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STATISTICAL, NONLINEAR, AND SOFT MATTER PHYSICS

Localization of Excitations near a Thin Structured Spacer between Linear and Nonlinear Crystals

S. E. Savotchenko

Belgorod State Technological University, Belgorod, 308012 Russia

e-mail: savotchenkose@mail.ru

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Abstract—It has been shown that localized and semi-localized stationary states exist near a thin structured defect layer between a linear medium and a Kerr nonlinear medium. Localized states are described by a monotonically decreasing amplitude of the field on the both sides of the interface between the media. Semi-localized states are characterized by the field that has the form of a standing wave in the linear medium and decreases monotonically in the nonlinear medium. Kerr media with self-focusing and defocusing are considered. The proposed model is described by a system of the linear and nonlinear Schrödinger equations with a specific potential simulating a thin structured defect layer. It has been shown that localized and semi-localized states exist in different energy ranges in the case of contact of the linear medium with the self-focusing medium. In the case of contact of the linear medium with the defocusing medium, two types of localized and semi-localized states differing in energy and field profile can exist in different energy ranges. In particular cases, expressions for energies of states of these types have been obtained and conditions of their applicability have been indicated.

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1. INTRODUCTION

Localization of excitations of various physical fields is one of the contact phenomena at an interface between crystals with different characteristics. Such processes play an important role when developing various technical applications involving nonlinear optical media, multilayer structures with various magnetic properties, layered crystals with a multiatomic unit cell, etc.

Theoretical studies of the features of propagation of nonlinear waves are of particular interest for the development of such systems. Nonlinear surface waves as vibrational states localized near defects in crystals have long been studied [1].

In most cases, nonlinear waves are analytically described by the nonlinear Schrödinger equation. This equation includes a term with the third power of the wavefunction corresponding to the so-called Kerr nonlinearity. Nonlinear optical waves localized near the interface between Kerr nonlinear media and in layered structures were theoretically described in detail in [2]. Such studies were continued in [3, 4], where various models of interfaces between nonlinear crystals were used. The existence of nonlinear surface waves at the interface between the linear and nonlinear media was revealed in [5]. Localized nonlinear states at the interface between nonlinear media with spatial dispersion were considered in [6]. The features of the interaction between bound soliton states referred to different states of a two-level system near a defect were analyzed in [7].

The interaction of nonlinear excitations with defects plays an important role in nonlinear dynamics. Mathematical models based on the nonlinear Schrödinger equation make it possible to qualitatively analyze effects of localization of an excitation. Localization of the excitation is due to the character of the interaction between the excitation and defects. A defect is usually simulated by a potential appearing in the nonlinear Schrödinger equation. In the standard approximation for a short-range potential in a one-dimensional model, this potential has the form

$$U(x) = U_0\delta(x), \tag{1}$$

where $\delta(x)$ is the Dirac delta function and U_0 is the intensity of the interaction of the excitation with the defect located at the origin of the coordinate system (sometimes called the "power" of the defect). At $U_0 > 0$ and $U_0 < 0$, the excitation repels and attracts a defect, respectively.

The model of the defect based on the potential given by Eq. (1) is insufficient to completely analyze the effect of the intrinsic characteristics of the defect on localization of excitations. In order to study the possibility of controlling the propagation of waves in layered structures through interfaces between media, which present the effect of long-range forces, a modified potential was proposed in [8] for simulation of a structured planar defect. To theoretically describe the features of scattering of waves in a linear medium with the spatial dispersion, the modified potential was used in [9], where the linear Schrödinger equation including higher-order derivatives with such a potential was solved. Peculiarities of the localization of nonlinear excitations near the structured defect were described in [10]. A solution of the nonlinear Schrödinger equation with the modified potential was found for nonlinear media containing the structured defect.

It is noteworthy that there are other theoretical approaches to the description of thin planar defects with the use of nonstandard potentials, in particular, nonlinear with respect to the desired field [11].

It is well known that the types of solutions of the nonlinear Schrödinger equation are determined by the sign of nonlinearity. Free solitons propagate in a medium without a defect ($U(x) \equiv 0$). In the case of positive nonlinearity ($\gamma > 0$) and when $E < \Omega$, the nonlinear Schrödinger equation ($\hbar = 1$ is accepted),

$$i\psi_t' = -\frac{1}{2m}\psi_{xx}'' + \Omega\psi - \gamma|\psi|^2\psi,$$

where m, Ω, and γ are constants, has the stationary solution

$$\psi(x) = A\frac{e^{-iEt}}{\cosh[q(x-x_0)]},$$

$$A^2 = q^2/m\gamma, \quad q^2 = 2m(\Omega - E).$$

As is known, the nonlinear Schrödinger equation with negative nonlinearity ($\gamma < 0$) has two types of stationary solutions:

(i)

$$\psi(x) = A\frac{e^{-iEt}}{\sinh[q(x-x_0)]},$$

with

$$A^2 = q^2/m|\gamma|, \quad q^2 = 2m(\Omega - E)$$

at $E < \Omega$;

(ii)

$$\psi(x) = A\frac{e^{-iEt}}{\tanh[q(x-x_0)]},$$

with

$$A^2 = q^2/m|\gamma|, \quad q^2 = m(E - \Omega)$$

at $E > \Omega$.

