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Order Parameter in Electron System: Its Fluctuations and Oscillations

K. B. Efetov[a,b,c]

[a] *Ruhr University Bochum, Faculty of Physics and Astronomy, Bochum, 44780 Germany*
[b] *National University of Science and Technology "MISiS," Moscow, 119049 Russia*
[c] *International Institute of Physics, UFRN, Natal, 59078-400 Brazil*
e-mail: Konstantin.B.Efetov@ruhr-uni-bochum.de

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Abstract—The concept of the order parameter is extremely useful in physics. Here, I discuss extensions of this concept to cases when the order parameter is no longer a constant but fluctuates or oscillates in space and time. This allows one to describe in an unified manner diverse physical phenomena including coexisting superconductivity and insulators in (quasi)one-dimensional systems, superconductivity and Coulomb blockade in granular superconductors and Josephson networks, Anderson localization and mesoscopic effects in disordered and chaotic systems, and thermodynamic quantum time-space crystals.

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1. INTRODUCTION

The concept of the order parameter introduced by Landau in his theory of phase transitions [1] plays the central role in condensed matter and statistical physics. It has become clear later with development of the scaling theory that, in order to describe a phase transition, one should integrate over the order parameter with a weight determined by a Ginzburg–Landau–Wilson free energy functional [2]. This field of research has attracted a lot of interest at the Landau Institute in particular because scaling ideas had been proposed previously by Patashinskii and Pokrovskii [3] and by Kadanoff [4], and first renormalization group study of the phase transitions had been performed by Larkin and Khmel'nitski [5]. I started my scientific activity at the Landau Institute in the time when all these ideas had just appeared and one could hear hot discussions on that topic.

Simultaneously, it was getting clear that fluctuations of the order parameter could be very important not only near a phase transition by also in low-dimensional systems. The dimension of the system was determined by geometry of the sample, while the dimension of the electron bands was not always important. My attention to this class of problems was drawn by Anatoly Larkin, with whom I made my PhD. We have understood quite generally that the behavior of two-point correlation functions of one-dimensional electron systems at large distances or times were completely determined by sound-like gapless quantum fluctuations [6]. Although many thermodynamical physical quantities could be calculated microscopically using rather sophisticated Bethe Ansatz methods, one could not determine the correlation functions using that technique. Instead, we have demonstrated that it was sufficient to compute correlations of superconducting or insulating order parameters taking into account their fluctuations. Actually, this was one of the first step in the subsequent development of powerful bosonization techniques [7] (for review, see, e.g. [8]).

The idea that many interesting effects can be efficiently described by considering low energy fluctuations of the order parameter motivated me later to study-physics of granular superconductors [9]. In these materials, the superconductivity in a single grain can be well described by a phase of the order parameter fluctuating in time. The modulus of the order parameter was assumed to be a constant and there was no need to consider space variations of the phase inside the grain. In order to describe the macroscopic superconductivity in the array of the grains one had to account for the Josephson coupling between the grains. A Coulomb interaction turns out to be very important enhancing the phase fluctuations in time and eventually leading to a superconductor-insulator transition dubbed later "Coulomb blockade." Since then, I have made several other works on granular

superconductors, and one can get more details in the review [10]. Physics of granular superconductors is the same as physics of artificially designed Josephson networks and the description developed in [9] has been used and further developed later in a huge number of publications.

Surprisingly, the idea of fluctuations of the order parameter turned out to be fruitful in problems of Anderson localization and mesoscopics. Originally, it was suggested in a "bosonic" replica reformulation of models with disorder [11] followed by "fermionic" replica representation [12]. Within this approach one reduces summation of certain classes of diagrams (so called "diffusons" and "cooperons") to study of fluctuations of a matrix that looked like an order parameter. The matrix looked formally as an order parameter and a "free energy functional" looked very similar to the one describing fluctuations of the phase in superconductor. The free energy functional had a form of a non-linear σ-model.

It turned out very soon that, although the σ-model allowed one to perform very efficiently perturbation theory and renormalization group calculations, it was not possible to do non-perturbative calculations. In order to circumvent this difficulty I have derived a super-matrix σ-model [13] that was free of these problems and allowed one to perform essentially non-perturbative calculations. I continued to work in this direction for quite a long time because the method worked not only for the localization problems but also in mesoscopics, quantum chaos, random matrices, etc. [14, 15].

The supermatrix "order parameter" Q that appears in the σ-model approach must be averaged with the free energy functional, and it has no physical sense without carrying out this procedure. Actually, the average $\langle Q\rangle$ with the action of the σ-model is not an interesting quantity because it is the average density of state and the latter quantity is a smooth function of energy. Conductivity, level correlations, density-density correlations, etc. can be written in terms of a product of several Q like, e.g. $\langle QQ\rangle$. Therefore, the matrix Q is not an order parameter in its usual sense. In order to obtain an interesting physical quantity, one should integrate a product of several Q over all configurations.

To my great surprise, I have encountered a rather similar situation in my investigation of a possibility of existence of a thermodynamically stable "Time-crystal." The time crystal is expected to demonstrate an oscillating behavior of physical quantities in time. The concept of a time-crystal has been proposed several years ago [16] in a simple model but later it turned out that the time crystal state proposed there was not the ground state and therefore could not be stable. Moreover, it was even argued that the thermodynamically stable quantum crystal could not exist at all [17]. The "no-go" theorem has been proven for systems with time-independent wave functions of the ground state, which is usually the case.

However, in a recent preprint [18], I have suggested and investigated a model that can undergo a transition into a state with an order parameter b depending on both real and imaginary times. As a result, wave functions also depend on the real and imaginary time. The position of this non-trivial order parameter in time is arbitrary and the averaging over the positions gives zero. At the same time, the average of a product of the order parameters can be finite and can be measured experimentally. For example, the average $\langle bb\rangle$ can be measured in quantum scattering experiments.

A more detailed presentation of the results listed in the introduction will be given in the next sections. It is not a review of publications in several different fields of physics. I simply want to emphasize considering several examples that many new results can be obtained using the generalized concept of the order parameter. Using this approach one can considerably simplify calculations because it is sufficient to consider large distances without going into details of band structures and interactions. Very often this route gives a possibility to solve problems that have not been solved before and predict new physical phenomena. I have realized the efficiency of this approach during my years at the Landau Institute and used it later in many works.

Although I have spent at the Landau Institute as Master and PhD student and later as researcher only 17 years of my already rather long scientific carrier, these years were decisive in forming my scientific profile and scientific tastes. It was typical for scientists working there to develop general fruitful concepts, and scientific criteria were very high. I myself have been following these principles all over the years after leaving the Institute.

Isaak Markovich Khalatnikov played an absolutely-outstanding role in creating the Institute, selecting researchers from different fields. Of course, there have been other people who played a very important role in the development of the Institute but his role was unique. The main criterion of selecting new members of the Institute was their ability to do outstanding original research, and different political reasons did not play a considerable role. It is amazing how Isaak Markovich tried to understand results of all works done at the Institute. Even now he really listens talks at seminars and meetings of the Scientific Council. He was always extremely proud of good works performed at the Institute, although it usually did not mean that he was coauthor of those publications. I definitely have nostalgic reminiscences about the time of my work at the Institute.

