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Spatially Periodic Inhomogeneous States in a Nonlinear Crystal with a Nonlinear Defect

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Abstract—Spatially periodic inhomogeneous stationary states are shown to exist near a thin defect layer with nonlinear properties separating nonlinear Kerr-type crystals. The contacts of nonlinear self-focusing and defocusing crystals have been analyzed. The spatial field distribution obeys a time-independent nonlinear Schrödinger equation with a nonlinear (relative to the field) potential modeling the thin defect layer with nonlinear properties. Both symmetric and asymmetric states relative to the defect plane are shown to exist. It has been established that new states emerge in a self-focusing crystal, whose existence is attributable to the defect nonlinearity and which do not emerge in the case of a linear defect. The dispersion relations defining the energy of spatially periodic inhomogeneous stationary states have been derived. The expressions for the energies of such states have been derived in an explicit analytical form in special cases. The conditions for the existence of periodic states and their localization, depending on the defect and medium characteristics, have been determined.

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1. INTRODUCTION

Studying the nonlinear surface states existing near the interfaces between dielectric media with different characteristics seems topical in connection with their wide application in various electron-optical devices, including optical data storage systems [1].

At present, there are a lot of theoretical works devoted to analyzing the field distribution and the dispersion relations for waves propagating along the interfaces between linear and nonlinear media [2, 3], two nonlinear media [4–7], and multilayered structures [8–10]. Modifications of the models of interfaces between nonlinear crystals were considered in [11]. The localization of nonlinear waves at the interface between nonlinear media with a spatial dispersion was analyzed in [12]. The interaction of coupled soliton states referring to different states of a two-level system near a defect was considered in [13] for the case of an interface between nonlinear media and [14] for the case of an interface between linear and nonlinear media.

The nonlinear Schrödinger equation (NLSE), which contains a cubic (with respect to the sought-for field) term when describing Kerr media, is commonly used to theoretically study nonlinear waves [15]. The NLSE is applied for the description of physically different nonlinear surface waves, for instance, elastic [6], magnetic [16], and electric [17] ones.

The excitations of fields are often considered in the form of solitons. The solitons in various magnetic systems were theoretically studied long ago and fundamentally [18–21]. In optical systems the solitons were considered both in simple waveguides and in the presence of defects and optical superlattices [22, 23].

When the effects related to the interactions of nonlinear excitations with a planar or point defect are studied, the potential of the field produced by such a defect is introduced into the NLSE [15]. The one-dimensional models in which the NLSE contains a short-range potential in the form

$$U(x) = U_0\delta(x), \quad (1)$$

where $\delta(x)$ is Dirac's delta function and U_0 is the intensity of the interaction between the excitation and the defect located at the coordinate origin (occasionally this quantity is called the defect "power"), are most popular. The excitation is repelled from the defect at $U_0 > 0$ and is attracted to it at $U_0 < 0$.

It should be noted that using a potential in the form (1) does not allow the interaction due to the physical properties of the defect with the local perturbations of the medium parameters forming near it to be analyzed in full. A potential with a quadratic nonlinearity relative to the sought-for field was applied to describe the excitation localization effects in layered structures with nonlinear properties of the layer interfaces [17, 24–26]. In the presence of a weak coupling between parallel-plate waveguides, in which the field ampli-

tude exceeds significantly the average field amplitude in the entire crystal, the nonlinear terms in the NLSE were taken into account inside the waveguides themselves.

In this paper we suggest a generalization of the model of a thin defect layer that is characterized by a Kerr nonlinearity inside and separates crystals with a Kerr nonlinearity proposed in [24, 25]. We will consider the spatially inhomogeneous distribution of the field of excitations of such a nonlinear defect. Our analysis will be performed for Kerr-type nonlinear media with different signs of the nonlinearity corresponding to self-focusing and defocusing. Our main goal is to find all types of spatially inhomogeneous stationary states emerging in the system under consideration and to determine their energy and existence conditions. A peculiarity of this work is that we investigate the periodic states near an interface with nonlinear properties between media with different characteristics. Particular attention will be given to revealing the effects due to the defect nonlinearity, i.e., such that do not emerge (are negligible) near a simple defect in the linear approximation. The formation of a special type of spatially inhomogeneous stationary states that do not emerge in the case of a defect modeled by a potential in the form (1), i.e., in the simplest (linear) approximation for the defect potential, can be attributed to such effects.

2. MODEL EQUATIONS

Consider the contact of two crystalline media. We will assume their interface to be thin compared to the localization distances of the perturbations of medium characteristics produced by it and to be planar. The interface may be considered as a planar crystalline defect, for instance, a twin boundary. Let us choose the coordinates in such a way that the defect plane passes through the coordinate origin and is located in the yz plane perpendicularly to the x axis. The contacting crystals are characterized by an anharmonic interaction of elementary excitations and, therefore, below we will call them nonlinear.

We will consider the excitations uniformly distributed along the defect plane and inhomogeneous in a direction perpendicular to it based on a one-dimensional model, when the dynamics is described by the one-dimensional NLSE

$$i\psi_t' = -\frac{1}{2m}\psi_{xx}'' + \Omega(x)\psi - \gamma(x)|\psi|^2\psi + U(x)\psi, \quad (2)$$

where m is the effective excitation mass,

$$\Omega(x) = \begin{cases}\Omega_1, & x < 0,\\ \Omega_2, & x > 0,\end{cases}$$

$\Omega_{1,2}$ are constants. The nonlinearity parameter in the NLSE will be

$$\gamma(x) = \begin{cases}\gamma_1, & x < 0,\\ \gamma_2, & x > 0,\end{cases}$$

where $\gamma_{1,2}$ are constants.

We will describe the nonlinear properties of the planar defect by a one-dimensional potential in the form [17, 24, 25]

$$U(x) = \{U_0 + W_0|\psi|^2\}\delta(x), \quad (3)$$

where W_0 is the defect nonlinearity parameter whose positive and negative values correspond, respectively, to defocusing and self-focusing in the thin defect layer.