Localized states in nonlinear media with positive and negative nonlinearities, which are described by wavefunctions with hyperbolic cosine and sine, and with the interface between them simulated by the short-range potential given by Eq. (1) have been unambiguously described in the literature. Solutions of such type vanish at infinity; i.e., $|\psi| \to 0$ at $x \to \infty$. The solution of the nonlinear Schrödinger equation with negative nonlinearity, which is described by the wavefunction with hyperbolic tangent and is called kink, does not satisfy this condition, but it is widely used to describe various physical phenomena (in paticular, in a superconductor).

In this work, it is shown that there are several types of localized states near the structured planar defect between linear and nonlinear media. The cases of contact of the linear medium with Kerr nonlinear media with positive and negative nonlinearities are considered. The main aims of the work are to find the energies of localized states of all types appearing in the considered system and to analyze the effect of the internal structure of the defect on the features of localization of excitations.

2. EQUATIONS OF THE MODEL

Let a thin spacer separating crystals with harmonic and anharmonic interactions between elementary excitations be located in the yz plane perpendicular to the x axis. The thickness of the spacer is much smaller than the characteristic localization length of excitations.

The linear (harmonic) and nonlinear (anharmonic) crystals occupy the $x < 0$ and $x > 0$ half-spaces, respectively. Correspondingly, the parameter of nonlinearity in the nonlinear Schrödinger equation has the form

$$\gamma(x) = \begin{cases} 0, & x < 0, \\ \gamma, & x > 0. \end{cases}$$

The interface as a planar defect produces a perturbation of the characteristics of media, which is localized at distances much smaller than the localization length of the considered excitations.

The model of the structured defect was proposed in [8] and was used for the case of a linear medium with spatial dispersion [9] and for a nonlinear crystal [10]. Such a defect is mathematically described by a short-range potential. The modified potential includes the effect of the interaction not only between the nearest neighbors in the crystal lattice but also between the next-nearest neighbors in the long-wavelength approximation. Such a description is important at the transition from discrete lattice models to continuum medium models.

It is proposed to describe the structured defect by the limiting case of a double-hump potential (potential with two symmetric peaks). Such a description is possible for a potential well with a quasistationary energy level. The potential well can be specified by Eq. (1) with an additional term with the second derivative of the Dirac delta function in the limiting case of an infinitely deep crater:

$$U(x) = U_0\delta(x) + V_0\delta''(x), \qquad (2)$$

where V_0 is the second parameter characterizing the interaction of the internal structure of the interface between media with excitations.

Let the interaction of nonlinear excitations localized near the structured defect be described by the one-dimensional nonlinear Schrödinger equation ($\hbar = 1$ is accepted):

$$i\psi'_t = -\frac{1}{2m}\psi''_{xx} + \Omega(x)\psi - \gamma(x)|\psi|^2\psi + U(x)\psi. \quad (3)$$

Here, m is the effective mass of an excitation and

$$\Omega(x) = \begin{cases} \Omega_1, & x < 0, \\ \Omega_2, & x > 0, \end{cases}$$

where Ω_1 and Ω_2 are constants.

To find stationary states with the energy E, it is appropriate to substitute the wavefunction

$$\psi(x,t) = \psi(x)e^{-iEt}$$

into the nonlinear Schrödinger equation (3). This substitution reduces Eq. (3) to the time-independent nonlinear Schrödinger equation

$$E\psi = -\frac{1}{2m}\psi''_{xx} + \Omega(x)\psi - \gamma(x)|\psi|^2\psi + U(x)\psi. \quad (4)$$

The solution of the nonlinear Schrödinger equation (4) with the potential (2) is equivalent to the solution of the nonlinear Schrödinger equation without potential:

$$E\psi = -\frac{1}{2m}\psi''_{xx} + \Omega(x)\psi - \gamma(x)|\psi|^2\psi, \quad (5)$$

with two matching boundary conditions at the point $x = 0$ lying in the plane of the defect. The first boundary condition corresponds to the continuity of the wavefunction:

$$\psi(-0) = \psi(+0) = \psi(0). \quad (6)$$

As was described in [8], to obtain the second boundary condition, it is necessary to integrate both sides of Eq. (4) with the potential (2) with respect to x over a narrow interval $[-\varepsilon; \varepsilon]$ and to tend ε to zero. Since the derivatives of the wavefunction are not continuous at the point $x = 0$, the second boundary condition is obtained in the form

$$\begin{aligned} &\psi'(+0) - \psi'(-0) \\ &= m\{2U_0\psi(0) + V_0[\psi''(+0) + \psi''(-0)]\}. \end{aligned} \quad (7)$$

Condition (7) with $V_0 = 0$ yields the well-known boundary condition used to describe the localization and scattering of excitations on a point defect, which corresponds to the short-range potential (1).

