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Rossby Waves in the Magnetic Fluid Dynamics of a Rotating Plasma in the Shallow-Water Approximation

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Abstract—We have studied rotating magnetohydrodynamic flows of a thin layer of astrophysical plasma with a free boundary in the β-plane. Nonlinear interactions of the Rossby waves have been analyzed in the shallow-water approximation based on the averaging of the initial equations of the magnetic fluid dynamics of the plasma over the depth. The shallow-water magnetohydrodynamic equations have been generalized to the case of a plasma layer in an external vertical magnetic field. We have considered two types of the flow, viz., the flow in an external vertical magnetic field and the flow in the presence of a horizontal magnetic field. Qualitative analysis of the dispersion curves shows the presence of three-wave nonlinear interactions of the magnetic Rossby waves in both cases. In the particular case of zero external magnetic field, the wave dynamics in the layer of a plasma is analogous to the wave dynamics in a neutral fluid. The asymptotic method of multiscale expansions has been used for deriving the nonlinear equations of interaction in an external vertical magnetic field for slowly varying amplitudes, which describe three-wave interactions in a vertical external magnetic field as well as three-wave interactions of waves in a horizontal magnetic field. It is shown that decay instabilities and parametric wave amplification mechanisms exist in each case under investigation. The instability increments and the parametric gain coefficients have been determined for the relevant processes.

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1. INTRODUCTION

This study is devoted to the development of the nonlinear theory of the Rossby waves in astrophysical and space plasma layers in the shallow-water approximation on the β-plane. Here, the Rossby waves is the term used for the large-scale waves emerging due to nonuniformity of the Coriolis force depending on the latitude on a sphere, which propagate due to the conservation of the total vorticity in a rotating plasma analogously to a neutral fluid. Such waves determine the large-scale dynamics of the Sun and stars [1–4], as well as dynamics of magnetoactive atmospheres of exoplanets trapped by tides of a carrier star [5] and the flows in accretion disks of neutron stars [6]. In spite of complexity of observation of the Rossby waves in astrophysical plasmas, such waves have been recently detected on the Sun [7]. The Rossby waves have also been discovered indirectly on the Sun in some studies [8–12].

Large-scale Rossby waves in a neutral fluid determine the global dynamics of planetary atmospheres. Analysis of such waves is the subject of numerous investigations in geophysical fluid dynamics [13, 14]. In this case, the waves are considered against the background of the trivial stationary (rest) state, and the theory of such waves is developed using the shallow-water approximation or the geostrophic approximation. A direct analog of Rossby waves in geophysical fluid dynamics are drift waves in a plasma [13, 15]. In the case of astrophysical plasma flows, the theory of Rossby waves is substantially complicated due to the existence of nontrivial stationary states of the magnetic fields (e.g., the toroidal and poloidal fields or an external vertical magnetic field) [3, 4, 16, 17]; for this reason, the main results concerning the magnetic Rossby waves were obtained in the linear approximation.

In this study, we develop a weakly nonlinear theory of magnetohydrodynamic Rossby waves. It is worth noting that the dynamics of Rossby waves in the presence and absence of an external vertical magnetic field differ significantly. The main difference lies in the fact that the magnetic field continuity equation plays an important role in the shallow-water approximation in a magnetic field; when this equation is taken into account correctly, traditional concepts concerning the 2D nature of shallow-water equations and the possibility of using the 2D magnetic fluid dynamics for explaining large-scale processes should be substantially modified.

The magnetohydrodynamic equations in the shallow-water approximation also play equally important role in the space and astrophysical plasmas like classical shallow-water equations in the fluid dynamics of a neutral fluid. The system of equations can be obtained

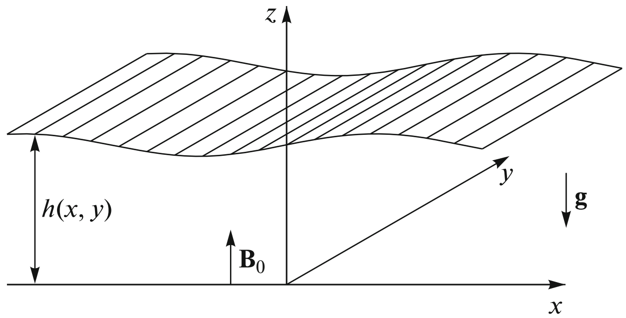

Fig. 1. Geometry of the problem.

from the classical equations of magnetic fluid dynamics of an incompressible plasma by averaging over the depth under the assumption of the hydrostatic nature of pressure distribution and the smallness of the layer thickness as compared to the characteristic horizontal linear size of the problem [18–24]. Such an approximation was used for developing the theory of the three-wave interaction of the Poincaré waves in magnetic fluid dynamics [25, 26]. It should be noted that a significant difference in the magnetohydrodynamic equations for a rotating plasma and for a neutral fluid is associated with the existence of the condition of a nondivergent magnetic field, which is satisfied identically. It follows, hence, that approximate shallow-water equations must obey the corollary of the zero-divergence condition in the initial system of magnetohydrodynamic equations, which must also be satisfied identically for the resultant simplified system. Therefore, the shallow-water equations in the presence of an external vertical magnetic field [26] differ from the traditional equations. Averaging of the magnetohydrodynamic equations over the depth in the presence of an external magnetic field indicates the fundamental three-component nature of the magnetic field along with the two-component nature of the velocity field [26]. Consequently, the use of the modified shallow-water equations provides a detailed description of the magnetic field, explains its 3D structure, and gives a more comprehensive explanation for both linear and nonlinear effects.

In this study, considerable advances have been made in analysis of Rossby waves. We write the magnetohydrodynamic shallow-water equations on the β-plane in an external magnetic field taking into account the identities ensuring the nondivergence of the magnetic field in the approximate equations. In zero magnetic field, the system of shallow-water equations can be reduced to the traditional system and has time-independent solutions in the form of horizontal (poloidal, toroidal, and their sum) magnetic fields. For each stationary state, we have developed a weakly nonlinear theory of waves, derived the equations for Rossby wave packets, and analyzed parametric instabilities [27–29].