The present paper is organized as follows:

Sections 2–4 are devoted to a short review of several works that I have done during my work at the Landau Institute that demonstrate how one can use the

concept of fluctuating order parameter. Section 2 contains discussion of how one can calculate non-trivial correlation functions using a hypothesis that their form is determined by gapless excitations, in Section 3 it is shown that Coulomb interaction can destroy the macroscopic superconductivity and make the system insulating, while Section 4 is devoted to development of the supersymmetry method for studying Anderson localization, physics of mesoscopic systems, quantum chaotic motion, etc. In Section 5 I present results of a recent work where the thermodynamic quantum time-space crystal is proposed. The order parameter of a such a crystal oscillates not only in space but also in both real and imaginary time. Conclusions are made in Section 6.

2. BOSONIZATION OF FERMIONIC MODELS IN 1D

The idea of my first PhD works [6, 19] with Anatoly Larkin of reducing calculations for rather complicated 1D models of fermions to simplified models describing low energy fluctuations of the order parameter was motivated by interest to studying quasi-one-dimensional materials. In the limit of weak coupling between the chains, fluctuations become very strong and one cannot use conventional mean field theory. In the first work [19], we suggested an idea to study the system by making a mean field theory for the interaction between the chains, while taking into account the interaction inside a single chain exactly. Without the interchain interaction the transition temperature had to be zero. Introducing the interchain interaction and taking it into account in the mean field approximation we have derived for the critical temperature a mean field equation containing two-particles correlation functions for a single isolated chain.

Although this was a considerable simplification, methods of calculation of two-particles correlation functions in 1D had not been developed for an arbitrary interaction (Tomonaga-Luttinger models with long-range interaction was an exception). Well-developed Bethe Ansatz methods did not allow to calculate the correlation functions and the problem looked quite non-trivial, although solutions had been found in models with a linear spectrum and with a long-range interaction [20, 21], as well as with a special value of the backward scattering [22].

In [19] we have calculated correlation functions of superconducting order parameters in wires considering gapless fluctuations of the phase ϕ, while in [6] calculated correlation functions of a one-dimensional Fermi gas with a strong attraction. The strong attraction lead to formation of bosonic electron pairs with a repulsion between them. Using a Jordan–Wigner transformation we reduced the thermodynamics of the system to the one of spinless fermions, while the correlation function of the superconducting order parameters was written in terms of a Toeplitz determinant. We have calculated these functions at large distances or times and compared them with those calculated in different models in [19–22]. As a result, we discovered that all of them had the form

$$\Pi_s(R,\tau) \propto \frac{T^\alpha}{|\sinh \pi T(R/v_s + i\tau)|^\alpha}, \tag{1}$$

where R is the distance between two points in space, τ is the distance in the imaginary time, T is temperature, v_s is the velocity of excitations, and α is a constant depending on the model considered. All this has allowed us to propose a hypothesis that the form of the correlation functions at large distances and times is formed by gapless excitations, and in order to calculate, e.g., the superconducting correlation function

$$\Pi_s(R,\tau) = \langle \psi^+_\uparrow(R,\tau)\psi^+_\downarrow(R,\tau)\psi_\downarrow(0,0)\psi_\uparrow(0,0)\rangle, \tag{2}$$

where $\psi^+(x,\tau)$ and $\psi(x,\tau)$ are creation and anihillation operators, one has to replace the pairs of the operators by the following operator

$$\psi^+_\uparrow(x,\tau)\psi^+_\downarrow(x,\tau) \to ae^{2i\hat\phi(R,\tau)}, \tag{3}$$

where a is a constant, and calculate the correlation function $\Pi_s(R,\tau)$ representing it in the form

$$\Pi_s(R,\tau) \propto \langle e^{2i\hat\phi(R,\tau)}e^{-2i\hat\phi(0,0)}\rangle_{\text{eff}}. \tag{4}$$

The angular brackets in Eq. (4) stand for quantum mechanical averaging with an effective Hamiltonian

$$\hat H_{\text{eff}} = \frac{1}{2}\int\left[\frac{(\hat\rho(x)-\overline{\rho})^2}{K} + Kv_s^2\left(\frac{\partial\hat\phi(x)}{\partial x}\right)^2\right]dx. \tag{5}$$

In Eq. (5) $\hat\phi(x)$ and $\hat\rho(x)$ are the phase and density operators satisfying the following commutations relations

$$[\hat\rho(x),\hat\phi(x')] = \delta(x-x'), \tag{6}$$

the constant $K = \partial\overline{\rho}/\partial\mu$ is compressibility ($\overline{\rho}$ is the electron density). The imaginary time dependence of the operators $\hat\phi(x,\tau)$ and $\hat\rho(x,\tau)$ is determined by usual quantum mechanical relations

$$\begin{aligned}\hat\phi(x,\tau) &= e^{H_{\text{eff}}\tau}\hat\phi(x)e^{-H_{\text{eff}}\tau},\\ \hat\rho(x,\tau) &= e^{H_{\text{eff}}\tau}\hat\rho(x)e^{-H_{\text{eff}}\tau}.\end{aligned} \tag{7}$$

Calculating the correlation function $\Pi_s(R,\tau)$, Eq. (2), with the Hamiltonian $\hat H_{\text{eff}}$, Eq. (5), one comes to Eq. (1) with

$$\alpha = 2(\pi K v_s)^{-1}. \tag{8}$$

Remarkably, one can calculate in the same way not only the superconducting correlations but also correlation function $\Pi_d(R,\tau)$ of $2p_F$-components $\tilde\rho(x,\tau)$ of the electron density. One introduces this function as

$$\Pi_d(R,\tau) \propto \sum_{\alpha,\beta} \psi^+_\alpha(R,\tau)\psi_\alpha(R,\tau)\psi^+_\beta(0,0)\psi_\beta(0,0) - \bar{\rho}^2. \quad (9)$$

Then, one can make a replacement

$$\psi^+_\alpha(x,\tau)\psi_\alpha(x,\tau) \to b\cos(\pi\int \hat{\rho}(x)dx). \quad (10)$$

Calculation of the correlation function $\Pi_d(R, \tau)$, Eqs. (9), (10), is similar to the calculation of the correlation function $\Pi_s(R, \tau)$, Eq. (2) and one comes to the following formula

$$\Pi_d(R,\tau) \propto \frac{T^{1/\alpha}}{|\sinh \pi T(R/v_s + i\tau)|^{1/\alpha}}, \quad (11)$$

demonstrating an interesting duality with Eq. (1).