It should be noted that $U_0 = 0$ was adopted in [24, 25]. The nonlinear equation with a term in the form (3) with its two parameters U_0 and W_0 was used in formulating the model of an optical system in which the periodic modulation of the linear refractive index is combined with the spatially inhomogeneous nonlinearity represented by a periodic Kronig–Penney lattice with a single nonlinear defect, a thin-layer nonlinear waveguide [26]. We analyzed the solitons generated by such a lattice approximated by a piecewise constant function in the cases corresponding to the possible combinations of signs of the defect parameters U_0 and W_0. In particular, for positive defects localized modes exist already in the linear regime. The regimes arising in both semi-infinite and the first finite forbidden gaps generated by bifurcations from the corresponding linear states are stable in a weakly nonlinear regime, but are destabilized as their amplitudes increase. In the case of a focusing nonlinearity and a negative defect, all of the modes with the same symmetry, along with those that were stable in the linear regime, become unstable, but a new stable mode appears in the first finite gap. In the case of a defocusing nonlinearity and a negative defect, unique modes of this type can exist and be stable in the first finite gap. Positive defects with a defocusing nonlinearity generate localized waves emerging through bifurcations from the linear modes in both semi-infinite and the first finite gaps. The modes that exist above a certain threshold power needed to change the combined defect response from focusing to defocusing also emerge [27].

We will consider a simple model describing only the stationary states with energy E determined from the time-independent NLSE derived from (2) after substituting a wave function in the form $\psi(x, t) = \psi(x)\exp(-iEt)$ into it:

$$E\psi = -\frac{1}{2m}\psi'' + \Omega(x)\psi - \gamma(x)\psi^3 + U(x)\psi. \quad (4)$$

Solving the NLSE (4) with potential (3) is equivalent to finding the solution of a contact boundary-value problem for the NLSE without a potential:

$$\psi'' + 2m(E - \Omega(x) + \gamma(x)\psi^2)\psi = 0, \quad (5)$$

with two boundary conditions for conjugation at the point $x = 0$ through which the defect plane passes. The continuity of the wave function determines the first (standard) boundary condition:

$$\psi(-0) = \psi(+0) = \psi(0). \quad (6)$$

If we integrate both parts of Eq. (5) with potential (3) over x on a small interval $[-\varepsilon, \varepsilon]$ and subsequently let ε approach zero, then, as a result, we can obtain the second boundary condition [24, 25]:

$$\psi'(+0) - \psi'(-0) = 2m\psi(0)\{U_0 + W_0\psi^2(0)\}. \quad (7)$$

Free waves with a quadratic dispersion law can propagate in a linear crystal without a defect. In the presence of a simple defect described by the short-range potential (1), a symmetric state localized on either side of the defect exists in a linear medium in the case of an attractive defect. Localized states also exist in a nonlinear medium both with a simple defect and with a nonlinear one [4, 17, 24, 25]. Nonlinear waves described by elliptic functions can exist in a nonlinear medium without a defect.

3. THE TYPES OF PERIODIC STATES

For convenience, let us introduce the notation for the wave functions to the left and the right of the interface between the crystals:

$$\psi(x) = \begin{cases} \psi_1(x), & x < 0, \\ \psi_2(x), & x > 0. \end{cases}$$

3.1. Periodic States of the First Type in a Self-Focusing Crystal

In the case of a self-focusing crystal, which corresponds to a positive nonlinearity ($\gamma > 0$) in Eq. (5), when the excitation energy lies in the range $E < \min\{\Omega_1, \Omega_2\}$, the NLSE (5) has a spatially periodic solution in the form

$$\psi_j(x) = A_{cj}\mathrm{cn}(q_{cj}(x - x_{cj}), k), \quad (8)$$

where k is the modulus of the elliptic function cn ($1 > k^2 > 1/2$). Here and below, the indices $j = 1$ and $j = 2$ correspond to the quantities referring to the crystal characteristics to the left and the right of the defect plane at $x < 0$ and $x > 0$, respectively.

Substituting the solution (8) into Eq. (5) gives expressions for the wave numbers and amplitudes in the form

$$q_{cj}^2 = \frac{2m(\Omega_j - E)}{2k^2 - 1}, \quad (9)$$

$$A_{cj}^2 = \frac{k^2 q_{cj}^2}{m\gamma_j}. \quad (10)$$

Substituting the solution (8) into the continuity condition (6) leads to the expression

$$\eta q_{c1}\mathrm{cn}(q_{c1}x_{c1}, k) = q_{c2}\mathrm{cn}(q_{c2}x_{c2}, k), \quad (11)$$

where $\eta = (\gamma_2/\gamma_1)^{1/2}$.

Substituting the solution (8) into the nonlinear boundary condition (7) leads to the expression

$$D_{c2} - D_{c1} = mU_0 + W_0 k^2 q_{c1}^2 \frac{\mathrm{cn}^2(q_{c1}x_{c1}, k)}{\gamma_1}, \quad (12)$$

where

$$D_{cj} = \frac{1}{2}\frac{q_{cj}\mathrm{sn}(q_{cj}x_{cj}, k)}{\mathrm{sn}(q_{cj}x_{cj} + K(k), k)},$$

$K(k)$ is the complete elliptic integral of the first kind.

Below we will call the pair of relations (11) and (12) the dispersion ones. For instance, from (11) we can determine, for example, x_{c2} and substitute it into (12), thereby eliminating it. Then, from (12) we find the energy as a function of the system's parameters: $E = E(m, U_0, W_0, \gamma_{1,2}, \Omega_{1,2}, x_{c1}, k)$. As a result, once E has been found, from (11) we determine x_{c2} as a function of the system's parameters: $x_{c2} = x_{c2}(m, U_0, W_0, \gamma_{1,2}, \Omega_{1,2}, x_{c1}, k)$. Therefore, (8) is a two-parameter solution of the NLSE with two free parameters, k and x_{c1}.

3.2. Periodic States of the Second Type in a Self-Focusing Crystal

In the case where the excitation energy lies in the range $E < \min\{\Omega_1, \Omega_2\}$, the NLSE (5) has a different spatially periodic solution:

$$\psi_j(x) = A_{dj}\mathrm{dn}(q_{dj}(x - x_{dj}), k), \quad (13)$$

which is expressed via the elliptic function dn.