In Section 2, we write the system of magnetohydrodynamic equations in the shallow-water approximation in an external vertical magnetic field. The dispersion relation for magnetohydrodynamic Rossby waves in a vertical magnetic field and in a horizontal (toroidal and poloidal) magnetic field are analyzed qualitatively, and conclusions concerning the possibility of three-wave interactions for these waves are formulated.

In Section 3, we derive the system of equations for slowly varying amplitudes of the three-wave interactions of the Rossby waves in an external vertical magnetic field using the method of multiscale expansions. Parametric instabilities are analyzed. In Section 4, the same is done for a horizontal magnetic field. The results are discussed in Conclusions.

2. SHALLOW-WATER MAGNETOHYDRODYNAMIC EQUATIONS. LINEAR WAVES. QUALITATIVE ANALYSIS OF DISPERSION RELATIONS

2.1. Modified Shallow-Water Magnetohydrodynamic Equations on the β-Plane

The magnetohydrodynamic equations in the shallow-water approximation describe the flows of a thin layer of a plasma (magnetic fluid) with a free boundary in a uniform gravity field in a rotating frame of reference in an external vertical magnetic field (Fig. 1) [26]:

$$\partial_t h + \mathrm{div}(h\mathbf{v}) = 0, \tag{2.1}$$

$$\begin{gathered}\partial_t(h\mathbf{v}) + (\mathbf{v}\cdot\nabla)(h\mathbf{v}) + h\mathbf{v}\,\mathrm{div}\mathbf{v} - (\mathbf{B}\cdot\nabla)(h\mathbf{B}) \\ -h\mathbf{B}\,\mathrm{div}\mathbf{B} + \mathrm{grad}\left(\frac{1}{2}gh^2\right) + B_0\mathbf{B} + fh[\mathbf{e}_z\times\mathbf{v}] = 0,\end{gathered} \tag{2.2}$$

$$\partial_t(h\mathbf{B}) + \mathrm{curl}(h[\mathbf{v}\times\mathbf{B}]) - B_0\mathbf{v} = 0, \tag{2.3}$$

$$\partial_t B_z + B_0\mathrm{div}\mathbf{v} = 0, \tag{2.4}$$

$$\mathrm{div}(h\mathbf{B}) + B_z = 0. \tag{2.5}$$

In these equations, h is the layer thickness, $\mathbf{v}(v_x, v_y)$ is the horizontal velocity averaged over the thickness, $\mathbf{B}(B_x, B_y)$ is the horizontal magnetic field averaged over the layer thickness, B_z is the vertical magnetic field component averaged over the layer thickness, g is the free-fall acceleration, f is the Coriolis parameter, and B_0 is the external vertical magnetic field. System (2.1)–(2.5) is the result of integration of 3D magnetohydrodynamic equations along the vertical coordinate z. We assume that the total pressure (sum of the hydrodynamic and magnetic pressures) is hydrostatic. Equation (2.1) is the continuity equation; Eq. (2.2) was obtained by averaging of the momentum variation equation over the layer thickness; Eqs. (2.3) and (2.4) describe the magnetic field variation, and Eq. (2.5) is the magnetic field nondivergence condition averaged over the height. If vertical magnetic field is $B_0 = 0$, Eqs. (2.1)–(2.5) can be reduced to the well-known magnetohydrodynamic shallow-water equations [18, 23, 30, 31]. The system of equations (2.1)–(2.3) is closed and is used in analysis of linear waves and nonlinear interactions [17, 25].

Let us clarify the physical meaning of additional equations (2.4) and (2.5) [26]. In the traditional derivation of magnetohydrodynamic shallow-water equations from the complete system of 3D magnetohydrodynamic equations, the vertical magnetic field component is assumed to be zero. It should be noted that the presence of a vertical magnetic field leads to significant modifications of the horizontal magnetic field dynamics in the shallow-water approximation. The horizontal magnetic field in the case of flows in zero magnetic field is sinusoidal, but the presence of an external vertical magnetic field changes the situation. Vertical changes of the magnetic field differ from zero, and the nondivergence condition contains vertical component (2.5). Therefore, we must add Eq. (2.4) for the vertical variation of the magnetic field to describe the magnetic field dynamics. Thus, the magnetic field is basically three-dimensional, and each of its components is a function of only horizontal coordinates. Nondivergence condition (2.5) is satisfied identically as a corollary of Eqs. (2.3) and (2.4) for the magnetic field and is used for specifying correct initial conditions. Equations (2.4) and (2.5) are important in the magnetohydrodynamic shallow-water approximation in an external magnetic field not only as technical details; these equations also indicate the existence of the z component of the magnetic field, the equation for which differs from the shallow-water equations [26]. For describing large-scale magnetic Rossby waves, we will use the β-plane approximation for the Coriolis force analogously to rotating flows of a neutral fluid on a sphere.

We assume that the vertical component of the velocity of rotation changes with latitude θ. The waves induced by the latitude dependence of the Coriolis force are analogs of the Rossby waves in geophysical fluid dynamics and are referred to as magnetic Rossby waves. In the β-plane approximation, we assume that the variations of Coriolis parameter f are small for small variations of the latitude and write this parameter in the form

$$f = 2\Omega\sin\theta \approx 2\Omega\sin\theta_0 + 2\Omega(\theta - \theta_0)\cos\theta_0 = f_0 + \beta y. \qquad (2.6)$$