We have proposed that the constants v_s, K and, hence, the exponent α can be calculated using the Bethe-Ansatz method. The knowledge of the correlation functions $\Pi_s(R, \tau)$ and $\Pi_d(R, \tau)$ has allowed us to estimate the transition temperatures to both superconducting and dielectric states in quasi-one-dimensional systems. Our hypothesis about the crucial role of the gapless excitations has in fact been confirmed later by the development of the theory of the Luttinger liquid by Haldane [7]. That theory is also based of considering gapless excitations in 1D systems but is more rigorous and allows one to calculate many details.

The superconducting transition temperature T_c can be calculated in the mean field approximation with respect to different chains. The mean field equation can be written within this scheme as

$$1 = W\int \int_0^{1/T} \Pi_s(R,\tau)dRd\tau, \quad (12)$$

where W is proportional to the square of the tunneling amplitude from chain to chain. An equation for the transition into a charge density wave can be written analogously using the correlation function $\Pi_d(R, \tau)$. One should keep in mind, though, that not only the interchain tunneling but also Coulomb interaction enters the mean field equation.

3. GRANULAR SUPERCONDUCTORS AND JOSEPHSON NEWORKS

The importance of the phase fluctuations of the superconducting order parameter has inspired me to apply several years later the same idea for description of granular superconductors. Certain granular materials could be produced in that time, although they were not very homogeneous with respect to the size of the grains and their arrangement. Artificially designed networks of Josephson junctions appeared considerably later but, from the point of view of the theory developed in [9], these systems are equivalent. At first glance, one could use the a similar scheme as described in the previous section by replacing the quasi-one-dimensional system by a "quasi-zero-dimensional" one.

However, in one-dimensional systems the long-range Coulomb interaction does not play an important role. Everything is completely different in a granular system. Indeed, in order to carry supercurrent a Cooper pair has to tunnel from a grain to another grain. However, this change of the charge configuration costs a considerable electrostatic energy and the supercurrent can simply be blocked. Fortunately, the electrostatic Coulomb interaction can rather easily be incorporated into an effective Hamiltonian containing only phase fluctuations. In contrast to the one-dimensional systems where theory based on the phase Hamiltonian H_{eff}, Eq. (5), has been introduced semi-phenomenologically, the derivation of the proper Hamiltonian for the granular system could be well justified unless the size of the grains was too small, such that the condition

$$\delta \ll \Delta_0, \quad (13)$$

where δ is the mean level spacing and Δ_0 is the modulus of the superconducting order parameter, could not be fulfilled.

Actually, each grain can be considered as zero dimensional with respect to variations of the order parameter Δ. In other words, $\Delta = \Delta_0 e^{i\phi}$ does not depend on coordinates inside the grains but its values vary from grain to grain. The Coulomb interaction can be taken into account writing the electrostatic energy of the tunneling of the Cooper pair from grain i to grain j as

$$B_{ij} = 2e(C^{-1})_{ij}, \quad (14)$$

where e is the electron charge, and C_{ij} is the capacitance matrix.

Microscopic derivation leads to the following effective Hamiltonian $\hat{H}_{\text{eff}}$ containing only the phases ϕ_i of the order parameter in the grain i

$$\hat{H}_{\text{eff}} = \sum_{i,j}\left[\frac{1}{2}B_{ij}\hat{\rho}_i\hat{\rho}_j + J_{ij}(1-\cos(\phi_i - \phi_j))\right], \quad (15)$$

where the operator $\hat{\rho}_i$ of the number of the Cooper pairs in the grain i equals

$$\hat{\rho}_i = -i\frac{\partial}{\partial\phi_i}, \quad (16)$$

and J_{ij} are Josephson couplings between grains i and j. The eigenvalues of the operators $\hat{\rho}_i$ are integers.

Again, the Hamiltonian $\hat{H}_{\text{eff}}$ describes low energy fluctuations of the order parameter. Higher energy degrees of freedom has been integrated out. The model described by the Hamiltonian $\hat{H}_{\text{eff}}$ is a version of the quantum XY-model. It has been demonstrated [9] making a mean field theory for the model specified in

Eq. (15) that at low temperatures the system undergoes a superconductor-insulator transition. The mean field equation has been written in the form

$$1 = \frac{J}{2}\int_0^{1/T} \Pi(\tau)d\tau, \tag{17}$$

where $\Pi(\tau)$ is the correlation function of the superconducting order parameters

$$\Pi(\tau) = \langle e^{i\phi(0)}e^{-i\phi(\tau)}\rangle_0, \tag{18}$$

$J = \sum_j J_{ij}$, and

$$\phi(\tau) = e^{i\hat{H}_0\tau}\phi e^{-i\hat{H}_0\tau}.$$

In Eqs. (18),

$$\hat{H}_0 = \frac{1}{2}\sum_{i,j} B_{ij}\hat{\rho}_i\hat{\rho}_j \tag{19}$$

is the effective Hamiltonian of isolated grains and the angular brackets $\langle ...\rangle_0$ stand for the averaging with this Hamiltonian. Equations (17)–(19) show that the phase diagram is completely determined by the correlation function of the order parameters $\Pi(\tau)$. It is important that $\Pi(\tau)$ satisfies the bosonic periodic boundary condition

$$\Pi(\tau) = \Pi(\tau + 1/T). \tag{20}$$

Calculation of the average in Eq. (18) is not difficult and one comes to the following formula

$$\Pi(\tau) = Z^{-1}\sum_{n_1,n_2,n_3...n_N} \exp\left[-\frac{\tau B_{11}}{2} - \tau\sum_j B_{1j}n_j\right] \times \exp\left[-\frac{1}{2T}\sum_{i,j} B_{ij}n_in_j\right], \tag{21}$$

where

$$Z = \sum_{n_1,n_2,n_3...n_N} \exp\left[-\frac{1}{2T}\sum_{i,j} B_{ij}n_in_j\right]. \tag{22}$$

In Eqs. (21), (22) summation is performed over $n_i = 0, \pm 1, \pm 2, ...$

One can see easily that the function $\Pi(\tau)$ satisfies the periodicity condition (20). At $T = 0$, the correlation function $\Pi(\tau)$ reduces to a simple form

$$\Pi(\tau) = \exp\left[-\frac{\tau B_{11}}{2}\right]. \tag{23}$$

Substituting Eq. (23) into Eq. (17) one obtains the critical value J_c of the Josephson coupling at which the macroscopic superconductivity appears

$$J_c = B_c = \frac{B_{11}}{2}. \tag{24}$$

In Eq. (24) the energy E_c is the energy of adding a Cooper pair into the grain. Nowadays, the effect of the destruction of superconductivity or conductivity by the Coulomb interaction is known under the name "Coulomb blockade."