Substituting the solution (13) into Eq. (5) gives expressions for the wave numbers and amplitudes in the form

$$q_{dj}^2 = \frac{2m(\Omega_j - E)}{2 - k^2}, \quad (14)$$

$$A_{dj}^2 = \frac{q_{dj}^2}{m\gamma_j}. \quad (15)$$

Substituting the solution (13) into the continuity condition (6) leads to the expression

$$\eta q_{d1}\mathrm{dn}(q_{d1}x_{d1}, k) = q_{d2}\mathrm{dn}(q_{d2}x_{d2}, k). \quad (16)$$

Substituting the solution (13) into the nonlinear boundary condition (7) leads to the expression

$$D_{d2} - D_{d1} = mU_0 + W_0 q_{d1}^2 \frac{\mathrm{dn}^2(q_{d1}x_{d1}, k)}{\gamma_1}, \quad (17)$$

where

$$D_{dj} = \frac{1}{2}k^2 q_{dj}\mathrm{sn}(q_{dj}x_{dj}, k)\mathrm{sn}(q_{dj}x_{dj} + K(k), k).$$

We will call relations (16) and (17) the dispersion ones. From (16) we can express, for example, x_{d2} and substitute it into (17), thereby eliminating it. Then, from (17) we find the energy as a function of the system's parameters: $E = E(m, U_0, W_0, \gamma_{1,2}, \Omega_{1,2}, x_{d1}, k)$. As a result, once E has been found, from (16) we determine x_{d2} as a function of the system's parameters: $x_{d2} = x_{d2}(m, U_0, W_0, \gamma_{1,2}, \Omega_{1,2}, x_{d1}, k)$. The function (13) will then be a two-parameter solution of the NLSE with two free parameters, k and x_{d1}.

The states in a self-focusing crystal described by the functions (8) and (13) were considered in [28].

3.3. Periodic States in a Defocusing Crystal

In the case of a defocusing crystal, which corresponds to a negative nonlinearity ($\gamma < 0$) in (5), where the excitation energy lies in the range $E > \max\{\Omega_1, \Omega_2\}$, the NLSE (5) has a spatially periodic solution

$$\psi_j(x) = A_{sj}\mathrm{sn}(q_{sj}(x - x_{sj}), k), \tag{18}$$

expressed via the elliptic sine sn with a modulus $0 < k < 1$.

Substituting the solution (18) into Eq. (5) gives expressions for the wave numbers and amplitudes in the form

$$q_{sj}^2 = \frac{2m(E - \Omega_j)}{1 + k^2}, \tag{19}$$

$$A_{sj}^2 = \frac{q_{sj}^2}{mg_j}, \tag{20}$$

where, for convenience, we designated the nonlinear parameters of the defocusing medium: $g_j = -\gamma_j > 0$.

Substituting the solution (18) into the continuity condition (6) leads to the expression

$$\eta q_{s1}\mathrm{sn}(q_{s1}x_{s1}, k) = q_{s2}\mathrm{sn}(q_{s2}x_{s2}, k). \tag{21}$$

Substituting the solution (18) into the nonlinear boundary condition (7) leads to the expression

$$D_{s2} - D_{s1} = mU_0 + W_0 q_{s1}^2 \frac{\mathrm{sn}^2(q_{s1}x_{s1}, k)}{g_1}, \tag{22}$$

where

$$D_{sj} = \frac{1}{2}\frac{k_1 q_{sj}\mathrm{cn}(q_{sj}x_{sj}, k)}{\mathrm{cn}(q_{sj}x_{sj} + K(k), k)},$$

$k_1^2 = 1 - k^2$ is the additional modulus of the elliptic functions.

We will call relations (21) and (22) the dispersion ones. From (21) we can express, for example, x_{s2} and substitute it into (22), thereby eliminating it. Then, from (22) we find the energy as a function of the system's parameters: $E = E(m, U_0, W_0, \gamma_{1,2}, \Omega_{1,2}, x_{s1}, k)$. As a result, once E has been found, from (21) we determine x_{s2} as a function of the system's parameters: $x_{s2} = x_{s2}(m, U_0, W_0, \gamma_{1,2}, \Omega_{1,2}, x_{s1}, k)$. The function (18) will then be a two-parameter solution of the NLSE with two free parameters, k and x_{s1}.

3.4. Periodic States of the First Type at the Contact of Self-Focusing and Defocusing Crystals

Consider now the contact of two semi-bounded crystals with opposite signs of the nonlinearity. Let the half-space to the left of the interface (planar defect) at $x < 0$ be occupied by a self-focusing crystal (where $\gamma_1 > 0$) and the half-space to the right of the interface at $x > 0$ be occupied by a defocusing crystal (where $\gamma_2 < 0$).

In that case, when the excitation energy lies in the range $\Omega_2 < E < \Omega_1$, the NLSE (5) has a spatially periodic solution described by functions in the form (8) for $j = 1$ and (18) for $j = 2$:

$$\begin{aligned}\psi_1(x) &= A_{c1}\mathrm{cn}(q_{c1}(x - x_{c1}), k),\\ \psi_2(x) &= A_{s2}\mathrm{sn}(q_{s2}(x - x_{s2}), k).\end{aligned} \tag{23}$$

The expressions for the wave numbers and amplitudes of these solutions are specified by Eqs. (9) and (10) for $j = 1$ and (19) and (20) for $j = 2$, respectively.

Substituting the functions (23) into the continuity condition (6) leads to the expression

$$\tilde{\eta}kq_{c1}\mathrm{cn}(q_{c1}x_{c1}, k) = -q_{s2}\mathrm{sn}(q_{s2}x_{s2}, k) \tag{24}$$

where $\tilde{\eta} = (g_2/\gamma_1)^{1/2}$.

Substituting the functions (23) into the nonlinear boundary condition (7) leads to the expression

$$D_{s2} - D_{c1} = mU_0 + W_0 k^2 q_{c1}^2 \frac{\mathrm{cn}^2(q_{c1}x_{c1}, k)}{\gamma_1}. \tag{25}$$

We will call relations (24) and (25) the dispersion ones. From (24) we can express, for example, x_{s2} and substitute it into (25), thereby eliminating it. Then, from (25) we find the energy as a function of the system's parameters: $E = E(m, U_0, W_0, \gamma_{1,2}, \Omega_{1,2}, x_{c1}, k)$. As a result, once E has been found, from (25) we determine x_{s2} as a function of the system's parameters: $x_{s2} = x_{s2}(m, U_0, W_0, \gamma_{1,2}, \Omega_1, x_{c1}, k)$. Then, (23) will define a two-parameter solution of the NLSE with two free parameters, k and x_{c1}.