Here, Ω is the angular velocity of the layer, $f_0 = 2\Omega\sin\theta_0$, and $\beta = \partial f/\partial y$. With allowance for the dependence (2.6) of the Coriolis parameter on the latitude, momentum variation equations (2.2) describe the rotating flows on a sphere in the Cartesian system of coordinates. Therefore, we obtain from Eqs. (2.1)–(2.3) the following system as the initial equations for analysis of magnetic Rossby waves:

$$\partial_t h + \mathrm{div}(h\mathbf{v}) = 0, \qquad (2.7)$$

$$\partial_t(h\mathbf{v}) + (\mathbf{v}\cdot\nabla)(h\mathbf{v}) + h\mathbf{v}\mathrm{div}\mathbf{v} - (\mathbf{B}\cdot\nabla)(h\mathbf{B}) - h\mathbf{B}\mathrm{div}\mathbf{B} + \mathrm{grad}\left(\frac{1}{2}gh^2\right) + B_0\mathbf{B} + (f_0 + \beta y)h[\mathbf{e}_z \times \mathbf{v}] = 0, \qquad (2.8)$$

$$\partial_t(h\mathbf{B}) + \mathrm{curl}(h[\mathbf{v}\times\mathbf{B}]) - B_0\mathbf{v} = 0, \qquad (2.9)$$

$$\partial_t B_z + B_0\mathrm{div}\mathbf{v} = 0. \qquad (2.10)$$

This system will be used below for studying the magnetic Rossby waves in an external vertical magnetic field. For $B_0 = 0$, initial system (2.7)–(2.10) is transformed into the system of magnetohydrodynamic equations in the shallow-water approximation on the β-plane and will be used for analyzing nonlinear processes in horizontal (toroidal and poloidal) fields. The complete solution of the linear problem will be given below for an external vertical magnetic field as well as for zero field, which is necessary for constructing the perturbation theory in the weakly nonlinear approximation.

2.2. Linear Rossby Waves in a Vertical Magnetic Field

Linearizing the initial system (2.7)–(2.9) relative to the state of rest ($h = H = \mathrm{const}$, $v_x = v_y = B_x = B_y = B_z = 0$), we obtain

$$\partial_t h + H\partial_x v_x + H\partial_y v_y = 0, \qquad (2.11)$$

$$H\partial_t v_x + gH\partial_x h + B_0 B_x - (f_0 + \beta y)Hv_y = 0, \qquad (2.12)$$

$$H\partial_t v_y + gH\partial_y h + B_0 B_y + (f_0 + \beta y)Hv_x = 0, \qquad (2.13)$$

$$H\partial_t B_x - B_0 v_x = 0, \qquad (2.14)$$

$$H\partial_t B_y - B_0 v_y = 0. \qquad (2.15)$$

Let us differentiate Eq. (2.12) with respect to y:

$$H\partial_y\partial_t v_x + gH\partial_y\partial_x h + B_0\partial_y B_x - f_0 H\partial_y v_y - \beta H v_y = 0. \qquad (2.16)$$

Further, we use the β-effect approximation [14] in the same way as for neutral fluid flows. We seek the solution to system (2.11)–(2.15) in the form

$$\begin{pmatrix} h \\ v_x \\ v_y \\ B_x \\ B_y \end{pmatrix} = \begin{pmatrix} h_0 \\ v_{x0} \\ v_{y0} \\ B_{x0} \\ B_{y0} \end{pmatrix} \exp[i(k_x x + k_y y - \omega t)]. \qquad (2.17)$$

Substituting this solution into system (2.11)–(2.15) and into Eq. (2.16), we obtain the system of linear equations

$$A\begin{pmatrix} h_0 \\ v_{x0} \\ v_{y0} \\ B_{x0} \\ B_{y0} \end{pmatrix} = 0, \qquad (2.18)$$

in which linear operator A has the form

$$A = \begin{pmatrix} -i\omega & ik_xH & ik_yH & 0 & 0 \\ -gHk_xk_y & \omega k_yH & -ik_yf_0H - \beta H & ik_yB_0 & 0 \\ igHk_y & f_0H & -i\omega H & 0 & B_0 \\ 0 & B_0 & 0 & -i\omega H & 0 \\ 0 & 0 & B_0 & 0 & -i\omega H \end{pmatrix}. \quad (2.19)$$

System (2.18) has nontrivial solution when the condition $\det A = 0$ holds. After simple transformations, we obtain from Eq. (2.18) the dispersion relation

$$\omega^4 - \omega^2\left[f_0^2 + C_0^2k^2 + 2\left(\frac{v_A}{H}\right)^2\right] - \omega C_0^2\beta k_x - \left(\frac{v_A}{H}\right)^2\left[C_0^2k^2 + \left(\frac{v_A}{H}\right)^2\right] = 0, \quad (2.20)$$

where $v_A = B_0$ is the Alfven velocity and $C_0 = \sqrt{gH}$. In the high-frequency approximation, the dependence of the Coriolis parameter on the latitude in expression (2.20) disappears, and the dispersion relation describes the magnetic Poincaré mode in magnetic fluid dynamics in the shallow-water approximation. This approximation leads to the dispersion relation, in which the wave dynamics is determined by the gravity force, rotation, and external vertical magnetic field:

$$\omega^2 \approx f_0^2 + C_0^2k^2 + 2\left(\frac{v_A}{H}\right)^2. \quad (2.21)$$

In the low-frequency approximation, dispersion relation (2.20) describes large-scale movements of Rossby waves. In the shallow-waver magnetic fluid dynamics, the dispersion relation for Rossby waves in the presence of an external vertical magnetic field is modified and assumes the form

$$\omega = \frac{(v_A/H)^2[C_0^2k^2 + (v_A/H)^2]}{C_0^2\beta k_x}. \quad (2.22)$$

It should be noted that in the interval between high-frequency solutions to the dispersion equation in the form of magnetic Poincaré modes and low-frequency solutions for slow magnetic Rossby waves, the dispersion relation permits the modes of fast magnetic Rossby waves [4].