3. LOCALIZED STATES

3.1. Localized States at the Interface between a Linear Crystal and a Self-Focusing Medium

In the case of contact of a linear crystal with a self-focusing medium, i.e., with a crystal with positive nonlinearity ($\gamma > 0$), when the energy of the excitation is in the range $E < \min\{\Omega_1, \Omega_2\}$, the nonlinear Schrödinger equation (5) has the solution

$$\psi(x) = \begin{cases} \psi_{0c}\exp(q_1 x), & x < 0, \\ A_c/\cosh[q_2(x - x_0)], & x > 0. \end{cases} \quad (8)$$

The parameters of the solution (8) are determined after its substitution into Eq. (5) and continuity condition (6):

$$q_{1,2}^2 = 2m(\Omega_{1,2} - E), \quad (9)$$

$$A_c^2 = \frac{q_2^2}{m\gamma}, \quad (10)$$

$$\psi_{0c} = \frac{q_2}{\sqrt{m\gamma}\cosh(q_2 x_0)}. \quad (11)$$

The parameter x_0 specifies the position of the "center" of the soliton in the nonlinear crystal to the right of the defect. It is related to the energy of localization of the excitation, which is determined from the dispersion relation obtained after the substitution of solution (8) into the boundary condition (7):

$$\begin{aligned} &q_2\tanh(q_2 x_0) - q_1 \\ &= m\{2U_0 + V_0[q_2^2(2\tanh^2(q_2 x_0) - 1) + q_1^2]\}. \end{aligned} \quad (12)$$

One of the wavenumbers (any of q_1 and q_2 because they are related to each other) can be found from Eq. (12); as a result, the energy is determined as a function of the parameters $E = E(m, U_0, V_0, \gamma, x_0)$. The position x_0 of the center of the soliton is a free parameter. Dispersion relation (12) will be analyzed in various particular cases where its solution can be obtained in an explicit form.

In the case of the structureless defect, Eq. (12) with $V_0 = 0$ gives the dispersion relation

$$q_2\tanh(q_2 x_0) - q_1 = 2mU_0. \quad (13)$$

In the long-wavelength approximation at $q_2 x_0 \ll 1$, the energy of the localized state can be obtained from Eq. (13) in the explicit form

$$E = \Omega_2 - \frac{\Omega_2 - \Omega_1 + 2mU_0^2}{1 + 4mU_0 x_0}. \quad (14)$$

For the structured defect at $V_0 \neq 0$, the energy of the localized state whose center lies in the plane of the defect, i.e., $x_0 = 0$, can be obtained from Eq. (12) in the explicit form. In this case, the spatial damping of the excitation in the linear crystal is determined from Eq. (12) in the form

$$q_1 = -2m[U_0 + mV_0(\Omega_1 - \Omega_2)]. \quad (15)$$

Since $q_1 > 0$, the parameters of the defect should satisfy the condition

$$U_0 < mV_0(\Omega_2 - \Omega_1).$$

The energy of this localized state can be obtained from Eqs. (9) and (15):

$$E = \Omega_1 - 2m[U_0 + mV_0(\Omega_1 - \Omega_2)]^2. \quad (16)$$

It is noteworthy that the spatial damping and energy of such a state with $x_0 = 0$ at $\Omega_1 = \Omega_2$ will be the same as for the structureless defect, as follows from Eqs. (14) and (16).

In the long-wavelength approximation at $q_2x_0 \ll 1$ and $V_0 \neq 0$, the energy of the localized state is obtained from Eq. (12) in the form

$$E = \Omega_2 - \frac{d^2 + 2m(\Omega_2 - \Omega_1)}{2m(1 - 2dx_0)}, \quad (17)$$

where $d = 2m[U_0 + mV_0(\Omega_1 - \Omega_2)]$. The long-wavelength approximation ($q_2x_0 \ll 1$) means that the energy of the excitation is close to the edge of the spectrum, i.e.,

$$|\Omega_2 - E| \ll 1/2mx_0^2.$$

In the long-wavelength approximation or at $x_0 = 0$, Eq. (11) gives the amplitude of oscillations of the defect layer in the form

$$\psi_{0c} = \frac{q_2}{\sqrt{m\gamma}}.$$

3.2. Localized States at the Interface between a Linear Crystal and a Defocusing Medium

In the case of contact of a linear crystal with a defocusing medium, i.e., with a crystal with negative nonlinearity ($\gamma < 0$), when the energy of the excitation lies in the range $E < \min\{\Omega_1, \Omega_2\}$, the nonlinear Schrödinger equation (5) has the solution

$$\psi(x) = \begin{cases} \psi_{0s}\exp(q_1x), & x < 0, \\ A_s/\sinh[q_2(x - x_0)], & x > 0. \end{cases} \quad (18)$$