3.5. Periodic States of the Second Type at the Interface between Self-Focusing and Defocusing Crystals

In the case of a contact between self-focusing (at $x < 0$) and defocusing (at $x > 0$) crystals, when the excitation energy lies in the range $\Omega_2 < E < \Omega_1$, the

NLSE (5) has a different spatially periodic solution described by functions in the form (13) for $j = 1$ and (28) for $j = 2$:

$$\psi_1(x) = A_{d1}\mathrm{dn}(q_{d1}(x - x_{d1}), k),$$
$$\psi_2(x) = A_{s2}\mathrm{sn}(q_{s2}(x - x_{s2}), k). \quad (26)$$

The expressions for the wave numbers and amplitudes of these solutions are specified by Eqs. (14) and (15) for $j = 1$ and (19) and (20) for $j = 2$, respectively.

Substituting the functions (26) into the continuity condition (6) leads to the expression

$$\tilde{\eta} q_{d1}\mathrm{dn}(q_{d1}x_{d1}, k) = -q_{s2}\mathrm{sn}(q_{s2}x_{s2}, k). \quad (27)$$

Substituting the functions (26) into the nonlinear boundary condition (7) leads to the expression

$$D_{s2} - D_{d1} = mU_0 + W_0 q_{d1}^2 \frac{\mathrm{dn}^2(q_{d1}x_{d1}, k)}{\gamma_1}. \quad (28)$$

We will call relations (27) and (28) the dispersion ones. From (27) we can express, for example, x_{s2} and substitute it into (28), thereby eliminating it. Then, from (28) we find the energy as a function of the system's parameters: $E = E(m, U_0, W_0, \gamma_{1,2}, \Omega_{1,2}, x_{d1}, k)$. As a result, once E has been found, from (27) we determine x_{s2} as a function of the system's parameters: $x_{s2} = x_{s2}(m, U_0, W_0, \gamma_{1,2}, \Omega_{1,2}, x_{d1}, k)$. Then, (26) will be a two-parameter solution of the NLSE with two free parameters, k and x_{d1}.

4. THE ENERGIES OF PERIODIC STATES

4.1. The Energy of a Periodic State of the First Type in a Self-Focusing Crystal

Consider the case where the excitation energy lies in the range $E < \min\{\Omega_1, \Omega_2\}$, the periodic state is described by Eq. (8), and its energy is found from the dispersion relations (11) and (12). Their analysis can be performed in an explicit analytical form in various special cases.

The energy of a state for which $x_{c2} = x_{c1} = 0$ can be determined in an exact form. Then, the relations $q_{c2} = \eta q_{c1}$ and $q_{c1}^2 = -\gamma_1 mU_0/W_0k^2$ follow from (11) and (12), respectively. Hence it follows that such a state is possible only when the signs of the defect parameters are opposite. Given (9), from these expressions we derive the energy

$$E = \Omega_1 + \frac{\gamma_1 U_0(2k^2 - 1)}{2W_0k^2}, \quad (29)$$

and the modulus of the elliptic function that is defined via the crystal and defect parameters:

$$k^2 = \frac{U_0(\gamma_2 - \gamma_1)}{2\{U_0(\gamma_2 - \gamma_1) + W_0(\Omega_2 - \Omega_1)\}}. \quad (30)$$

States of this type with energy (29) are possible only when a planar defect separates crystals with nonlinearity characteristics different in magnitude (but not in sign) ($\gamma_1 \neq \gamma_2$). Moreover, their existence is attributable exclusively to the fact that the defect possess nonlinear properties, because they do not emerge at $W_0 = 0$.

We will now assume that the characteristics of the crystals on either side of the defect plane are identical, i.e., $\Omega_1 = \Omega_2 = \Omega$ and $\gamma_1 = \gamma_2 = \gamma$. It then follows from (9) and (10) that $q_{c1} = q_{c2} = q_c$ and the defect oscillation amplitude $A_{c1} = A_{c2} = \psi_{c0}$, respectively.

First consider a state for which $x_{c2} = -x_{c1} = x_{c0}$. The states under such conditions are characterized by two free parameters, k and x_{c0}. In this case, Eq. (11) is satisfied automatically. The energy can be found in an explicit analytical form in the "long-wavelength" approximation at $q_c x_{c0} \ll 1$. It should be noted that the condition of the long-wavelength approximation $q_c x_{c0} \ll 1$ implies that the excitation energy is close to the edge of the spectrum when the requirement $|\Omega - E| \ll (2k^2 - 1)/2mx_{c0}^2$ is met. In this approximation the energy is found from (12):

$$E = \Omega - \frac{(2k^2 - 1)\gamma U_0}{2(\gamma x_{c0} - W_0k^2)}. \quad (31)$$

The defect oscillation amplitude is determined from (10):

$$\psi_{c0} = k\left(\frac{U_0}{\gamma x_{c0} - W_0k^2}\right)^{1/2}. \quad (32)$$

States of this type with energy (31) can exist when one of the pairs of conditions is satisfied: (1) $U_0 > 0$ and $W_0 < \gamma x_{c0}/k^2$, (2) $U_0 < 0$ and $W_0 > \gamma x_{c0}/k^2$. In other words, a state of the type under consideration with energy (31) can exist both for an attractive thin defect layer with nonlinear properties and for a repulsive one. An additional constraint arises for the position of the maximum of the perturbation amplitude: $x_{c0} \neq W_0k^2/\gamma$.