Relation (2.22) describes the Rossby waves propagating in the k direction in the shallow-water approximation. The main mechanism of their formation is the shift of the rotating flow due to the latitude dependence of the Coriolis force. The general form of dispersion curves is shown in Fig. 2. The solutions to the linearized systems are amplitude waves $\alpha(\mathbf{k})$. In the general form, solution $\mathbf{u}_1$ to the linear system of differential equations (2.11)–(2.15) can be written as

$$\mathbf{u}_1 = \int dk_x dk_y \mathbf{a}\alpha(\mathbf{k}) \times \exp[i(k_xx + k_yy - \omega(\mathbf{k})t)] + \text{c.c.}, \quad (2.23)$$

where $\omega(\mathbf{k})$ is determined by expression (2.20) and $\mathbf{a}$ is the eigenvector of linear operator A,

$$A\mathbf{a} = 0. \quad (2.24)$$

System (2.24) has nontrivial solutions $\mathbf{a} \neq 0$ because condition (2.20) holds. Therefore, the eigenvector has the form

$$\mathbf{a} = \begin{pmatrix} \omega^2k_xH - k_xH\left(\frac{B_0}{H}\right)^2 - i\omega k_yfH \\ \omega^3 - \omega\left(\frac{B_0}{H}\right)^2 - \omega k_y^2gH \\ \omega k_xk_ygH - i\omega^2 f \\ -\omega^2\left(\frac{B_0}{H}\right) - \left(\frac{B_0}{H}\right)^3 - k_y^2gH\left(\frac{B_0}{H}\right) \\ -k_xk_ygH\left(\frac{B_0}{H}\right) + i\omega f\left(\frac{B_0}{H}\right) \end{pmatrix} = \mathbf{a}(\mathbf{k}), \quad (2.25)$$

where $\omega = \omega(\mathbf{k})$ in accordance with condition (2.20).

To estimate the possibility of interwave interactions for the wave described above, we must analyze its dispersion relation and determine the asymptotic forms of dispersion curves. The synchronism condition required for

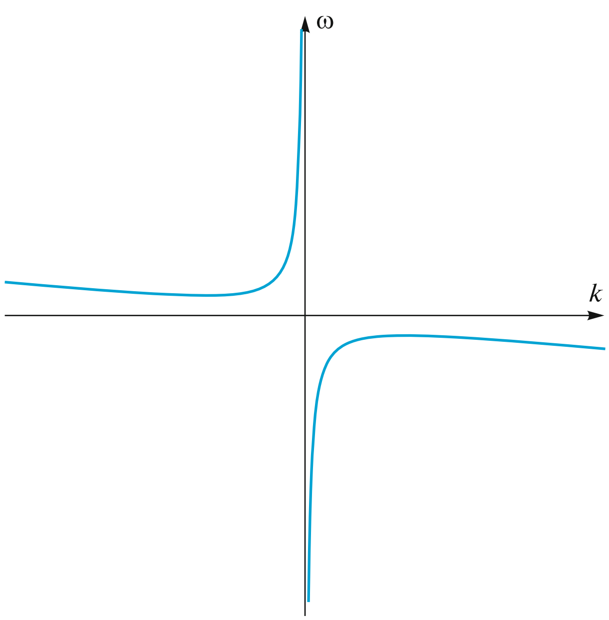

Fig. 2. (Color online) Dispersion curves for Rossby waves in a vertical external magnetic field in magnetic fluid dynamics.

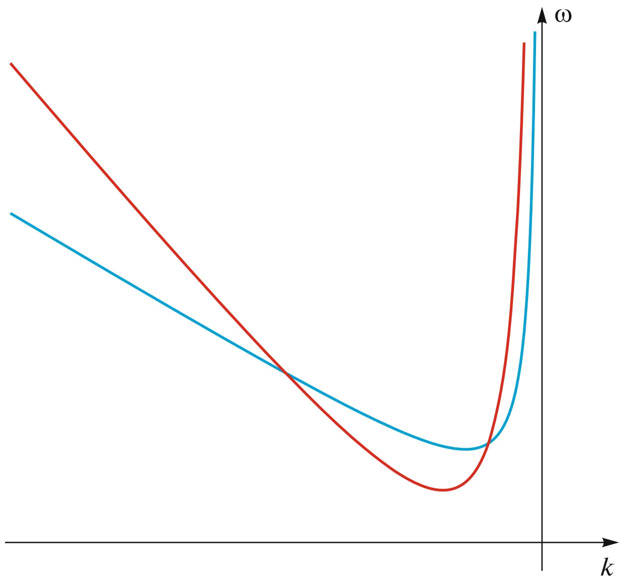

Fig. 3. (Color online) Dispersion curves for the Rossby waves in magnetic fluid dynamics in the absence of a large-scale magnetic field.

the interaction between waves with different wavevectors and frequencies can be written in general form as

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_1 + \mathbf{k}_1).$$

Let us illustrate this condition graphically (Fig. 3). The first term defines point ($\mathbf{k}_1$, $\omega(\mathbf{k}_1)$) on the dispersion curve of one of the solutions, while the second term defines point ($\mathbf{k}_2$, $\omega(\mathbf{k}_2)$) on the dispersion curve displaced by ($\mathbf{k}_1$, $\omega(\mathbf{k}_1)$). The intersection indicates the existence of a set of three waves, which satisfies the synchronism condition. Qualitative analysis of dispersion curves makes it possible to find out whether the above synchronism condition is satisfied for the Rossby waves in a vertical magnetic field. It can be seen from Fig. 3 that dispersion curve $\omega(\mathbf{k})$ intersects the identical curve displaced by ($\mathbf{k}_1$, $\omega(\mathbf{k}_1)$) and, hence, there exist three such waves for which the synchronism condition is satisfied. Therefore, three-wave interactions occur in the case of a vertical external magnetic field $\mathbf{B}_0$.