For the solution (18) to be bounded, the condition $x_0 < 0$ should be satisfied. The parameters of the solution (18) are determined after its substitution into Eq. (5) and the continuity condition (6). Let $g = -\gamma > 0$ for convenience. The parameters $q_{1,2}$ are determined by Eq. (9), and the amplitudes have the form

$$A_s^2 = \frac{q_2^2}{mg}, \quad (19)$$

$$\psi_{0s} = -\frac{q_2}{\sqrt{mg}\sinh(q_2x_0)}. \quad (20)$$

The substitution of the solution (18) into the boundary condition (7) gives the dispersion relation

$$\begin{aligned} &q_2\coth(q_2x_0) + q_1 \\ &= m\{V_0[q_2^2(1 - 2\coth^2(q_2x_0)) - q_1^2] - 2U_0\}. \end{aligned} \quad (21)$$

In the case of the structureless defect, Eq. (21) with $V_0 = 0$ provides the dispersion relation in the form

$$q_2\coth(q_2x_0) + q_1 = -2mU_0. \quad (22)$$

In the long-wavelength approximation ($q_2x_0 \ll 1$), the spatial damping in the linear crystal can be obtained from Eq. (22) in the form

$$q_1 = -\left(2mU_0 + \frac{1}{x_0}\right). \quad (23)$$

Since $q_1 > 0$, the parameters of the defect should satisfy the condition $U_0 < -1/2mx_0$. To be bounded, the localized state (18) should satisfy the condition $x_0 < 0$; consequently, $U_0 > 0$. In other words, the localized state under consideration exists only for repulsive defects. The energy of such a state is determined after the substitution of Eq. (23) into Eq. (9):

$$E = \Omega_1 - \frac{(2mU_0 + 1/x_0)^2}{2m}. \quad (24)$$

In the long-wavelength approximation ($q_2x_0 \ll 1$), the spatial damping of the excitation in the linear crystal with the structured defect at $V_0 \neq 0$ is obtained from Eq. (21) in the form

$$q_1 = 2m\left\{V_0\left[m(\Omega_2 - \Omega_1) - \frac{1}{x_0^2}\right] - U_0\right\} - \frac{1}{x_0}. \quad (25)$$

Since the spatial damping in the linear crystal is positive, the parameters of the defect should satisfy the condition

$$U_0 < V_0\left[m(\Omega_2 - \Omega_1) - \frac{1}{x_0^2}\right] - \frac{1}{2mx_0}.$$

Thus, because of the existence of the internal structure of the defect, the localized state described by the solution given by Eq. (18) can exist for both attractive and repulsive defects. The energy of this state is determined after the substitution of Eq. (25) into Eq. (9):

$$E = \Omega_1 - 2m\left\{V_0\left[m(\Omega_2 - \Omega_1) - \frac{1}{x_0^2}\right] - U_0 - \frac{1}{2mx_0}\right\}^2. \quad (26)$$

In the long-wavelength approximation at $q_2x_0 \ll 1$, Eq. (20) yields the amplitude of oscillations of the defect layer in the form

$$\psi_{0s} = -1/x_0\sqrt{mg}.$$

In the case of contact of the linear crystal with the crystal with negative nonlinearity, when the energy of the excitation lies in the range $\Omega_2 < E < \Omega_1$, the nonlinear Schrödinger equation (5) has another solution

$$\psi(x) = \begin{cases} \psi_{0t}\exp(q_1 x), & x < 0, \\ A_t\tanh[q_t(x - x_0)], & x > 0. \end{cases} \quad (27)$$

For the existence of such a solution, the condition $\Omega_2 < \Omega_1$ should be satisfied, which was not required for the existence of the solutions described above.

The parameters of the solution (27) are determined after its substitution into Eq. (5) and the continuity condition (6). The q_1 value is given by Eq. (9) and the other characteristics have the form

$$q_t^2 = m(E - \Omega_1), \quad (28)$$

$$A_t^2 = \frac{q_t^2}{mg}, \quad (29)$$

$$\psi_{0t} = -\frac{q_t\tanh(q_t x_0)}{\sqrt{mg}}. \quad (30)$$

The substitution of the solution (27) into the boundary condition (7) gives the dispersion relation

$$\frac{2q_t}{\sinh(2q_t x_0)} + q_1 = m\left\{V_0\left[\frac{2q_t^2}{\cosh^2(q_t x_0)} - q_1^2\right] - 2U_0\right\}. \quad (31)$$

In the case of the structureless defect at $V_0 = 0$, the dispersion relation (31) acquires the form

$$\frac{2q_t}{\sinh(2q_t x_0)} + q_1 = -2mU_0. \quad (32)$$

In the long-wavelength approximation ($q_t x_0 \ll 1$), the same expressions for the spatial damping of the excitation in the linear crystal (23) and energy (24) as for the localized state described by the wavefunction (18) are obtained from Eq. (32).