Now consider an asymmetric state for which $x_{c2} \neq x_{c1}$. As before, we will assume that $\Omega_1 = \Omega_2 = \Omega$ and the nonlinearity characteristics of the crystals on either side of the defect plane are different, i.e., $\gamma_1 \neq \gamma_2$. Then, from (12) the energy can be represented in the long-wavelength approximation at $q_c x_{cj} \ll 1$ as

$$E = \Omega - \frac{(2k^2 - 1)\gamma_1 U_0}{\gamma_1(x_{c2} - x_{c1}) - 2W_0k^2}. \quad (33)$$

In that case, given (33), from (11) we can express the parameter

$$x_{c2} = \frac{(1 - \eta)F_c(x_{c1})}{2mU_0}, \quad (34)$$

where

$$F_c(x_{c1}) = 1 \pm \left[1 - \frac{4mU_0}{\gamma_1(1-\eta)^2} \times ((1-\eta)(\gamma_1 x_{c1} + 2W_0k^2) - mU_0\gamma_1\eta x_{c1}^2)\right]^{1/2}.$$

To simplify the subsequent analysis, we will consider a state with $x_{c1} = 0$. Equation (34) is simplified in this case:

$$x_{c2} = \frac{(1-\eta)F_{c0}}{2mU_0}, \quad (35)$$

where

$$F_{c0} = F_c(x_{c1} = 0) = 1 \pm \left[1 - \frac{8mU_0W_0k^2}{\gamma_1(1-\eta)}\right]^{1/2}.$$

Then, given (35), from (33) we can completely define the energy via the crystal and defect parameters:

$$E = \Omega - \Omega_{c0}F_{c0}^2,$$

where

$$\Omega_{c0} = \frac{\gamma_1^2(1-\eta)(2k^2-1)}{16mW_0^2k^4}.$$

The condition $U_0 < \gamma_1(1-\eta)/8mW_0k^2$ must be satisfied for the existence of such a local state.

4.2. The Energy of a Periodic State of the Second Type in a Self-Focusing Crystal

Consider the case where the excitation energy lies in the range $E < \min\{\Omega_1, \Omega_2\}$, the periodic state is described by Eq. (13), and its energy is found from the dispersion relations (16) and (17).

The energy of a state for which $x_{d2} = x_{d1} = 0$ can be determined in an exact form. Then, the relations $q_{d2} = \eta q_{d1}$ and $q_{d1}^2 = -\gamma_1 mU_0/W_0$ follow from (16) and (17), respectively. Hence it follows that, as for the states of the first type, such a state is possible only when the signs of the defect parameters are opposite. Given (14), from these expressions we derive the energy

$$E = \Omega_1 + \frac{\gamma_1 U_0(2-k^2)}{2W_0}, \quad (36)$$

and the modulus of the elliptic function

$$k^2 = 2\frac{U_0(\gamma_1 - \gamma_2) + W_0(\Omega_1 - \Omega_2)}{U_0(\gamma_1 - \gamma_2)}. \quad (37)$$

States of this type, just as those of the first type, are possible only when a planar defect separates crystals with different nonlinearity characteristics; such a defect must necessarily possess nonlinear properties ($W_0 \neq 0$).

We will now assume that the characteristics of the crystals on either side of the defect plane are identical, i.e., $\Omega_1 = \Omega_2 = \Omega$ and $\gamma_1 = \gamma_2 = \gamma$. Then, it follows from (14) and (15) that $q_{d1} = q_{d2} = q_d$ and the defect oscillation amplitude $A_{d1} = A_{d2} = \psi_{d0}$, respectively.

First consider a state for which $x_{d2} = -x_{d1} = x_{d0}$. The states under such conditions are characterized by two free parameters, k and x_{d0}. In this case, Eq. (16) is satisfied automatically. The energy can be found in an explicit analytical form in the long-wavelength approximation at $q_d x_{d0} \ll 1$. It should be noted that the condition of the long-wavelength approximation $q_d x_{d0} \ll 1$ implies that the excitation energy is close to the edge of the spectrum when the requirement $|\Omega - E| \ll (2-k^2)/2mx_{d0}^2$ is met. In this approximation the energy is found from (17):

$$E = \Omega - \frac{(2-k^2)\gamma U_0}{2(\gamma x_{d0}k^2 - W_0)}. \quad (38)$$

The defect oscillation amplitude is determined from (15):

$$\psi_{d0} = \left(\frac{U_0}{\gamma x_{d0}k^2 - W_0}\right)^{1/2}. \quad (39)$$

States of this type with energy (38) can exist when one of the pairs of conditions is satisfied: (1) $U_0 > 0$ and $W_0 < \gamma x_{d0}k^2$, (2) $U_0 < 0$ and $W_0 > \gamma x_{d0}k^2$. Consequently, states of the type under consideration with energy (38) can exist both for an attractive thin defect layer with nonlinear properties and for a repulsive one.

Now consider an asymmetric state for which $x_{d2} \neq x_{d1}$. As before, we will now assume that $\Omega_1 = \Omega_2 = \Omega$ and the nonlinearity characteristics of the crystals on either sides of the defect plane are different, i.e., $\gamma_1 \neq \gamma_2$. Then, from (17) the energy can be represented in the long-wavelength approximation at $q_d x_{dj} \ll 1$ as

$$E = \Omega - \frac{(2-k^2)\gamma_1 U_0}{\gamma_1 k^2(x_{d2} - x_{d1}) - 2W_0}. \quad (40)$$

In that case, given (40), from (16) we can express the parameter

$$x_{d2} = (1-\eta)F_d(x_{d1})/2mU_0, \quad (41)$$

where

$$F_d(x_{d1}) = 1 \pm \left[1 - \frac{4mU_0}{\gamma_1 k^2(1-\eta)^2} \times ((1-\eta)(\gamma_1 x_{d1}k^2 + 2W_0) - mU_0\gamma_1\eta x_{d1}^2k^2)\right]^{1/2}.$$

To simplify the subsequent analysis, we will consider a state with $x_{d1} = 0$. Equation (34) is simplified in that case:

$$x_{d2} = \frac{(1-\eta)F_{d0}}{2mU_0}, \tag{42}$$

where

$$F_{d0} = F_d(x_{d1} = 0) = 1 \pm \left[1 - \frac{8mU_0W_0}{\gamma_1 k^2(1-\eta)}\right]^{1/2}.$$

Then, given (42), from (40) we can completely define the energy via the crystal and defect parameters:

$$E = \Omega - \Omega_{d0}F_{d0}^2, \tag{43}$$

where

$$\Omega_{d0} = \frac{\gamma_1^2 k^2(1-\eta)(2-k^2)}{16mW_0^2}.$$

The condition $U_0 < \gamma_1 k^2(1-\eta)/8mW_0$ must be met for the existence of such a local state with energy (43).