2.3. Linear Rossby Waves in a Horizontal Magnetic Field

Let us now analyze the linear problem in zero external magnetic field. As noted above, system (2.7)–(2.9) in this case has steady solutions in the form of a horizontal (toroidal, poloidal, or their sum) magnetic field. As the initial equations for the Rossby waves in a horizontal magnetic field, we consider system (2.7)–(2.9) for $B_0 = 0$. We linearize this system relative to the steady solution describing horizontal field B^0: $h = H$, $v_x = v_y = 0$, $B_x = B_x^0$, and $B_y = B_y^0$:

$$\partial_t h + H\partial_x v_x + H\partial_y v_y = 0, \quad (2.26)$$

$$\partial_t\partial_y v_x - B_x^0\partial_x\partial_y B_x - B_y^0\partial_y^2 B_x + g\partial_x\partial_y h - f_0\partial_y v_y - \beta v_y = 0, \quad (2.27)$$

$$\partial_t v_y - B_x^0\partial_x B_y - B_y^0\partial_y B_y + g\partial_y h + f_0 v_x = 0, \quad (2.28)$$

$$\partial_t B_x - B_x^0\partial_x v_x - B_y^0\partial_y v_x = 0, \quad (2.29)$$

$$\partial_t B_y - B_y^0\partial_y v_y - B_x^0\partial_x v_y = 0. \quad (2.30)$$

The last two equations, (2.29) and (2.30), lead to the magnetic field nondivergence condition:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0.$$

Taking this condition into account and substituting the solution in the form of a wave, we obtain the following system of linear equations

$$A_h\begin{pmatrix} h_0 \\ v_{x0} \\ v_{y0} \\ B_{x0} \\ B_{y0} \end{pmatrix} = 0, \quad (2.31)$$

in which operator A_h has the form

$$A_h = \begin{pmatrix} -i\omega & ik_xH & ik_yH & 0 & 0 \\ -gk_xk_y & \omega k_y & -ik_yf_0 - \beta & k_y(\mathbf{k}\cdot\mathbf{B}^0) & 0 \\ ik_yg & f_0 & -i\omega & 0 & -i(\mathbf{k}\cdot\mathbf{B}^0) \\ 0 & -i(\mathbf{k}\cdot\mathbf{B}^0) & 0 & -i\omega & 0 \\ 0 & 0 & -i(\mathbf{k}\cdot\mathbf{B}^0) & 0 & -i\omega \end{pmatrix}. \quad (2.32)$$

Introducing notation $\mathbf{v}_A = \mathbf{B}^0$ and $C_0^2 = gh$, we can write the dispersion relation in the form

$$\omega^4 - \omega^2[2(\mathbf{k}\cdot\mathbf{v}_A)^2 + f_0^2 + C_0^2k^2] - \omega k_x C_0^2\beta + (\mathbf{k}\cdot\mathbf{v}_A)^2[C_0^2k^2 + (\mathbf{k}\cdot\mathbf{v}_A)^2] = 0. \quad (2.33)$$

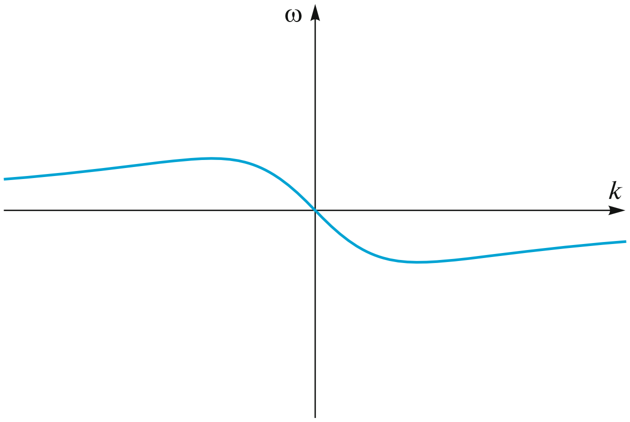

Fig. 4. (Color online) Synchronism condition for three Rossby waves in an external vertical field in magnetic fluid dynamics.

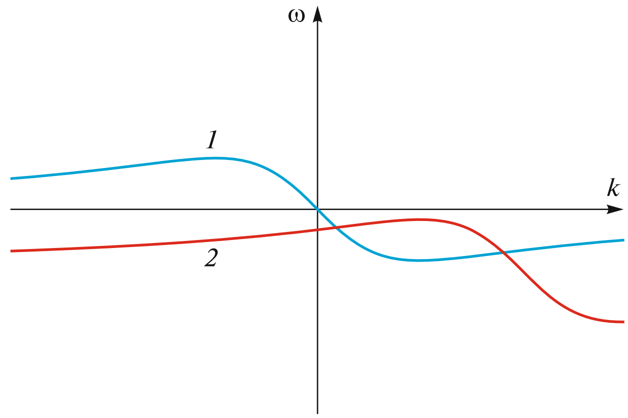

Fig. 5. (Color online) Synchronism condition for three Rossby waves in magnetic fluid dynamics in the absence of a large-scale field.

Analogously to the case of (2.22) with a vertical external magnetic field, the dispersion relation for low frequencies and long-wavelength flows assumes the form

$$\omega = \frac{C_0^2 k^2 (\mathbf{k} \cdot \mathbf{v}_A)^2 + (\mathbf{k} \cdot \mathbf{v}_A)^4}{C_0^2 k_x \beta} \tag{2.34}$$

and describes the mode of slow magnetic Rossby waves in the magnetic fluid dynamics in the shallow-water approximation [4]. The general view of dispersion curves was given in Fig. 2.

Let us analyze qualitatively the resultant dispersion curves to explore the possibility of realization of weakly nonlinear interactions. The necessary condition for the three-wave interaction of Rossby waves is that the three waves must satisfy the synchronism condition

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_1 + \mathbf{k}_2). \tag{2.35}$$

Let us consider three waves with dispersion relation (2.34). Their three-wave interactions are possible when condition (2.35) is satisfied, which corresponds to the intersection of the dispersion curves in Fig. 3. The existence of intersection indicates the possibility of three-wave interactions of magnetohydrodynamic Rossby waves in horizontal magnetic field $\mathbf{B}^0 = (B_x^0, B_y^0)$.

In the particular case of a toroidal magnetic field, the dispersion relation has the form

$$\omega = \frac{k_x (v_{Ax})^2 [C_0^2 k^2 + (k_x v_{Ax})^2]}{C_0^2 \beta}. \tag{2.36}$$

Such a magnetic field configuration is typical of flows in the solar tachocline [2, 3].