In the long-wavelength approximation at $q_t x_0 \ll 1$ and under the additional condition $mV_0 q_1 \ll 1$, the spatial damping of the excitation in the linear crystal with the structured defect at $V_0 \neq 0$ is obtained from Eq. (31) in the form

$$q_1 = -\left\{2m[U_0 + mV_0(\Omega_2 - \Omega_1)] + \frac{1}{x_0}\right\}. \quad (33)$$

The long-wavelength approximation $q_t x_0 \ll 1$ means that the energy of the excitation is close to the edge of the spectrum; i.e.,

$$|E - \Omega_2| \ll \frac{1}{mx_0^2}.$$

The additional requirement means that $|\Omega_1 - E| \ll 1/2m^3V_0^2$. Such conditions can be satisfied simultaneously in a sufficiently narrow band when Ω_2 and Ω_1 are close to each other:

$$\Omega_1 - \frac{1}{2m^3V_0^2} \ll E \ll \Omega_2 - \frac{1}{mx_0^2}.$$

Without the additional requirement $mV_0 q_1 \ll 1$, in the long-wavelength approximation, the spatial damping of the excitation in the linear crystal is obtained from Eq. (31) in the form

$$q_1 = \frac{1}{4mV_0}\left\{\sqrt{1 + 8mV_0[mV_0(\Omega_1 - \Omega_2) - U_0] - \frac{1}{x_0}} - 1\right\}. \quad (34)$$

The energy of this state is determined after the substitution of Eq. (34) into Eq. (9):

$$E = \Omega_2 - \Omega_a\left(1 \pm \sqrt{1 + \frac{\Omega_b}{\Omega_a}}\right), \quad (35)$$

where

$$\Omega_a = \frac{1}{32m^2V_0^2},$$

$$\Omega_b = \frac{1}{2V_0}\left[mV_0(\Omega_1 - \Omega_2) - U_0 - \frac{1}{2mx_0}\right].$$

For the existence of such a state, the condition $\Omega_a > -\Omega_b$ should be satisfied; i.e.,

$$U_0 < mV_0(\Omega_1 - \Omega_2) + \frac{1}{2m}\left(\frac{1}{2mV_0} - \frac{1}{x_0}\right).$$

Under this condition, the considered state can exist for different signs of the parameters of the defect and the free parameter.

In the long-wavelength approximation at $q_t x_0 \ll 1$, Eq. (30) yields the amplitude of oscillations of the defect layer in the form

$$\psi_{0t} = -\frac{q_t^2 x_0}{\sqrt{mg}}.$$

4. SEMI-LOCALIZED STATES

4.1. Semi-Localized States at the Interface between a Linear Crystal and a Self-Focusing Medium

In the case of contact of a linear crystal with a crystal with positive nonlinearity ($\gamma > 0$), when the energy of the excitation lies in the range $\Omega_1 < E < \Omega_2$, the nonlinear Schrödinger equation (5) has the solution

$$\psi(x) = \begin{cases} B_c\cos(kx + \varphi), & x < 0, \\ A_c/\cosh[q_2(x - x_0)], & x > 0. \end{cases} \quad (36)$$

For the existence of such a solution, the condition $\Omega_1 < \Omega_2$ should be satisfied. The parameters of the solution (36) are determined after its substitution into Eq. (5) and continuity condition (6). The parameter q_2 is given by Eq. (9), the amplitude A_c is specified by Eq. (10), and the characteristics of the wave in the linear crystal have the form

$$k^2 = 2m(E - \Omega_1), \tag{37}$$

$$B_c = \frac{q_2}{\sqrt{mg}\cos\varphi\cosh(q_2 x_0)}. \tag{38}$$

The substitution of the solution (36) into the boundary condition (7) yields the dispersion relation

$$\begin{gathered} k\tan\varphi + q_2\tanh(q_2 x_0) \\ = m\{2U_0 + V_0[q_2^2(2\tanh^2(q_2 x_0) - 1) - k^2]\}. \end{gathered} \tag{39}$$

One of the wavenumbers (any of k and q_2 because they are related to each other) can be found from Eq. (39); as a result, the energy is determined as a function of the parameters $E = E(m, U_0, V_0, \gamma, \varphi, x_0)$. The position x_0 of the center of the soliton and the phase φ are now free parameters.

The solution (36) describes the state in which the linear wave after the transition through the thin defect layer damps deep in the anharmonic crystal; i.e., waves are localized. Since the energy of such a stationary state is in the spectrum of linear waves and the excitation is localized on one side of the planar defect, states of such a type can be called semi-localized.

In the case of the structureless defect at $V_0 = 0$, the dispersion relation is obtained from Eq. (39) in the form

$$k\tan\varphi + q_2\tanh(q_2 x_0) = 2mU_0. \tag{40}$$

The energy of the solution for which $x_0 = 0$ is determined from Eq. (40) in the form

$$E = \Omega_1 + 2mU_0^2\cot^2\varphi. \tag{41}$$

The considered state and states that will be obtained below exist for certain values of the phase φ.