4.3. The Energy of a Periodic State in a Defocusing Crystal

Consider the case where the excitation energy lies in the range $E > \max\{\Omega_1, \Omega_2\}$, the periodic state is described by (18), and its energy is found from the dispersion relations (21) and (22).

We will now assume that the characteristics of the crystals on either side of the defect plane are identical, i.e., $\Omega_1 = \Omega_2 = \Omega$ and $g_1 = g_2 = g$. Then, it follows from (19) and (20) that $q_{s1} = q_{s2} = q_s$ and the defect oscillation amplitude $A_{s1} = A_{s2} = \psi_{s0}$, respectively.

First consider a state for which $x_{s2} = -x_{s1} = x_{s0}$. The states under such conditions are characterized by two free parameters, k and x_{s0}. The energy can be found in an explicit analytical form in the long-wavelength approximation at $q_s x_{s0} \ll 1$. It should be noted that the condition of the long-wavelength approximation $q_s x_{s0} \ll 1$ implies that the excitation energy is close to the edge of the spectrum when the requirement $|E - \Omega| \ll (1 + k^2)/2mx_{s0}^2$ is met. In this approximation the energy is found from (22):

$$E = \Omega + \frac{1+k^2}{2m}\left(-\frac{g(1+mU_0x_{s0})}{W_0x_{s0}^3}\right)^{1/2}. \tag{44}$$

The defect oscillation amplitude is determined from (20)

$$\psi_{s0} = \left(-\frac{1+mU_0x_{s0}}{mW_0x_{s0}}\right)^{1/2}. \tag{45}$$

States of this type with energy (44) can exist when one of the sets of conditions is satisfied: (1) $U_0 > -1/mx_{s0}$ and $W_0x_{s0} < 0$, (2) $U_0 < -1/mx_{s0}$, $W_0x_{s0} > 0$. Consequently, states of the type under consideration with energy (44) can exist both for an attractive thin defect layer with nonlinear properties and for a repulsive one; the signs of both defect parameters must be the same.

Now consider an asymmetric state for which $x_{s2} \neq x_{s1}$. As before, we will assume that $\Omega_1 = \Omega_2 = \Omega$ and the nonlinear characteristics of the crystals on either side of the defect plane are different, i.e., $\gamma_1 \neq \gamma_2$. Then, in the long-wavelength approximation at $q_s x_{sj} \ll 1$ it follows from (21) that $x_{s2} = \eta x_{s1}$, while the energy can be found from (22):

$$E = \Omega + \frac{1+k^2}{2mx_{s1}}\left(g_1\frac{\eta(1-2mU_0x_{s1})-1}{2\eta W_0x_{s1}}\right)^{1/2}. \tag{46}$$

States of this type with energy (46) can exist when one of the sets of conditions is satisfied: (1) $U_0 > (\eta - 1)/2m\eta x_{s1}$ and $W_0x_{s1} < 0$, (2) $U_0 < (\eta - 1)/2m\eta x_{s1}$ and $W_0x_{s1} > 0$. As in the case described above, states of the type under consideration with energy (46) can exist both for an attractive thin defect layer with nonlinear properties and for a repulsive one.

Finally, consider an asymmetric state for which $x_{s2} \neq x_{s1}$ and the characteristics of the crystals on either side of the defect plane are different, i.e., $\Omega_1 \neq \Omega_2$ and $\gamma_1 \neq \gamma_2$. Then, in the long-wavelength approximation at $q_s x_{sj} \ll 1$ the energy can be found from (22):

$$E = \Omega_1 + \Omega_{s0}\left\{1 \pm \left(1 - \frac{\Omega_{sa}}{\Omega_{s0}}\right)^{1/2}\right\}, \tag{47}$$

where

$$\Omega_{s0} = \frac{g_1(\eta-1)(1+k^2)}{4m\eta W_0x_{s1}^3},$$

$$\Omega_{sa} = \frac{4\eta^2x_{s1}}{\eta-1}\left\{U_0 + \frac{2\eta(\Omega_1-\Omega_2)}{1+k^2}\right\}.$$

Given (47), from (21) we can derive the expression

$$x_{s2} = \eta\frac{E-\Omega_1}{E-\Omega_2}. \tag{48}$$

It follows from (48) that $x_{s2} > 0$. States of this type with energy (47) can exist when the following condition is satisfied:

$$U_0 < \frac{g_1(\eta-1)^2}{16m\eta^2W_0x_{s1}^4} - \frac{2\eta(\Omega_1-\Omega_2)}{1+k^2}.$$

The signs of the defect parameters can be opposite.

The fundamental difference of defocusing crystals from self-focusing crystals is that for spatially inhomogeneous periodic states both parameters x_{sj} characterizing the distributions of the maxima of the perturbation amplitudes in space cannot be zero in them.

5. THE LOCALIZATION OF STATIONARY STATES

It is well known that the elliptic functions pass into the hyperbolic ones in the limit $k \to 1$. Apart from the spatially periodic states, this also allows the localized states near the interface between media to be described in terms of the proposed model.