An interesting property of the function $\Pi(\tau)$ is that it is periodic not only in the imaginary time, Eq. (20), but also in the real one t. Making a Wick rotation $\tau \to it$ one obtains for the function $K(t) = \Pi(it)$,

$$K(t) = Z^{-1}\sum_{n_1,n_2,n_3...n_N} \exp\left[-\frac{itB_{11}}{2} - it\sum_j B_{1j}n_j\right] \times \exp\left[-\frac{1}{2T}\sum_{i,j} B_{ij}n_in_j\right]. \tag{25}$$

In the limit of low temperatures $T \to 0$ one comes to a simple formula

$$K_0(t) = \exp\left[-\frac{itB_{11}}{2}\right] \tag{26}$$

demonstrating the periodicity of the function $K_0(t)$ with the period t_0,

$$t_0 = \frac{2\pi}{E_c}. \tag{27}$$

In this limit one can consider the grain as a two-level systems ("no Cooper pairs" and "one Cooper pair").

The real time correlation function determines a frequency dependent response $K(\omega)$ to an external electric field. At $T = 0$, its imaginary part has a δ-functional form

$$\mathrm{Im}\, K(\omega) \propto \delta\left(\omega - \frac{1}{2}(B_{11} + B_{22} - 2B_{12})\right). \tag{28}$$

Equation (28) can be used when the tunneling between the two grains is small but finite.

It is interesting to note that the same correlation function $\Pi(\tau)$, Eq. (21), arises in grains fabricated from normal metal [23], which is not accidental because this is also an effect of the charge quantization. One can read more about the granular electron systems in the review [10].

4. SUPERSYMMETRY IN DISORDER AND CHAOS

4.1. Prehistory

The prediction of a new phenomenon of the Anderson localization [24] has strongly stimulated both theoretical and experimental study of disordered materials. At the same time, one could see from the Anderson's work that quantitative description of the disordered systems was not a simple task and many conclusions were based on semi-qualitative arguments. Development of theoretical methods for quantitative study of quantum effects in disordered systems was clearly very demanding.

The most straightforward way to take into account disorder is using perturbation theory in the strength of

the disorder potential [25]. However, the phenomenon of the localization is not easily seen within this method and the conventional classical Drude formula for conductivity was considered in [25] as the final result for the dimensionality $d > 1$. This result is obtained after summation of diagrams without intersection of impurity lines. Diagrams with intersection of the impurity lines give a small contribution if the disorder potential is not strong, so that $\varepsilon_0\tau \gg 1$, where ε_0 is the energy of the particles (Fermi energy in metals) and τ in the elastic scattering time.

Although there was a clear understanding that the diagrams with the intersection of the impurity lines were not small for one dimensional chains, $d = 1$, performing explicit calculations for those systems was difficult. This step has been done considerably later by Berezinsky [26] who demonstrated localization of all states in 1D chains by summing complicated series of the perturbation theory. As concerns higher dimensional systems, $d > 1$, the Anderson transition was expected at a strong disorder but it was clear that the perturbation theory could not be applied in that case.

So, the classical Drude theory- was considered as a justified way of the description of disordered metals in $d > 1$ and $\varepsilon_0\tau \gg 1$. At the same time, several results for disordered systems could not be understood within this simple generally accepted picture.

In 1965 Gorkov and Eliashberg [27] suggested a description of level statistics in small disordered metal particles using the random matrix theory (RMT) of Wigner and Dyson [28, 29]. At first glance, the diagrammatic method of [25] had to work for such a system but one could not see any indication on how the formulae of RMT could be obtained diagrammatically. Of course, the description of [27] was merely a hypothesis and the RMT had not been used in the condensed matter before but nowadays it looks rather strange that this problem did not attract an attention. Apparently, the diagrammatic methods were not very widely used in that time and therefore not so many people were interested in resolving such problems.

Actually, the discrepancies were not discussed in the literature until 1979, the year when the publication [30] appeared. In this work, localization of all states for any disorder already in $2D$ was predicted. This result has attracted much attention and it was simply unavoidable that people started thinking about how to confirm it diagrammatically. The only possibility could be that there were some diverging quantum corrections to the classical conductivity, and soon the mechanism of such divergencies has been discovered [31–33].

It turns out that the sum of a certain class of the diagrams with intersecting impurity lines diverges in the limit of small frequencies $\omega \to 0$ in a low dimension $d \le 2$ and it can be considered as a new effective mode. This mode has a form of the diffusion propagator and its contribution to the conductivity $\sigma(\omega)$ can be written in the form

$$\sigma(\omega) = \sigma_0\left(1 - \frac{1}{\pi\nu}\int \frac{1}{D_0\mathbf{k}^2 - i\omega}\frac{d^d\mathbf{k}}{(2\pi)^d}\right), \qquad (29)$$

where $D_0 = v_0^2\tau/3$ is the classical diffusion coefficient and $\sigma_0 = 2e^2\nu D_0$ is the classical conductivity. The parameters v_0 and ν are the Fermi velocity and density of states on the Fermi surface.

Similar contributions arise also in other quantities. Equation (29) demonstrates that in the dimensions $d = 0$, 1, 2 the correction to conductivity diverges in the limit $\omega \to 0$. It is very important that the dimension is determined by the geometry of the sample. In this sense, small disordered particles correspond to zero dimensionality, $d = 0$, and wires to $d = 1$.

In this way, one can reconcile the hypothesis about the Wigner–Dyson level statistics in disordered metal particles and assertion about the localization in thick wires and $2D$ films with the perturbation theory in the disorder potential. The divergences due to the contribution of the diffusion modes make the perturbation theory inapplicable in the limit $\omega \to 0$ and therefore one does not obtain just the classical conductivity using this approach.

Unfortunately, the divergence of the quantum corrections to the conductivity in the limit $\omega \to 0$ makes the direct analytical consideration very difficult for small ω because even the summation of all orders of the perturbation theory does not necessarily lead to the correct result. For example, the formulae for the level-level correlation functions [28, 29] contain oscillating parts that cannot be obtained in any order of the perturbation theory.

All this meant that a better tool had to be invented for studying the localization phenomena and quantum level statistics. Analyzing the perturbation theory one could guess that a low energy theory explicitly describing the diffusion modes rather than single electrons might be an adequate method.

The first formulation of such a theory was proposed by Wegner [11] who has introduced a non-linear σ-model based on a replica representation of electron Green functions in a form of functional integrals over complex fields.

Working with this model one has to integrate over $N \times N$ matrices Q obeying the constraint $Q^2 = 1$. The σ-model is renormalizable and renormalization group equations were written in [11]. These equations agreed with the perturbation theory of Eq. (29) and with the scaling hypothesis of [30].

However, the saddle point approximation was not carefully worked out in [11] because the saddle points were in the complex plane, while the original integration had to be done over the real axis. This question was addressed in the subsequent publications [12, 34].

In the work [34], the initial derivation of [11] was done more carefully and the authors have come to the conclusion that the matrices Q varied on a hyperboloid.

In contrast, the derivation of [12] was based on a representation of the electron Green functions in a form of an integral over Grassmann anticommuting variables. As a result, the σ-model derived in [12] has lead to the result that the $N \times N$ matrix Q varied on a sphere. As in all these works the replica trick was used, there was no contradiction between the approaches because one expected that in the limit $N \to 0$ both the models would give the same results. Indeed, the perturbation theory and renormalization group calculations lead to identical formulas.