In the long-wavelength approximation at $q_2 x_0 \ll 1$, the expression for the energy of the semi-localized state can be obtained from Eq. (40) in the explicit form

$$E = \frac{2mU_0(U_0 - 2x_0\Omega_2) + \Omega_1\tan^2\varphi}{\tan^2\varphi - 4mU_0 x_0}. \tag{42}$$

According to Eqs. (41) and (42), a separate state for $\varphi = 0$ can exist only at $x_0 \neq 0$.

For the structured defect at $V_0 \neq 0$ and $x_0 = 0$, the energy is obtained from Eq. (39) in the form

$$E = \Omega_1 + 2m[U_0 - mV_0(\Omega_2 - \Omega_1)]^2\cot^2\varphi. \tag{43}$$

In the long-wavelength approximation at $q_2 x_0 \ll 1$ and $V_0 \neq 0$, the energy of the semi-localized state is obtained from Eq. (39) in the form

$$E = \Omega_2 - \Omega_c^{\varphi}\left(1 \pm \sqrt{1 + \frac{\Omega_c}{\Omega_c^{\varphi}}}\right), \tag{44}$$

where

$$\Omega_c^{\varphi} = \frac{\tan^2\varphi}{8mx_0^2},$$

$$\Omega_c = \frac{1}{x_0}[U_0 - (mV_0 + x_0)(\Omega_2 - \Omega_1)].$$

States with the energy given by Eq. (44) exist at phases satisfying the condition $\tan^2\varphi > 8mx_0^2\Omega_c$.

4.2. Semi-Localized States at the Interface between a Linear Crystal and a Defocusing Medium

In the same energy range, $\Omega_1 < E < \Omega_2$, in the case of contact of a linear crystal with a crystal with negative nonlinearity ($\gamma = -g < 0$), the nonlinear Schrödinger equation (5) has another solution:

$$\psi(x) = \begin{cases} B_s\cos(kx + \varphi), & x < 0, \\ A_s/\sinh[q_2(x - x_0)], & x > 0. \end{cases} \tag{45}$$

The parameters of the solution (45) are determined after its substitution into Eq. (5) and the continuity condition (6). The parameter q_2 is given by Eq. (9), the amplitude A_s is specified by Eq. (19), the wavenumber k is given by Eq. (37), and the amplitude of the wave in the linear crystal has the form

$$B_s = -\frac{q_2}{\sqrt{mg}\cos\varphi\sinh(q_2 x_0)}. \tag{46}$$

The substitution of the solution (45) into the boundary condition (7) yields the dispersion relation

$$\begin{gathered} k\tan\varphi - q_2\coth(q_2 x_0) \\ = m\{2U_0 + V_0[q_2^2(1 - 2\coth^2(q_2 x_0)) - k^2]\}. \end{gathered} \tag{47}$$

The position x_0 of the center of the soliton, which should be negative for the solution (45) to be bounded, and the phase φ are free parameters for such a semi-localized state.

In the case of the structureless defect at $V_0 = 0$, Eq. (47) gives the dispersion relation

$$k\tan\varphi - q_2\coth(q_2 x_0) = 2mU_0. \tag{48}$$

In the long-wavelength approximation ($q_2 x_0 \ll 1$), the energy of the semi-localized state can be obtained from Eq. (48) in the explicit form

$$E = \Omega_1 + \frac{1}{2m}\left(2mU_0 + \frac{1}{x_0}\right)^2\cot^2\varphi, \tag{49}$$

In the long-wavelength approximation at $q_2 x_0 \ll 1$ and $V_0 \neq 0$, the energy of the semi-localized state is obtained from Eq. (47) in the form

$$\begin{gathered} E = \Omega_1 \\ + 2m\left\{U_0 + V_0\left[\frac{1}{x_0^2} - m(\Omega_2 - \Omega_1)\right] + \frac{1}{2mx_0}\right\}^2\cot^2\varphi. \end{gathered} \tag{50}$$

In the case of contact of a linear crystal with a crystal with negative nonlinearity ($\gamma = -g < 0$), the nonlinear Schrödinger equation (5) in the energy range $E > \max\{\Omega_1, \Omega_2\}$ has another solution

$$\psi(x) = \begin{cases} B_t \cos(kx + \varphi), & x < 0, \\ A_t \tanh[q_t(x - x_0)], & x > 0. \end{cases} \quad (51)$$

The parameters of the solution (51) are determined after its substitution into Eq. (5) and the continuity condition (6). The parameter q_t is given by Eq. (28), the amplitude A_t is specified by Eq. (29), the wavenumber k is given by Eq. (37), and the amplitude of the wave in the linear crystal has the form

$$B_t = -\frac{q_t \tanh(q_t x_0)}{\sqrt{mg}\cos\varphi}. \quad (52)$$

The substitution of the solution (51) into the boundary condition (7) gives the dispersion relation

$$k\tan\varphi - \frac{2q_t}{\sinh(2q_t x_0)} = m\left\{2U_0 - V_0\left[\frac{2q_t^2}{\cosh^2(q_t x_0)} + k^2\right]\right\}. \quad (53)$$

In the case of the structureless defect at $V_0 = 0$, Eq. (53) provides the dispersion relation

$$k\tan\varphi - \frac{2q_t}{\sinh(2q_t x_0)} = 2mU_0. \quad (54)$$

The energy of such a state obtained from Eq. (54) in the long-wavelength approximation ($q_t x_0 \ll 1$) coincides with the energy given by Eq. (49).