5.1. The Localization of States in a Self-Focusing Crystal

In the case of $\gamma > 0$ and $E < \min\{\Omega_1, \Omega_2\}$, the periodic state described by the solution (8) at $k = 1$ passes into a localized state described by a solution that decreases as we recede from the interface to infinity:

$$\psi_j(x) = \frac{A_j}{\cosh(q_j(x - x_j))}. \quad (49)$$

Here, in the limit $k \to 1$ it follows from (9) and (10) that $q_{cj}^2 \to q_j^2 = 2m(\Omega_j - E)$ and $A_{cj}^2 \to A_j^2 = q_j^2/(m\gamma_j)$, respectively. Relation (11) will take the form

$$\eta q_1 \cosh(q_2 x_2) = q_2 \cosh(q_1 x_1), \quad (50)$$

while the dispersion relation (12) will be

$$q_2 \tanh(q_2 x_2) - q_1 \tanh(q_1 x_1) = 2\left\{mU_0 + \frac{W_0 q_1^2 \cosh^{-2}(q_1 x_1)}{\gamma_1}\right\}. \quad (51)$$

For the case where the characteristics of the crystals on either side of the defect plane are identical ($\Omega_1 = \Omega_2 = \Omega$ and $\gamma_1 = \gamma_2 = \gamma$), $q_1 = q_2 = q$, and $A_1 = A_2$, at $x_2 = x_1 = x_0$ from (51) we derive the dispersion relation defining the energy of such localized states:

$$mU_0\gamma\cosh^2(qx_0) = -W_0 q_0^2. \quad (52)$$

For localized states in the long-wavelength approximation at $qx_0 \ll 1$ from (52) we derive the expression $q^2 = -\gamma mU_0/W_0$, which gives the localization energy

$$E = \Omega + \frac{\gamma U_0}{2W_0}. \quad (53)$$

We see that, in agreement with the results of Section 4.1, a localized state of the type under consideration exists only when the signs of the defect parameters are opposite. It should be noted that Eq. (53) is also derived from (31) at $k = 1$ and $x_{c0} = 0$.

Now consider a symmetric localized state for which $x_2 = -x_1 = x_0$. The dispersion relation defining the energy of such localized states is then derived from (51):

$$q\tanh(qx_0) = mU_0 + \frac{W_0 q^2 \cosh^{-2}(qx_0)}{\gamma}. \quad (54)$$

In the long-wavelength approximation at $qx_0 \ll 1$ from (54) we derive the expression $q^2 = \gamma mU_0/(\gamma x_0 - W_0)$, which allows the localization energy to be written as

$$E = \Omega - \frac{\gamma U_0}{2(W_0 - \gamma x_0)}. \quad (55)$$

Localized states of this type with energy (55) can exist when one of the pairs of conditions is satisfied: (1) $U_0 > 0$ and $W_0 < \gamma x_0$, (2) $U_0 < 0$ and $W_0 > \gamma x_0$, in agreement with the results of Section 4.1.

In the case where the characteristics of the crystals on either side of the defect plane differ ($\Omega_1 \neq \Omega_2$ and $\gamma_1 \neq \gamma_2$), the existence of a state for which $x_2 = x_1 = 0$ is pointed out in Section 4.1. Then, $q_2 = \eta q_1$ follows from (50), while $q_1^2 = -\gamma_1 mU_0/W_0$ is found from (51). A localized state of this type can exist only for a certain relation between the crystal and defect parameters that follows from (30) at $k = 1$ and defines the localization condition: $U_0/W_0 = 2(\Omega_1 - \Omega_2)/(\gamma_1 - \gamma_2)$.

The periodic state of the second type in a self-focusing crystal described by the function (13) passes into the same localized state described by the function (49) in the limit $k \to 1$, because $\mathrm{dn}x \to 1/\cosh z$ and it follows from (14) and (15) that $q_{dj}^2 \to q_j^2$ and $A_{dj}^2 \to A_j^2$, respectively. Thus, two periodic states degenerate into one localized state in the limit $k \to 1$.

5.2. The Localization of States in a Defocusing Crystal

In the case of $\gamma < 0$ and $E > \max\{\Omega_1, \Omega_2\}$, the periodic state described by the solution (18) at $k = 1$ passes into a localized state described by a kink-type solution of the NLSE:

$$\psi_j(x) = A_{tj}\tanh(q_{tj}(x - x_j)). \quad (56)$$

Here, in the limit $k \to 1$ $q_{sj}^2 \to q_{tj}^2 = m(\Omega_j - E)$ and $A_{sj}^2 \to A_{tj}^2 = q_{tj}^2/(mg_j)$ follow from (19) and (20), respectively. Relation (21) will take the form

$$\eta q_{t1}\tanh(q_{t1}x_1) = \tanh(q_{t2}x_2), \quad (57)$$

while the dispersion relation (22) will be

$$\frac{q_{t1}}{\sinh(2q_{t1}x_1)} - \frac{q_{t2}}{\sinh(2q_{t2}x_2)} = mU_0 + \frac{W_0 q_{t1}^2 \tanh^2(q_{t1}x_1)}{g_1}. \quad (58)$$

For the case where all parameters of the media to the left and the right of the interface are identical ($\Omega_1 = \Omega_2 = \Omega$, $g_1 = g_2 = g$), $q_{t1} = q_{t2} = q_t$, and $A_{t1} = A_{t2}$,

at $x_2 = x_1 = x_0$ from (58) we obtain the dispersion relation

$$mgU_0 = -W_0 q_t^2 \tanh^2(q_t x_0). \quad (59)$$

For localized states in the long-wavelength approximation at $q_t x_0 \ll 1$ we derive the expression $q_t = (-mgU_0/W_0 x_0^2)^{1/4}$ from (59), which gives the localization energy

$$E = \Omega + \frac{1}{x_0}\left(-\frac{gU_0}{mW_0}\right)^{1/2}. \quad (60)$$

At identical parameters of the crystals on either side of the defect plane and at $A_{t1} = -A_{t2}$ and $x_2 = x_1 = x_0$ the dispersion relation is derived from (58):

$$-\frac{2q_t}{\sinh(2q_t x_0)} = mU_0 + \frac{W_0 q_t^2 \cosh^{-2}(q_t x_0)}{g}. \quad (61)$$

In the long-wavelength approximation at $q_t x_0 \ll 1$ from (61) we derive the expression $q_t = \{g(1 + mU_0 x_{s0}/W_0 x_{s0}^2\}^{1/4}$, which allows the localization energy to be written as

$$E = \Omega + \frac{1}{m}\left(-\frac{g(1 + mU_0 x_{s0})}{W_0 x_{s0}^3}\right)^{1/2}. \quad (62)$$

It should be noted that (62) is derived directly from (44) at $k = 1$. Accordingly, the satisfaction of the same conditions as those for the existence of a periodic state with energy (44) is required for the existence of such a localized state with energy (62).