The compact replica σ-model of [12] has several years later been extended by Finkelstein [35] to models of interacting electrons. An additional topological term was added to this model by Pruisken [36] for studying the Integer Quantum Hall Effect. So, after all, the compact replica σ-models have helped to solve interesting problems in the localization theory.

I was excited by all this development in particular because the description of the disorder problems could be reformulated in terms of field theories. One could speak of the matrix Q in terms of an order parameter and could consider the diffusion modes as Goldstone modes arising due to degeneracy of the ground state. This correlated very well with my tastes, and I decided to move into this field instead of continuing study of granular superconductors and Josephson networks.

4.2. Supermatrix "Order Parameter"

However, everything turned out to be considerably more complicated for non-perturbative calculations. Attempts to study the level-level statistics in a limited volume and localization in disordered wires using the replica σ-model of [12] have led me to the conclusion that the replica σ-models were not a convenient tool for studying non-perturbative problems.

At this point I would like to mention again the atmosphere at the Landau Institute created by Isaak Markovich. I had spent a year trying to calculate a level-level correlation function in a disordered metallic grain but I did not feel any pressure to publish something and could continue my work. Finally, it resulted in constructing another type of the σ-model that was not based on the replica trick. I called the proposed technique "supersymmetry method," although the word "supersymmetry" is often used in field theory in a narrower sense. The field theory derived for the disordered systems using this approach has the same form of the σ-model as the one obtained with the replica trick, and all perturbative calculations are similar [37].

The free energy functional $F[Q]$ of the σ-model has a standard form

$$F[Q] = \frac{\pi\nu}{8}\int S\mathrm{Tr}[D_0(\nabla Q)^2 + 2i(\omega + i\delta)\Lambda Q]d\mathbf{r}, \quad (30)$$

where $D_0 = v_0^2\tau/d$ is the classical diffusion coefficient (v_0 is the Fermi velocity and d is the dimensionality of the sample) and the 8×8 supermatrix Q obeys the constraint

$$Q^2 = 1, \quad (31)$$

and ω is the frequency and ν is the average density of states. The symbol "*Str*" stands for supertrace.

The matrix Λ equals

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (32)$$

and the supermatrix Q can be written in the form

$$Q = U\Lambda\bar{U},$$

where $\bar{U}U = 1$.

The matrix U has both compact sector on the sphere and non-compact one on the hyperboloid. They are glued with each other by anticommuting Grassmann fields.

Calculation of, e.g., density-density correlation function $K_\omega(\mathbf{r})$ reduces to calculation of a functional integral over Q

$$K_\omega(\mathbf{r}) = 2\int Q^{12}_{\alpha\beta}(0)Q^{21}_{\beta\alpha}(\mathbf{r})\exp(-F[Q])DQ, \quad (33)$$

while

$$\langle Q\rangle = \int Q\exp(-F[Q])DQ = 1 \quad (34)$$

is proportional to the average density of states.

The level-level correlation function $R(\omega)$ is given by relation

$$R(\omega) = \frac{1}{2} - \frac{1}{2}\mathrm{Re}\int Q^{11}_{11}Q^{22}_{11}\exp(-F_0[Q])dQ, \quad (35)$$

where ω is the distance between the levels. One should also notice absence of a weight denominator in Eqs. (33), (34) that simply equals to unity due to the supersymmetry. The superscripts of the matrix Q stand for blocks that are in the same space as those in Eq. (32) and the subscripts numerate elements in these blocks.

Equations (33)–(35) display a reformulation of the initial problem of disordered metals in terms of a field theory that does not contain disorder because the averaging over the initial disorder has already been carried out. The latter enters the theory through the classical diffusion coefficient D_0. The supermatrix σ-model described by Eq. (33) resembles σ-models used for calculating contributions of spin waves for magnetic materials. At the same time, the non-compactness of the symmetry group of the supermatrices Q makes this σ-model unique.

We see that the supermatrix Q plays to some extent the role of an order parameter and its fluctuations are similar to Goldstone modes. At the same time, its average is a constant and, actually, the Anderson metal-insulator transition occurs as a result of fluctuations.

The first attempt to calculate the level-level correlation function lead to a real surprise: the method worked [38]. For example, considering the unitary ensemble one could reduce calculation of the integral in Eq. (35) to the following integral over two variables

$$R(\omega) = 1 + \frac{1}{2}\mathrm{Re}\int_1^\infty\int_{-1}^1 \exp[i(x+i\delta)(\lambda_1-\lambda)]d\lambda_1 d\lambda, \quad (36)$$

where $x = \pi\omega/\Delta$, and Δ is the mean level spacing. Calculations for the orthogonal and symplectic ensembles could be reduced to integrals over three variables. The remaining integration is trivial for the unitary ensemble, Eq. (36), and doable for the orthogonal and symplectic ensembles leading to the famous formulae for level-level correlation functions

$$R_{\mathrm{orth}}(\omega) = 1 - \frac{\sin^2 x}{x^2} - \frac{d}{dx}\left(\frac{\sin x}{x}\right)\int_1^\infty \frac{\sin xt}{t}\,dt, \quad (37)$$

$$R_{\mathrm{unit}}(\omega) = 1 - \frac{\sin^2 x}{x^2}, \quad (38)$$

$$R_{\mathrm{sympl}}(\omega) = 1 - \frac{\sin^2 x}{x^2} + \frac{d}{dx}\left(\frac{\sin x}{x}\right)\int_0^1 \frac{\sin xt}{t}\,dt. \quad (39)$$

Equations (37)–(39) are known in the Wigner-Dyson theory [28, 29], and this result established the relevance of the latter to the disordered systems. Since then one could use the RMT for calculations of various physical quantities in mesoscopic systems or calculate directly using the zero-dimensional supermatrix σ-model. Actually, to the best of my knowledge, this was the first explicit demonstration that RMT could correspond to a real physical system. Its original application to nuclear physics was in that time phenomenological and confirmed by neither analytical nor numerical calculations.

A direct derivation of Eqs. (37)–(39) from Gaussian ensembles of the random matrices using the supermatrix approach was done in the review [39]. This allowed the authors to compute certain average compound-nucleus cross sections that could not be calculated using the standard RMT route.

The proof of the applicability of the RMT to the disordered systems was followed by the conjecture of Bohigas, Giannonni and Schmid [40] about the possibility of describing by RMT the level statistics in classically chaotic clean billiards. Combination of the results for clean and disordered small systems (billiards) has established the validity of the use of RMT in mesoscopic systems. Some researches use for explicit calculations methods of RMT but many others use the supermatrix zero-dimensional σ-model (for review see, e.g., [41, 42, 50]). At the same time, the σ-model is applicable to a broader class of systems than the Wigner–Dyson RMT because it can be used in higher dimensions as well. Actually, one can easily go beyond the zero dimensionality taking higher space harmonics in $F[Q]$, Eq. (30). In this case, the universality of Eqs. (37), (38) is violated. One can study this limit for $\omega \gg \Delta$ using also the standard diagrammatic expansions of [25] and this was done in [43].