In the case of a poloidal magnetic field, the dispersion relation has the form

$$\omega = \frac{C_0^2 k^2 (k_y v_{Ay})^2 + (k_y v_{Ay})^4}{C_0^2 k_x \beta}. \tag{2.37}$$

In the limit of a neutral fluid, $B_x^0 = 0$, $B_y^0 = 0$, the dispersion relations for the Rossby waves takes the form

$$\omega^3 - \omega(f_0^2 + C_0^2 k^2) - C_0^2 \beta k_x = 0. \tag{2.38}$$

In the long-wave approximation, dispersion equation (2.38) has a solution (Fig. 4)

$$\omega = -\frac{k_x C_0^2 \beta}{f_0^2 + C_0^2 k^2}. \tag{2.39}$$

It should be noted that for zero horizontal magnetic field, the linearized equations for the magnetic field can be reduced to the trivial form

$$\frac{\partial B_x}{\partial t} = 0, \quad \frac{\partial B_y}{\partial t} = 0.$$

Therefore, a wave is not magnetohydrodynamic any longer, and the dispersion relation describes the dynamics of only hydrodynamic parameters. The eigenvector in this case has the form

$$\mathbf{a} = \begin{pmatrix} \omega k_x H - i k_y f H \\ \omega^2 - k_y^2 g H \\ k_x k_y g H - i \omega f \end{pmatrix} = \mathbf{a}(\mathbf{k}), \tag{2.40}$$

where $\omega = \omega(\mathbf{k})$ in accordance with relation (2.39).

Let us analyze qualitatively the resultant dispersion curves to explore the possibility of realization of weakly nonlinear interactions. It is shown in Fig. 5 for zero horizontal magnetic field that there exist such values ω_1, $\mathbf{k}_1$, ω_2, $\mathbf{k}_2$, and ω_3, $\mathbf{k}_3$, for which the synchronism conditions $\omega_1 + \omega_2 = \omega_3$ and $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ hold. Indeed, the coordinates of points on curve *1* represent the wavevector and frequency of the first wave. Points on curve *2*, which is displaced relative to curve *1* by ω_1, $\mathbf{k}_1$, have coordinates $\mathbf{k}_1 + \mathbf{k}_2$ (sum of two wavevectors) and $\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)$ (sum of two frequencies). The fact that curve *2* intersects curve *1* means that the coordinates of the point of intersection of curves *1* and *2* coincide; i.e., the synchronism condition is satisfied. Therefore, in zero horizontal magnetic field, the three-wave interactions of Rossby waves take place in the resultant hydrodynamic problem [32].

3. THREE-WAVE INTERACTIONS AND PARAMETRIC INSTABILITIES OF ROSSBY WAVES IN AN EXTERNAL VERTICAL MAGNETIC FIELD

For deriving equations describing the three-wave interactions, we can use the asymptotic method of large-scale expansions [33]. The solution to system (2.7)–(2.10) can be written in the form of a series in small parameter ε:

$$\mathbf{u} = \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \ldots, \quad (3.1)$$

where $\mathbf{u}_1$ is the solution to linearized system (2.18) and $\mathbf{u}_2$ is a correction describing the effect of the quadratic nonlinearity. Writing the terms proportional to ε^2, we obtain a system of linear inhomogeneous differential equations in $\mathbf{u}_2$, containing on the right-hand side the resonance terms leading to a linear increase in the solution (in time and coordinate). Therefore, the condition $\varepsilon^2 u_2 \ll \varepsilon u_1$ is violated on large scales. To eliminate the influence of the resonance terms, we introduce the dependence of the wave amplitude on large temporal and spatial scales in the form

$$\mathbf{u}_1(T_1, X_1, Y_1)\exp[i(k_x X_0 + k_y Y_0 - \omega T_0)].$$

The evolution equation for a slowly varying amplitude ensures the uniform convergence of the asymptotic series. We pass from argument t, x, y to "fast" (T_0, X_0, Y_0) and "slow" (T_1, X_1, Y_1) arguments in accordance with the relations

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1}, \\ \frac{\partial}{\partial x} &= \frac{\partial}{\partial X_0} + \varepsilon\frac{\partial}{\partial X_1}, \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial Y_0} + \varepsilon\frac{\partial}{\partial Y_1}. \end{aligned} \quad (3.2)$$

Substituting expressions (3.1) and (3.2) into the initial system (2.7)–(2.10), we write the terms proportional to ε^2:

$$\begin{pmatrix} \dfrac{\partial h_2}{\partial T_0} + H\dfrac{\partial v_{x2}}{\partial X_0} + H\dfrac{\partial v_{y2}}{\partial Y_0} \\ H\dfrac{\partial^2 v_{x2}}{\partial T_0 \partial Y_0} + gH\dfrac{\partial^2 h_2}{\partial X_0 \partial Y_0} + B_0\dfrac{\partial B_{x2}}{\partial y_0} - f_0 H\dfrac{\partial v_{y2}}{\partial Y_0} - \beta H v_{y2} \\ H\dfrac{\partial v_{y2}}{\partial T_0} + gH\dfrac{\partial h_2}{\partial Y_0} + B_0 B_{y2} + f_0 H \partial v_{x2} \\ H\dfrac{\partial B_{x2}}{\partial T_0} - B_0 v_{x2} \\ H\dfrac{\partial B_{y2}}{\partial T_0} - B_0 v_{y2} \end{pmatrix}$$

$$= -\begin{pmatrix} \dfrac{\partial h_1}{\partial T_1} + H\dfrac{\partial v_{x1}}{\partial X_1} + H\dfrac{\partial v_{y1}}{\partial Y_1} \\ H\dfrac{\partial^2 v_{x1}}{\partial T_1 \partial Y_0} + H\dfrac{\partial^2 v_{x1}}{\partial T_0 \partial Y_1} + gH\dfrac{\partial^2 h_1}{\partial X_1 \partial Y_0} + gH\dfrac{\partial^2 h_1}{\partial X_0 \partial Y_1} + B_0\dfrac{\partial B_{x1}}{\partial Y_1} - f_0 H\dfrac{\partial v_{y1}}{\partial Y_1} \\ H\dfrac{\partial v_{y1}}{\partial T_1} + gH\dfrac{\partial h_{1\partial Y_1}}{\partial Y_1} \\ H\dfrac{\partial B_{x1}}{\partial T_1} \\ H\dfrac{\partial B_{y1}}{\partial T_1} \end{pmatrix} \quad (3.3)$$

