that this relation is equivalent to the Virasoro algebra

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{c}{12}(n^3-n)\delta_{n+m,0}.$$

whose generators are defined by the Laurent series $T_{zz}(z)=\sum_n L_n(z-z')^{-n-2}$ about a pont z'. The central extension of the Virasoro algebra appears to be $c=-1$.

11. QUADRUPOLE MODES

We conclude by a brief discussion of the global symmetry of the quantum flow.

Infinite dimensional algebra **SDiff** possesses a finite dimensional subgroup. This is $sl(2, \mathbb{R})$ algebra (isomorphic to $su(1, 1) \approx sp(2, \mathbb{R}) \approx so(2, 1)$). The maximal compact subgroup of $SL(2, \mathbb{R})$ is $SO(2)$, a group of planar rotations, the symmetry of the fluid. The generator of global rotations is the angular impulse (11) $L = m_v\sum_i (\mathbf{r}_i \times \mathbf{v}_i)$. Hence, the Hamiltonian and angular impulse act as diagonal operators in a module ($SL(2, \mathbb{R})$, $SO(2)$). The angular impulse is quantized in units of the Plank constant. We use it to label weight states $L|l\rangle = \hbar l|l\rangle$.

In geometry of the (punctured) sphere we can realize $sl(2, \mathbb{R})$ algebra by the quadrupole moment of vorticity, the rank-2 symmetric tensor

$$Q_{ij} = -\frac{\rho_A}{2}\int x_i x_j \omega dV.$$

In complex coordinates, we have

$$Q_{\bar{z}\bar{z}} = -\frac{\rho_A}{2}\int z^2\omega dV = \pi\hbar n_A\sum_i z_i^2,$$

$$Q_{z\bar{z}} = -\frac{\rho_A}{2}\int |z|^2\omega dV = \pi\hbar n_A\sum_i \frac{1}{2}(z_i z_i^\dagger + z_i^\dagger z_i).$$

The Lie–Poisson algebra of quadrupoles follows from (1)

$$\{Q_{ij}, Q_{kl}\} = \epsilon_{ik}Q_{jl} + \epsilon_{jk}Q_{il} + \epsilon_{il}Q_{jk} + \epsilon_{jl}Q_{ik}.$$

When we replace the Poisson brackets by the commutator we obtain $sl(2, \mathbb{R})$ Lie algebra. In complex coordinates it reads

$$[Q_{z\bar{z}}, Q_{zz}] = -\hbar Q_{zz}, \quad [Q_{z\bar{z}}, Q_{\bar{z}\bar{z}}] = \hbar Q_{\bar{z}\bar{z}},$$
$$[Q_{\bar{z}\bar{z}}, Q_{zz}] = 2\hbar Q_{z\bar{z}}.$$

The trace of the quadrupole moment is conserved $\frac{d}{dt}Q_{z\bar{z}} = (i/\hbar)[H, Q_{z\bar{z}}] = 0$ [6]. This is an immediate consequence of the Kirchhoff or Euler equations. Up to an additive number which depends on the choice of a frame, and an overall factor, $Q_{z\bar{z}}$ is equivalent to the angular momentum L. It generates in-plane rotation $SO(2)$. The eigenvalues of the angular impulse follow from the Kirchhoff Eq. (5)

$$Q_{z\bar{z}}|l\rangle = \hbar\left(k + \frac{l}{2}\right)|l\rangle,$$

where $k = \frac{1}{8}N_A(N_v - 1)$. This constant is called Bargmann index. It gives the value to the Casimir operator of $sl(2, \mathbb{R})$

$$C_2 = Q_{z\bar{z}}^2 - \frac{1}{2}(Q_{zz}Q_{\bar{z}\bar{z}} + Q_{\bar{z}\bar{z}}Q_{zz}) = \hbar^2 k(k-1). \quad (34)$$

The operators $Q_{z\bar{z}}$ and Q_{zz} generate clockwise and anticlockwise shear flows. They are ladder operators changing the angular impulse by $\pm 2\hbar$ [25]

$$Q_{zz}|l\rangle = -\frac{\hbar l}{4}|l-2\rangle, \quad Q_{\bar{z}\bar{z}}|l\rangle = \hbar\left(k + \frac{l}{2}\right)|l+2\rangle.$$

Therefore, the flow is the weight state with a given angular impulse raised and lowered by the quadrupole operators by the increment 2. Say, the shear operator $Q_{\bar{z}\bar{z}}$ acting on the ground state $|0\rangle$ creates a spin-2 state (called squeezed modes in optics). The "squeezing operator" $\exp(\bar{\xi}^2 Q_{\bar{z}\bar{z}} - \xi^2 Q_{zz})$ creates a flow equivalent to a coherent state.