The calculation of the level correlations in small disordered systems followed by the full solution of the localization problem in wires [44], on the Bethe lattice and in high dimensionality [45–49]. After that it has become clear that the supersymmetry technique is really an efficient tool suitable for solving various problems of theory of disordered metals.

By now several reviews and a book have been published [14, 15, 39, 50–53] where numerous problems of disordered, mesoscopic and ballistic chaotic system are considered and solved using the supersymmetry method. The interested reader can find all necessary references in those publications.

5. THERMODYNAMIC QUANTUM TIME-SPACE CRYSTAL

Space and time play in many respects a similar role in modern physics. At the same time, many materials have stable crystalline structures that are periodic in space but not in time. Are thermodynamic states with a periodic time dependence of physical quantities forbidden by fundamental laws of nature?

Several years ago Wilczek [16] proposed a concept of Quantum Time Crystals using a rather simple model that possessed a state with a current oscillating in time. Later a more careful consideration of the model [54] has led to the conclusion that this was not an equilibrium state. These publications were followed by a hot discussion of the possibility of realization of a thermodynamically stable quantum time crystal [55–60]. More general arguments against thermodynamically stable quantum time crystals have been presented later [17]. As a result, a consensus has been achieved that thermodynamically macroscopic quantum time crystals could not exist.

Slowly decaying oscillations in systems out of equilibrium were not forbidden by the "no-go" theorems, and their study is definitely interesting by its own. Recent theoretical [61–65] and experimental [66–68] works have clearly demonstrated that this research field is very interesting and is fast growing. At present, the term "Quantum Time Crystal" is usually used for non-equilibrium systems.

However, it turns out that thermodynamically stable quantum time crystals are nevertheless possible. The results of [17, 58] are correct for the models considered there but it was implied that the wave functions

of the ground state did not depend on time. Although this assumption is usually correct in the thermodynamic equilibrium, time dependence of the wave functions in the equilibrium is generally not forbidden.

Actually, a phase transition into a state with an order parameter oscillating in both imaginary τ and real t time is possible in a model of interacting fermions, and this is demonstrated in this section. Again, I start by introducing an order parameter b but, in contrast to the previous sections, I do not consider any fluctuations. All calculations are performed in the mean field approximation but the order parameter of a thermodynamically stable state oscillates in time. Here I give a short account of ideas and results, while a more detailed presentation is given in [18].

I start with a model with a Hamiltonian $\hat{H}$ already adopted for using the mean field approximation

$$\hat{H} = \sum_p c_p^+(\varepsilon_p^+ + \varepsilon_p^-\Sigma_3)c_p + \frac{1}{2V}\left[\tilde{U}_0\left(\sum_p c_p^+\Sigma_1 c_p\right)^2 - U_0\left(\sum_p c_p^+\Sigma_2 c_p\right)^2\right]. \quad (40)$$

Equation (40) describes interacting fermions of two bands 1 and 2, and $p = \{\mathbf{p}, \alpha\}$ stands for the momentum $\mathbf{p}$ and spin α. The energies $\varepsilon_a(\mathbf{p})$ are two-dimensional spectra in the bands counted from the chemical potential μ, $\varepsilon_p^{\pm} = \frac{1}{2}(\varepsilon_1(\mathbf{p}) \pm \varepsilon_2(\mathbf{p}))$, the interaction constants U_0 and $\tilde{U}_0$ are positive, while V is the volume of the system. Two-component vectors c_p, c_p^+ contain creation and annihilation operators for the fermions of the bands 1 and 2. Matrices Σ_1, Σ_2, Σ_3 are Pauli matrices in the space of numbers 1 and 2 numerating the bands.

The Hamiltonian $\hat{H}$ written for the electron-hole pairs resembles the Bardeen-Cooper-Schrieffer (BCS) Hamiltonian for Cooper pairs [69]. At the same time, one can imagine other physical systems described by Eq. (40). Hamiltonian $\hat{H}$ contains an inter-band attraction (term with matrix Σ_2) and repulsion (term with Σ_1). Taking into account only the term with the attraction one obtains in a spin-fermon model introduced earlier [70, 71] static loop currents oscillating in space with the double period of the lattice [72]. This corresponds to a hypothetical d-density wave (DDW) state [73].

In order to obtain the new thermodynamic quantum time-space crystal state, one should consider both the types of the interaction. It turns out [18] in this case that the novel state with an order parameter oscillating both in real and imaginary time is also possible in addition to the DDW state. The order parameters of both the states oscillate in space with a vector $\mathbf{Q} = (\pi, \pi)$ corresponding to the vector of antiferromagnetic modulations but their time dependence is drastically different. In order to prove the existence of this state one should calculate the free energy of the new state and compare it with that of the DDW state.

Thermodynamics of quantum systems can be very conveniently studied using imaginary time τ in the interval $(0, 1/T)$ [25] and I sketch here the main steps of the calculation of the free energy. The real-time behavior will be derived using a Wick rotation $\tau \to it$.

Following the mean field theory scheme one introduces imaginary-time order parameters $b(\tau)$ and $b_1(\tau)$ corresponding to the two interaction terms in Eq. (40), and computes trace over fermionic states. Making a rotation

$$c_p(\tau) = \mathcal{U}_0\tilde{c}_p(\tau), \quad \mathcal{U}_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad (41)$$

and using the relations

$$\mathcal{U}_0^+\Sigma_2\mathcal{U}_0 = \Sigma_3, \quad \mathcal{U}_0^+\Sigma_3\mathcal{U}_0 = -\Sigma_2, \quad \mathcal{U}_0^+\Sigma_1\mathcal{U}_0 = \Sigma_1, \quad (42)$$

one can write the free energy functional $\mathcal{F}$ of the model described by the Hamiltonian $\hat{H}$ (40) in the form

$$\frac{\mathcal{F}[b, b_1]}{T} = \int_0^{1/T}\left[-2\sum_{\mathbf{p}} \mathrm{tr}[\ln(h(\tau, \mathbf{p}) - ib_1(\tau)\Sigma_1)]_{\tau,\tau} + V\left(\frac{b^2(\tau)}{2U_0} + \frac{b_1^2(\tau)}{2\tilde{U}_0}\right)\right]d\tau, \quad (43)$$

where

$$h(\tau, \mathbf{p}) = \partial_\tau + \varepsilon^+(\mathbf{p}) - \varepsilon^-(\mathbf{p})\Sigma_2 - b(\tau)\Sigma_3,$$

symbol "tr" means trace in space of the bands 1, 2, and

$$b(\tau) = b(\tau + 1/T), \quad b_1(\tau) = b_1(\tau + 1/T). \quad (44)$$

The form of the interaction between the electron-hole pairs in Hamiltonian $\hat{H}$ (40) makes the mean field theory exact. Both the terms in the functional $\mathcal{F}[b, b_1]$ are proportional to the volume V, and one can obtain the physical free energy simply minimizing $\mathcal{F}[b, b_1]$ with respect to $b(\tau)$ and $b_1(\tau)$. Although $b(\tau) = \gamma$, $b_1(\tau) = 0$ obtained previously [72] provides a minimum of $\mathcal{F}[b, b_1]$, there is a region of parameters where the absolute minimum is reached at τ-dependent functions $b(\tau)$ and $b_1(\tau)$. Unfortunately, the exact minimization is difficult and certain additional approximations will be used.