In the case of the structured defect ($V_0 \neq 0$), the energy of the semi-localized state is obtained from Eq. (54) in the long-wavelength approximation ($q_t x_0 \ll 1$) in the form

$$E = \Omega_1 + \Omega_t^{\varphi}(\sqrt{1 + \Omega_t/\Omega_t^{\varphi}} - 1), \quad (55)$$

where

$$\Omega_t^{\varphi} = \frac{\tan^2\varphi}{32mx_0^2},$$

$$\Omega_t = \frac{1}{2U_0}[U_0 + mV_0(\Omega_2 - \Omega_1) + 1/2mx_0].$$

For the existence of such a state, the condition $\Omega_t < -\Omega_t^{\varphi}$ should be satisfied. Consequently,

$$\tan^2\varphi > -32m^2U_0^2\Omega_t.$$

Since Ω_t can have any sign, semi-localized states with the energy given by Eq. (55) can exist for both attractive and repulsive defects.

5. DISCUSSION

The effect of the internal structure of the defect on the features of localization of stationary states will be analyzed below.

5.1. Defect in the Linear Medium

The defect in the linear medium ($\gamma = 0$) is first considered under the simplifying assumption that the medium to the left and right of the defect has the same characteristics, in particular, $\Omega_1 = \Omega_2 = \Omega$. In this case, $q_1 = q_2 = q$. Free waves with the square dispersion relation $E = \Omega + k^2/2m$, where k is the wavenumber, propagate in the linear medium without defect ($\gamma(x) \equiv 0$ and $U(x) \equiv 0$ in the nonlinear Schrödinger equation (4)).

It is well known that a symmetric state exists in a linear medium with a simple defect ($U_0 \neq 0$, $V_0 = 0$), which is described by the short-range potential (1), and is localized on both sides of the defect. This state is described by the wavefunction

$$\psi(x) = \psi_0 \exp(-q|x|),$$

where $q = -mU_0$, and exists only for the attractive defect with $U_0 < 0$. The energy of such a local level is $E = \Omega - mU_0^2/2$.

It was shown in [8] that localized states exist in the linear medium with the structured defect and are described by the wavefunction exponentially decreasing on both sides of the defect for both the attractive and repulsive defects. To reveal the effect of the internal structure of the defect on the localization of excitations, the results will be analyzed below in detail for the case $U_0 = 0$ and $V_0 \neq 0$.

In this case, the localized state in the linear medium is described by the same wavefunction as in the linear medium with the simple defect (i.e., at $U_0 \neq 0$ and $V_0 = 0$), but the spatial damping is now $q = -1/mV_0$ and the energy is $E = -1/2m^3V_0^2$. Such a localized state exists at $V_0 < 0$. It is noteworthy that the localization length is $l = 1/q = m|V_0|$; i.e., the coefficient V_0 is proportional to the field localization length in this case.

A particular role of the internal structure of the defect leading to qualitatively new effects can be demonstrated on the example of scattering of the monochromatic plane wave from the defect simulated by the potential (2). The scattering wavefunction can be represented in the form

$$\psi(x) = \begin{cases} e^{ikx} + \mathrm{R}e^{-ikx}, & x < 0, \\ Te^{-ikx}, & x > 0. \end{cases} \quad (56)$$

The reflection and transmission coefficients were obtained in [8]. After the substitution of Eq. (56) into the boundary conditions (6) and (7) at $U_0 \neq 0$ and

$V_0 \neq 0$, these coefficients can be determined by the formulas

$$|R|^2 = \frac{m^2(U_0 - V_0k^2)^2}{k^2 + m^2(U_0 - V_0k^2)^2}, \quad (57)$$

$$|T|^2 = \frac{k^2}{k^2 + m^2(U_0 - V_0k^2)^2}. \quad (58)$$

Therefore, the total transmission when $|R|^2 = 0$ and $|T|^2 = 1$ at the wavenumber $k^2 = U_0/V_0$ is possible only at $V_0 \neq 0$. The energy of the total transmission is given by the expression

$$E_T = \Omega + U_0/2mV_0.$$

This effect does not occur for the structureless defect (i.e., at $V_0 = 0$).