Let us now turn to the dynamics of the quadrupole operators. From Kirchhoff equation we obtain

$$i\frac{d}{dt}Q_{zz} = \frac{\hbar\Omega}{\nu}\sum_{i>j} 2e^{-2i\theta_{ij}},$$

where $\theta_{ij} = \arg(z_i - z_j)$. The coarse-grained version of this expression follows from (17)

$$\frac{d}{dt}Q_{zz} - \frac{i}{2}\int u_z^2 dV = \frac{i}{2}\int T_{zz} dV. \quad (35)$$

The term in the lhs is the advection. The rhs is the effect of the quantum stress. In the classical fluid only advection deforms the quadrupole moment of vorticity. For example, the viscous stress does not contribute to the volume integral in (35). However, the quantum stress does (see (18)). Its effect stabilizes instabilities driven by the advection. In a near stationary flows with a small advection the quantum stress could be a dominant contribution. We omitted the boundary line integral in (35). It could be the major source of quadrupole modes when the bulk vorticity is uniform.

Summing up, we computed the quantum correction to the Euler equation for 2D chiral flows. The quantum correction appears in the form of the quantum stress, as an additional force exerted by vortices. It consists of higher derivatives, hence has a scale of fundamental vortex circulation Γ. The quantum correction destroys the scaling invariance of Euler equation, and the major law of 2D classical flows. The matrix elements of vorticity are no longer frozen into the flow. This is the price of maintaining a fundamental symmetry of fluid flows, the relabeling symmetry.

This effect is in parallel to the quantization of 2D gravity, where the invariance with respect to diffeomorphisms uniquely determines the Polyakov-Liouville action. In both cases upholding the diffeomorphism invariance is implemented through the gravitational anomaly.

Quantum corrections appear as an added stress. There is no surprise that the Lie-Poisson algebra of the quantum stress is the Virasoro algebra, the centrally extended algebra of holomorphic diffeomorphisms.

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REFERENCES

1. V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics*, Vol. 125 of *Applied Mathematical Sciences* (Springer, New York, 1999).
2. I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Perseus Books, 1989).
3. P. B. Wiegmann, Phys. Rev. B **88**, 241305 (2013).
4. P. Wiegmann, J. Exp. Theor. Phys. **117**, 538 (2013).
5. L. D. Landau, J. Phys. (USSR) **5**, 71 (1941).
6. P. G Saffman, *Vortex Dynamics* (Cambridge Univ. Press, Cambridge, 1992).
7. S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys. Rev. B **33**, 2481 (1986).
8. V. Bargmann, Commun. Pure Appl. Math. **14**, 187 (1961).
9. R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
10. A. Zabrodin and P. Wiegmann, J. Phys. A: Math. Gen. **39**, 8933 (2006).
11. R. P. Feynman, Prog. Low Temp. Phys. **1**, 17 (1955).
12. D. B. Fairlie and C. K. Zachos, Phys. Lett. B **224**, 101 (1989).
13. A. M. Polyakov, *Gauge Fields and Strings*, Contemporary Concepts in Physics (Harwood Academic, Switzerland, 1987).
14. I. L. Bekarevich and I. M. Khalatnikov, Sov. Phys. JETP **13**, 643 (1961).
15. M. P. Kemoklidze and I. M. Khalatnikov, Sov. Phys. JETP **19** (5) (1964).
16. H. E. Hall, Adv. Phys. **9**, 89 (1960).
17. H. E. Hall and W. F. Vinen, Proc. R. Soc. London, Sect. A **238**, 215 (1956).
18. P. Wiegmann and A. G. Abanov, Phys. Rev. Lett. **113**, 034501 (2014).
19. A. Bogatskiy and P. Wiegmann, Phys. Rev. Lett. **122**, 214505 (2019).
20. A. Bogatskiy, arXiv:1903.07607 (2019).
21. A. F. Andreev and M. Yu. Kagan, Sov. Phys. JETP **59**, 318 (1984).
22. G. E. Volovik and V. S. Dotsenko, Sov. Phys. JETP **78**, 132 (1980).
23. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).
24. T. Eguchi and H. Ooguri, Nucl. Phys. B **282**, 308 (1987).
25. A. Perelomov, *Generalized Coherent States and their Applications* (Springer, Berlin, Heidelberg, 2012).