Let us assume for a while that $b_1(\tau) = 0$ and study extrema of $\mathcal{F}[b(\tau), 0]$. Varying the functional $\mathcal{F}[b(\tau), 0]$ one comes to the following equation

$$b(\tau) = -U_0\mathrm{tr}\int \Sigma_3[h^{-1}(\tau, \mathbf{p})]_{\tau,\tau}\frac{d\mathbf{p}}{(2\pi)^2}. \quad (45)$$

Although Eq. (45) is quite non-trivial due to a possible dependence of $b(\tau)$ on τ, solutions $b_0(\tau)$ can be

written exactly in terms of a Jacobi double periodic elliptic function $\mathrm{sn}(x|k)$,

$$b_0(\tau) = k\gamma \mathrm{sn}(\gamma(\tau - \tau_0)|k), \quad (46)$$

where parameter k, $0 < k < 1$, is the modulus, γ is an energy, and τ_0 is an arbitrary shift of the imaginary time in the interval $0 < \tau_0 < 1/T$. In the limit $k \to 1$, the function has an asymptotic behavior $\mathrm{sn}(x|k) \to \pm\tanh x$, while in the limit $k \ll 1$ one obtains $\mathrm{sn}(x|k) \to \sin x$.

The period of the oscillations for an arbitrary k equals $4K(k)/\gamma$, where $K(k)$ is the elliptic integral of the first kind, and therefore the condition

$$\gamma = 4K(k)mT \quad (47)$$

with integer m must be satisfied to fulfill Eqs. (44). In the most interesting limit of small $1 - k$, the period $4K(k)/\gamma$ of $b_0(\tau)$ grows logarithmically as $\frac{1}{2}\ln\left(\frac{8}{1-k}\right)$, and the solution $b_0(\tau)$ consists of $2m$ well separated alternating instantons and anti-instantons with the shape $\pm\gamma\tanh\gamma\tau$. It is important that the integral over the period of the oscillations in Eq. (46) equals zero. Averaging over the position τ_0 of the instanton one obtains zero as well

$$\overline{b_0(\tau)} = 0. \quad (48)$$

The existence of the non-trivial local minima $\mathcal{F}[b, 0]$, Eq. (43), at $b_0(\tau)$ has been established previously by Mukhin [74–76] starting from a different model. Generally, there can be many solutions corresponding to different minima of $\mathcal{F}[b, 0]$ depending on the number m of instanton-antiinstanton pairs (IAP). However, the lowest value of the functional $\mathcal{F}[b, 0]$ is reached at $m = 0$ corresponding to the static order [18]. Coordinate-dependent Jacobi elliptic functions are also solutions of a mean field time-independent equation arising in 1D models of polymers, which has been discovered long ago [77].

The field $b_1(t)$ does not couple to the static order in the mean field approximation, and this is why this field was not considered previously [72]. The absence of the coupling is quite natural because the order parameter $b(\tau)$ describes loop currents oscillating in space with $\mathbf{Q} = (\pi, \pi)$, while $b_1(\tau)$ corresponds to a charge oscillations with the same vector. The situation changes when the field $b(\tau)$ varies in time. In order to understand what happens, one should calculate a linear term in $b_1(\tau)$ in the expansion of $\mathcal{F}[b, b_1]$, Eq. (43), in $b_1(\tau)$ for $b(\tau) = b_0(\tau)$.

Substituting $b_0(\tau)$ into Eq. (43) and expanding $\mathcal{F}[b, b_1]$ in $b_1(\tau)$ one obtains the following linear term

$$\frac{\mathcal{F}_{\mathrm{int}}[b_1]}{VT} = -\frac{J}{2}\int_0^{1/T} \dot{b}_0(\tau)b_1(\tau)d\tau, \quad (49)$$

where J is a constant.

Fluctuations of $b_1(\tau)$ generate an effective attraction between the instantons and anti-instantons and favor formation of τ-dependent structures. The mechanism of the attraction is similar to the one of the electron-phonon interaction in solids, which can be seen after a formal replacement of τ by a coordinate. The field $b_1(\tau)$ plays in this picture the role of phonons, and its fluctuations may result in a sufficiently strong attraction of instantons and anti-instantons. Eventually, an order parameter $b(\tau)$ depending on the imaginary time τ can provide the minima of the free energy.

Exact minimization of the free energy functional $\mathcal{F}[b, b_1]$ (43) is difficult. Therefore we simplify the study by considering the limit of low temperatures T when one can expect a large number of IAP in the system and of small $1 - k$ corresponding to a large period of the IAP lattice. In this limit, the difference ΔF between the total free energy F and the free energy F_{st} of the system with the static order parameter is proportional to $2m$. The case $\Delta F/(2mTV) > 0$ corresponds to the state with the static order parameter, while in the region of parameters where $\Delta F/V(2mT) < 0$ one expects a chain of alternating instantons and antiinstantons.

Computation of the free energy is performed choosing

$$\begin{aligned}\varepsilon_1(\mathbf{p}) &= \alpha p_x^2 - \beta p_y^2 + P - \mu,\\ \varepsilon_2(\mathbf{p}) &= \alpha p_y^2 - \beta p_x^2 - P - \mu\end{aligned} \quad (50)$$

corresponding to the spectrum of cuprates near the middle of the edges of the Brillouin zone (momenta $\mathbf{p}$ are counted from the middle of the edges), where P is a Pomeranchuk order parameter obtained previously in a spin-fermion model with overlapping hot spots [70], and μ is the chemical potential.

Numerical calculation of the free energy ΔF has been performed expanding the free energy functional $\mathcal{F}[b, b_1]$ up to the second order in $b_1(\tau)$ and $\delta b(\tau) = b(\tau) - b_0(\tau)$, and finding the minimum of the quadratic form of these variables. It has been demonstrated in [18] that for small $1 - k$ there is a region of parameters where the free energy ΔF becomes negative, which indicates that the time-independent DDW state is unstable. In this region a chain of alternating instantons and antiinstantons appears. The number m of the instanton-antiinstanton pairs depends on temperature T but this dependence has not been determined sofar.

Here, structures periodic in space (oscillations with vector $\mathbf{Q}_{AF}$ connecting the bands 1 and 2) are considered. Therefore the periodic in τ order parameter $b(\tau)$ providing the minimum of the free energy is at the same time the amplitude of the periodic oscillations in space. As the present consideration does not determine the number of the function of temperature, we calculate physical quantities without specifying the value of m or k related to each other by Eq. (47).