$$-\begin{pmatrix} \dfrac{h_1 v_{x1}}{\partial X_0} + \dfrac{h_1 v_{y1}}{\partial Y_0} \\ \dfrac{\partial^2 h_1 v_{x1}}{\partial T_0 \partial Y_0} + g\dfrac{\partial h_1}{\partial X_0}\dfrac{\partial h_1}{\partial Y_0} - f_0 h_1 \dfrac{\partial v_{y1}}{\partial Y_0} - \beta h_1 v_{y1} \\ \dfrac{\partial h_1 v_{y1}}{\partial T_0} + g h_1 \dfrac{\partial h_1}{\partial Y_0} + f_0 h_1 v_{x1} \\ \dfrac{\partial h_1 B_{x1}}{\partial T_0} \\ \dfrac{\partial h_1 B_{y1}}{\partial T_0} \end{pmatrix}.$$

The right-hand sides of these equations contain the terms including the solution of the problem obtained in the first approximation, which may induce the resonance with the operator on the left-hand side. To eliminate the resonance terms on the right-hand side, we can use the condition of orthogonality of the right-hand side of expression (3.3) to the kernel of operator A (2.19), which is known as the compatibility condition. We denote the eigenvector of operator A^* by $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5)^T$. The eigenvectors of operator A^* can be found from

$$\det\begin{pmatrix} -i\omega & ik_x H & ik_y H & 0 & 0 \\ -gHk_x k_y & \omega k_y H & -ik_y f_0 H - \beta H & ik_y B_0 & 0 \\ igHk_y & f_0 H & -i\omega H & 0 & B_0 \\ 0 & B_0 & 0 & -i\omega H & 0 \\ 0 & 0 & B_0 & 0 & -i\omega H \end{pmatrix} = 0. \tag{3.4}$$

This gives dispersion relation (2.20). Eigenvector $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5)^T$ satisfies the following system of equations:

$$-i\omega z_1 + ik_x H z_2 + ik_y H z_3 = 0, \tag{3.5}$$

$$\begin{gathered} -gHk_x k_y z_1 + \omega k_y H z_2 \\ -(ik_y f_0 H + \beta H) z_3 + ik_y B_0 z_4 = 0, \end{gathered} \tag{3.6}$$

$$igHk_y z_1 + f_0 H z_2 - i\omega H z_3 + B_0 z_5 = 0, \tag{3.7}$$

$$B_0 z_2 - i\omega H z_4 = 0, \tag{3.8}$$

$$B_0 z_3 - i\omega H z_5 = 0. \tag{3.9}$$

The expression for eigenvector $\mathbf{z}$ has the following form correct to a constant:

$$\mathbf{z} = \begin{pmatrix} igk_x H^4 f\lambda_0 + gk_y \lambda_0^2 H^4 + gk_y H^2 B_0^2 \\ iH^3 f\lambda_0^2 - k_x k_y g H^4 \lambda_0 \\ \lambda_0^3 H^3 + B_0^2 H \lambda_0 + k_x^2 g H^4 \lambda_0 \\ -H^2 f\lambda_0 B_0 - ik_x k_y g H^3 B_0 \\ iB_0 \lambda_0^3 H^2 + iB_0^3 \lambda_0 + ik_x^2 g B_0 H^3 \lambda_0 \end{pmatrix} c, \tag{3.10}$$

where c is an arbitrary constant. We can write the solution in the form of three magnetic Rossby waves satisfying synchronism condition (2.35), where $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$:

$$\begin{gathered} \mathbf{u}_1 = \phi \mathbf{a}(\mathbf{k}_1)\exp(i\theta_1) + \psi \mathbf{a}(\mathbf{k}_2)\exp(i\theta_2) \\ + \chi \mathbf{a}(\mathbf{k}_3)\exp(i\theta_3) + \text{c.c.}, \end{gathered} \tag{3.11}$$

where ϕ, ψ, χ are the amplitudes of the interacting waves, $\theta_i = -\omega(\mathbf{k}_i)T_0 + k_{xi}X_0 + k_{yi}Y_0$ are the phases of the waves, $\mathbf{a}$ is the eigenvector of operator A, and $\mathbf{a}^*$ is the complex-conjugate vector. After differentiation, the right-hand side of system of equations (3.3), which is proportional to $\exp(i\theta_1)$, takes the form

$$-\begin{pmatrix} a_1\frac{\partial\phi}{\partial T_1}+a_2H\frac{\partial\phi}{\partial X_1}+a_3H\frac{\partial\phi}{\partial Y_1} \\ a_2H\frac{\partial\phi}{\partial T_1}+a_1gH\frac{\partial\phi}{\partial X_1} \\ -i\omega(\mathbf{k}_1)a_3H\frac{\partial\phi}{\partial Y_1}+ik_{y1}a_3H\frac{\partial\phi}{\partial T_1}+2ik_{y1}a_1gH\frac{\partial\phi}{\partial Y_1}+a_5B_0\frac{\partial\phi}{\partial Y_1}+a_2f_0H\frac{\partial\phi}{\partial Y_1} \\ a_4H\frac{\partial\phi}{\partial T_1} \\ a_5H\frac{\partial\phi}{\partial T_1} \end{pmatrix} \quad (3.12)$$

$$-\begin{pmatrix} 2ik_{x1}a_1a_2\psi^*\chi+2ik_{y1}a_1a_3\psi^*\chi \\ -2i\omega(\mathbf{k}_1)a_1a_2\psi^*\chi+2ik_{x1}ga_1^2\psi^*\chi-f_0a_1a_3\psi^*\chi \\ \omega(\mathbf{k}_1)k_{y1}a_1a_3\psi^*\chi-2k_{y2}k_{y3}^2ga_1^2\psi^*\chi+2ik_{y1}f_0a_1a_2\psi^*\chi+\beta a_1a_2\psi^*\chi \\ -2i\omega(\mathbf{k}_1)a_1a_4\psi^*\chi \\ -2i\omega(\mathbf{k}_1)a_1a_5\psi^*\chi \end{pmatrix}.$$