5.2. Interface between Linear and Nonlinear Media

The case of contact of a linear crystal with a crystal with positive nonlinearity ($\gamma > 0$), where the energy of the excitation is in the range $E < \min\{\Omega_1, \Omega_2\}$ and the wavefunction has the form of Eq. (8), is first considered. For the sake of simplicity, the localized state in which $x_0 = 0$ is considered. In the case of interest at $U_0 = 0$ and $V_0 \neq 0$, Eq. (15) gives the spatial damping

$$q_1 = -2m^2V_0(\Omega_1 - \Omega_2),$$

which corresponds to the energy of the localized state

$$E = \Omega_1 - 2m^2[V_0(\Omega_1 - \Omega_2)]^2.$$

Consequently, if $\Omega_1 > \Omega_2$, localization occurs at $V_0 < 0$, whereas if $\Omega_1 < \Omega_2$, localization occurs at $V_0 > 0$. This means that the excitation can be localized near both the attractive and repulsive defects depending on the relation between the characteristics of the media (e.g., the chemical potentials).

The next case is contact of a linear crystal with a crystal having negative nonlinearity ($\gamma < 0$) when the energy of the excitation is in the range $E < \min\{\Omega_1, \Omega_2\}$ and the wavefunction has the form of Eq. (18). For simplicity, only the case $\Omega_1 = \Omega_2 = \Omega$ can be considered; in this case, $q_1 = q_2 = q$. In the case of interest at $U_0 = 0$ and $V_0 \neq 0$, Eq. (25) in the long-wavelength approximation $qx_0 \ll 1$ gives the spatial damping

$$q = -(x_0 + 2mV_0)/x_0^2.$$

Therefore, for the existence of a localized state, the condition $x_0 < -2mV_0$ should be satisfied. Since the condition $x_0 < 0$ is required for the solution (18) to be bounded, localization of the excitation is possible at $V_0 > 0$. At small V_0 values, the localization length of excitations is $l \approx x_0(1 - 2V_0/x_0)$; i.e., such an excitation almost damps at distances of about x_0.

The next case is contact of a linear crystal with a crystal having negative nonlinearity when the energy of the excitation is in the range $\Omega_2 < E < \Omega_1$ and the wavefunction is given by Eq. (27). In the case of interest at $U_0 = 0$ and $V_0 \neq 0$, in the long-wavelength approximation $qx_0 \ll 1$ and under the additional requirement $mV_0q_1 \ll 1$, Eq. (33) gives

$$q_1 = 2m^2V_0(\Omega_1 - \Omega_2) - 1/x_0.$$

For the existence of such a localized state, the condition $x_0 > 1/2m^2V_0(\Omega_1 - \Omega_2)$ should be satisfied. Consequently, since $\Omega_2 < \Omega_1$ and the sign of x_0 is not fixed, localization of the excitation is possible at any sign of V_0.

It can be similarly shown that semi-localized states described by functions (36), (45), and (51) can exist under the conditions $U_0 = 0$ and $V_0 \neq 0$.

Thus, the inclusion of the internal structure of the interface, which sometimes leads to the appearance of qualitatively new effects, can be important when studying the features of localization of excitations near the interface between media.

6. CONCLUSIONS

It has been shown that localized states of several types can appear near a structured planar defect between linear and nonlinear media. Such localized states are described by soliton solutions of the nonlinear Schrödinger equation.

The mathematical formulation of the model for the description of the structured defect requires the use of the modified potential including derivatives of the Dirac delta function. The solution of the nonlinear Schrödinger equation with this potential is reduced to the solution of the nonlinear Schrödinger equation without potential with boundary conditions. Solutions of the formulated contact boundary value problem with such conditions have been found. Explicit analytical expressions have been obtained for the energy. It has been shown that the inclusion of the internal structure of the defect modifies the profile of nonlinear localized excitations and the region of their existence.

Two types of stationary states exist in the considered system. States localized on both sides of the defect constitute the first type. The second type of the stationary state consists of a state localized in the nonlinear medium and a standing wave in the linear medium and is called semi-localized.

Both types of stationary states are implemented in three types determined by the sign of nonlinearity of the medium and by the range of the possible energy of excitations. In the case of contact of the linear crystal with the self-focusing medium, i.e., with the crystal with positive nonlinearity ($\gamma > 0$), the localized state occurs when the energy of the excitation is in the range $E < \min\{\Omega_1, \Omega_2\}$ and the semi-localized state appears when the energy of the excitation is in the range $\Omega_1 < E < \Omega_2$.

In the case of contact of the linear crystal with the defocusing medium, i.e., with the crystal with negative nonlinearity ($\gamma < 0$), two types of both localized and semi-localized states exist. Localized states of the first type are implemented in the excitation energy range $E < \min\{\Omega_1, \Omega_2\}$, whereas localized states of the second type exist in the range $\Omega_2 < E < \Omega_1$. Semi-localized states of the first and second types exist in the excitation energy ranges $\Omega_1 < E < \Omega_2$ and $E > \max\{\Omega_1, \Omega_2\}$, respectively. Thus, different types of localized states can be obtained by varying the localization energy.

The results obtained in this work supplement studies [8–10] of the features of localization of nonlinear excitations in media with defects to the case of the interface between linear and nonlinear media.

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