The periodic structure described by the Jacobi elliptic function $b_0(\tau)$ (46) is actually double periodic in the complex plane of τ and, hence, is periodic in real time t. Remarkably, $b_0(it)$ still satisfies Eq. (45) after the rotation $\tau \to it$.

It is convenient to introduce a function $B(t)$,

$$B(t) = b(it). \tag{51}$$

The Jacobi elliptic function $\mathrm{sn}(iu, k)$ of an imaginary argument iu is related to an antisymmetric elliptic function $\mathrm{sc}(u|k)$ with the period $2K(k)$ as [78]

$$\mathrm{sn}(iu|k) = k\,\mathrm{sc}(u|k'), \quad k^2 + k'^2 = 1.$$

Therefore, the order parameter $B(t)$ is an imaginary and antisymmetric in time (counted from t_0) function, while the function $B_1(t) = b_1(it)$ is real and symmetric. One can write $B(t)$ in the form

$$B(t) = b(it) = i\gamma k\,\mathrm{sc}(\gamma(t - t_0)|k'), \tag{52}$$

where t_0 is an arbitrary shift of time.

The oscillating behavior of the function $B(t)$ leads to oscillations of wave functions. In order to calculate observable physical quantities one should average the result over t_0 and this gives immediately

$$\overline{B(t)} = 0, \tag{53}$$

which means that the average order parameter vanishes.

In order to calculate a 2-times correlation function

$$N(t) = \overline{B(t)B^*(0)}, \tag{54}$$

one can use a Fourier series for the function $\mathrm{sc}(u|k)$. Then, the average over t_0 leads in the limit $1 - k \ll 1$ to the following result [18]

$$N(t) \approx 2\gamma^2 \sum_{n=1}^{\infty} f_n^2 \cos(2\gamma n t), \tag{55}$$

where

$$f_n = \left[1 - \frac{1}{2}\left(\frac{1-k}{8}\right)^{2n}\right]. \tag{56}$$

Function $N(t)$ shows an oscillating behavior with the frequencies $2\gamma n$ (we put everywhere $\hbar = 1$). The energy 2γ is the energy of the breaking of electron-hole pairs and one can interpret the form of $N(t)$ as oscillations between the static order and normal state. The oscillations of $N(t_1 - t_2)$ resemble those of the order parameter in the non-equilibrium superconductors [79–85] but, in contrast to the latter, the function $N(t)$ does not decay in time. The contribution of high harmonics n does not decay with n but apparently this is a consequence of the used approximations.

Non-decaying time oscillations of the two-time correlation function $N(t)$ together with the vanishing of the single time average (53) allow us to generalize the definition of a space crystal by including time in addition to the space coordinates. Therefore the physical state found here can be classified as "Thermodynamic Quantum Time-Space Crystal."

The correlation function $N(t)$, Eqs. (55), (56), was calculated by averaging over the position t_0. Remarkably, the same results for correlation functions can be obtained using an alternative description based on the notion of an "operator order parameter" $\hat{B}$. One can formally introduce a Hamiltonian $\hat{H}_{TC}$ for a harmonic oscillator

$$\hat{H}_{TC} = 2\gamma\left(a^+ a + \frac{1}{2}\right), \tag{57}$$

where a^+ and a are bosonic creation and annihilation operators (for simplicity, we consider here the limit $1 - k \ll 1$). Using the Hamiltonian $\hat{H}_{TC}$ one can write the correlation function $N(t_1 - t_2)$ in the form

$$N(t) = \gamma^2(\langle 0|A(t)A^+(0)|0\rangle + \langle 0|A(0)A^+(t)|0\rangle), \tag{58}$$

where

$$A^+(t) = e^{i\hat{H}_{TC}t} A^+ e^{-i\hat{H}_{TC}t}, \quad A^+ = \sum_{n=1}^{\infty} f_n \frac{(a^+)^n}{\sqrt{n!}},$$

and $|0\rangle$ stands for the wave function of the ground state of the Hamiltonian $\hat{H}_{TC}$ (57). At the same time, quantum averages of the operators A and A^+ vanish

$$\langle 0|A(t)|0\rangle = \langle 0|A^+(t)|0\rangle = 0.$$

One can interpret the operator A as an operator order parameter. This type of the order parameters extends the variety of conventional order parameters like scalars, vectors, matrices used in theoretical physics. The non-decaying time oscillations can be an important property for designing qubits.

Possibility of an experimental observation depends on systems described by the Hamiltonian (40). For cuprates, inelastic polarized neutron spectroscopy can be a proper tool for observations. It is important that the magnetic moments are basically perpendicular to the planes, which can help to distinguish them from the antiferromagnetism spin excitations at (π, π). Calculating the Fourier transform $N(\omega)$ of the function $N(t)$ and comparing it with the one for the hypothetical time-independent DDW state $2\pi\gamma^2\delta(\omega)$ one can write at low temperatures the ratio of the responses at (π, π) for these two states as

$$\chi(\omega, \mathbf{q}) = \chi_0 \sum_{n=1}^{\infty} f_n \delta(\omega - 2n\gamma)\delta(\mathbf{q} - \mathbf{Q}_{AF}), \tag{59}$$

where χ_0 determines the response χ_{DDW} of the DDW state, $\chi_{DDW}(\omega) = \chi_0\delta(\omega)$. In the absence of static magnetic moments the elastic scattering is not expected to bring interesting information. Actually, anisotropic magnetic (π, π) excitations have been observed [86] in

$YBa_2Cu_3O_{6.9}$ but more detailed experiments are necessary to clarify their origin.

The main conclusion of this section is the time-space crystals may exist as a thermodynamically stable state in macroscopic systems. The order parameter of the thermodynamic quantum crystals is periodic in both real and imaginary times as well as in space but its average over the phases of the oscillations vanish. The non-decaying oscillations can be seen, e.g., in two-time correlation functions that determine cross-section in inelastic scattering experiments. The frequency of the oscillations remains finite in the limit of infinite volume, $V \to \infty$. One can expect various experimental consequences and, in particular, one can suppose that the time crystal may be a good candidate for the pseudogap state in superconducting cuprates.

6. CONCLUSIONS

In this paper I tried to present rather different fields of research using a generalized concept of the order parameter. In contrast to the standard notion of a static long range order in an ordered phase, one may encounter situations when there is no static long-range order. One can see from the results of the investigation of several models of electrons with interaction or moving in a random potential that there can be interesting non-trivial physics. The properties of the models have been understood considering either fluctuations or oscillations in space and time of a generalized order parameter. Coulomb blockade, Anderson localization, space-time quantum crystals, etc., are clearly quite different phenomena but their theoretical description has many common features.

An essential part of my the results presented in this paper either has been done at the Landau Institute or followed from ideas developed there. I have started my scientific carrier and worked for many years at the Landau Institute at its best time, and I am personally very-grateful to Isaac Markovich for the creation of the Institute, for the support of my research, and for giving me the possibility to work at the Institute.

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