For crystals with different nonlinearities on either side of the defect plane ($\gamma_1 \neq \gamma_2$), but at $\Omega_1 = \Omega_2 = \Omega$ for different $x_2 \neq x_1$ in the long-wavelength approximation at $q_t x_j \ll 1$ it follows from (57) that $x_2 = \eta x_1$, while from (58) we derive the expression

$$q_t = \left\{\frac{g[\eta(1 - 2mU_0 x_1) - 1]}{2\eta W_0 x_1^3}\right\}^{1/4},$$

which gives the localization energy

$$E = \Omega + \frac{1}{2}\left(g_1 \frac{\eta(1 - 2mU_0 x_1) - 1}{2\eta W_0 x_1^3}\right)^{1/2}. \quad (63)$$

It should be noted that Eq. (63) is derived directly from (46) at $k = 1$.

The local states described by the functions (49) and (56) for a defect with $U_0 = 0$ were considered in [29].

5.3. The Localization of States at the Interface between Self-Focusing and Defocusing Crystals

If a planar defect separates self-focusing (at $x < 0$) and defocusing (at $x > 0$) crystals, then at $\Omega_2 < E < \Omega_1$ the periodic state described by (26) at $k = 1$ passes into a localized state described by functions of the form (49) for $j = 1$ and (56) for $j = 2$:

$$\psi_1(x) = \frac{A_1}{\cosh(q_1(x - x_1))}, \quad \psi_2(x) = A_{t2}\tanh(q_{t2}(x - x_2)). \quad (64)$$

In the limit $k \to 1$ relation (27) will take the form

$$\tilde{\eta} q_1 = -q_{t2}\tanh(q_{t2}x_2)\cosh(q_1 x_1), \quad (65)$$

while the dispersion relation (28) will be

$$\frac{2q_{t2}}{\sinh(2q_{t2}x_2)} + q_1\tanh(q_1 x_1) = -2\left\{mU_0 + \frac{W_0 q_1^2 \cosh^{-2}(q_1 x_1)}{\gamma_1}\right\}. \quad (66)$$

Here we can note a state for which $x_0 = 0$ and $x_2 \neq 0$. In this case, by combining (65) and (66), we obtain a cubic equation for q_1:

$$aq_1^3 + q_1^2 + bq_1 - c = 0, \quad (67)$$

where

$$a = \frac{2\eta W_0}{\gamma_1(\eta^2 + 1/2)}, \quad b = \frac{2\eta mU_0}{\eta^2 + 1/2}, \quad c = \frac{m(\Omega_1 - \Omega_2)}{\eta^2 + 1/2} > 0.$$

The positivity of the coefficient c guarantees the existence of at least one real root of Eq. (67). Substituting the roots of Eq. (67) into the expression

$$E = \Omega_1 - q_1^2/2m \quad (68)$$

allows the localization energy for states of this type to be determined.

The energy of localized states can be obtained in an explicit form from Eq. (67) in some limiting cases.

(1) If the energy band of the existence of localized states in the spectrum is very narrow, when Ω_1 is close to Ω_2, then we obtain the expression

$$q_1 = -\frac{\gamma_1(\eta^2 + 1/2)}{4\eta W_0}\times\left\{1 \pm \left[1 - \frac{16\eta mU_0 W_0}{\gamma_1(\eta^2 + 1/2)^2}\right]^{1/2}\right\}. \quad (69)$$

(2) If the excitation–defect interaction is assumed, in addition to the previous condition, to be very weak, then from (67) we obtain the quantity

$$q_1 = -\frac{\gamma_1(\eta^2 + 1/2)}{4\eta W_0}.$$

(3) If the energy band of the existence of localized states in the spectrum is finite, the intensity of the excitation–defect interaction is finite, and the defect

nonlinearity is very small, then from (67) we obtain the quantity

$$q_1 = -\frac{\eta m U_0}{\eta^2 + 1/2} \times\left\{1 \pm \left[1 + \frac{(\Omega_1 - \Omega_2)(\eta^2 + 1/2)}{m U_0^2 \eta^2}\right]^{1/2}\right\}. \quad (70)$$

(4) If the energy band of the existence of localized states in the spectrum is finite, but both defect parameters are small (which corresponds to the continuity of the normal derivative of the field when passing through the interface between the media), then from (67) we obtain the quantity

$$q_1^2 = \frac{m(\Omega_1 - \Omega_2)}{\eta^2 + 1/2},$$

which defines the localization energy of the state that does not interact with the defect.

Substituting the above expressions for q_1 into (68) allows the energy of the localized state in the corresponding case to be determined.

6. CONCLUSIONS

We showed that spatially inhomogeneous stationary states of several types could exist near a thin planar defect with nonlinear properties separating nonlinear crystals. Such states are produced by various types of periodic NLSE solutions.

The model of a planar defect with nonlinear properties included a potential with a quadratic term with respect to the sought-for field [24, 25]. Finding the solutions of the NLSE with a nonlinear potential is reduced to solving a contact boundary-value problem for the NLSE without a potential with nonlinear boundary conditions. As a result of solving such a boundary-value problem, we established that an interface with nonlinear properties between nonlinear crystals could produce various types of spatially inhomogeneous stationary periodic states describing the excitations of media asymmetric relative to the defect plane. For each type of such states we obtained the spatial distributions of the fields whose amplitude and shape are determined by the sign of the medium nonlinearity and the range of possible excitation energies.

An analysis of the model of a thin defect layer with nonlinear properties described by potential (3) introduces new structural features in the spectrum of spatially inhomogeneous periodic states, in contrast to the model of a simple defect described by potential (1). The main difference lies in the dispersion relations and, as a consequence, in the energy levels and the domains of existence of states. Furthermore, we managed to detect new types of spatially inhomogeneous periodic states whose existence is attributable exclusively to the nonlinear properties of the defect. The revealed new states emerge in the case of a combination of defocusing defect nonlinearity with defect attraction or in the case of a combination of self-focusing defect nonlinearity with defect repulsion.

The results obtained in this paper can serve as an extension of the studies of nonlinear excitations in media with nonlinear defects performed in [12–14, 17, 24–29] to the case of spatially inhomogeneous periodic perturbations of media. In view of the wide application of layered structures containing parallel-plate waveguides in nonlinear optics, studying the propagation of nonlinear surface waves in systems with such properties is of great importance.

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