Multiplying expression (3.12) by the eigenvector $\mathbf{z}$ (3.10) of the conjugate operator and using the compatibility condition $A^*\mathbf{z} = 0$, we obtain the following equation for amplitude ϕ of the first wave:

$$s_{v1}\phi = f_{v1}\psi^*\chi, \quad (3.13)$$

where the differentiation operator s_{v1} with respect to slow time and coordinates T_1, X_1, Y_1 and coefficient f_{v1} depend only on the initial conditions and characteristics of the interacting waves:

$$s_{v1} = r_{v1}\frac{\partial}{\partial T_1} + p_{v1}\frac{\partial}{\partial X_1} + q_{v1}\frac{\partial}{\partial Y_1}, \quad (3.14)$$

$$r_{v1} = z_1a_1 + z_2a_2H + ik_{y1}z_3a_3H + z_4a_4H + z_5a_5H, \quad (3.15)$$

$$p_{v1} = z_1a_2H + z_2a_1gH, \quad (3.16)$$

$$q_{v1} = z_1a_3H - i\omega(\mathbf{k}_1)z_3a_3H + 2ik_{y1}z_3a_1gH + z_3a_5B_0 + z_3a_2f_0H. \quad (3.17)$$

In these expressions, the eigenvector $\mathbf{a} = \mathbf{a}(\mathbf{k}_1)$,

$$\begin{aligned} f_{v1} &= f_{v1}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,f_0,\beta,g,H,B_0) \\ &= 2iz_1a_1(k_{x1}a_2 + k_{y1}a_3) \\ &+2iz_2a_1[\omega(\mathbf{k}_1)a_2 + k_{x1}ga_1 - f_0a_3] \\ &+z_3a_1[\omega(\mathbf{k}_1)k_{y1}a_3 - 2k_{y2}k_{y3}ga_1 + 2ik_{y1}f_0a_2 + \beta a_2] \\ &-2i\omega(\mathbf{k}_1)a_1(z_4a_4 + z_5a_5). \end{aligned} \quad (3.18)$$

In the latter expression, the product of the form $a_ia_j = [a_i^*(\mathbf{k}_2)a_j(\mathbf{k}_3) + a_i(\mathbf{k}_3)a_j^*(\mathbf{k}_2)]/2$.

Analogously, multiplying the right-hand side of system (3.3) by the eigenvector $\mathbf{z}$, we obtain the equation for amplitude ψ of the second wave for the terms proportional to $\exp(i\theta_2)$:

$$s_{v2}\psi = f_{v2}\phi^*\chi, \quad (3.19)$$

where

$$s_{v2} = r_{v2}\frac{\partial}{\partial T_1} + p_{v2}\frac{\partial}{\partial X_1} + q_{v2}\frac{\partial}{\partial Y_1}, \quad (3.20)$$

$$r_{v2} = z_1a_1 + z_2a_2H + ik_{y2}z_3a_3H + z_4a_4H + z_5a_5H, \quad (3.21)$$

$$p_{v2} = z_1a_2H + z_2a_1gH, \quad (3.22)$$

$$q_{v2} = z_1a_3H - i\omega(\mathbf{k}_1)z_3a_3H + 2ik_{y1}z_3a_1gH + z_3a_5B_0 + z_3a_2f_0H. \quad (3.23)$$

In these expressions, the eigenvector $\mathbf{a} = \mathbf{a}(\mathbf{k}_2)$. The coefficient f_{v2} has the form

$$\begin{aligned} f_{v2} &= f_{v2}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,f_0,\beta,g,H,B_0) \\ &= 2iz_1a_1(k_{x2}a_2 + k_{y2}a_3) \\ &+2iz_2a_1[\omega(\mathbf{k}_1)a_2 + k_{x2}ga_1 - f_0a_3] \\ &+z_3a_1[\omega(\mathbf{k}_2)k_{y1}a_3 - 2k_{y1}k_{y3}ga_1 + 2ik_{y3}f_0a_2 + \beta a_2] \\ &-2i\omega(\mathbf{k}_3)a_1(z_4a_4 + z_5a_5). \end{aligned} \quad (3.24)$$

In this expression, the product of the form $a_ia_j = [a_i^*(\mathbf{k}_1)a_j(\mathbf{k}_3) + a_i(\mathbf{k}_3)a_j^*(\mathbf{k}_1)]/2$.

Writing the terms proportional to $\exp(i\theta_3)$ from the right-hand side of expression (3.3), we obtain the equation for amplitude χ of the third of the interacting waves,

$$s_{v3}\chi = f_{v3}\phi\psi, \quad (3.25)$$

where

$$s_{v3} = r_{v3}\frac{\partial}{\partial T_1} + p_{v3}\frac{\partial}{\partial X_1} + q_{v3}\frac{\partial}{\partial Y_1}, \quad (3.26)$$

$$r_{v3} = z_1a_1 + z_2a_2H + ik_{y3}z_3a_3H + z_4a_4H + z_5a_5H, \quad (3.27)$$

$$p_{v3} = z_1a_2H + z_2a_1gH, \quad (3.28)$$

$$q_{v3} = z_1a_3H - i\omega(\mathbf{k}_1)z_3a_3H + 2ik_{y1}z_3a_1gH + z_3a_5B_0 + z_3a_2f_0H. \quad (3.29)$$

In these expressions, the eigenvector $\mathbf{a} = \mathbf{a}(\mathbf{k}_3)$. The interaction coefficient f_{v3} has the form

$$\begin{aligned} f_{v3} &= f_{v3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, f_0, \beta, g, H, B_0) \\ &= 2iz_1a_1(k_{x3}a_2 + k_{y3}a_3) \\ &+2iz_2a_1[\omega(\mathbf{k}_3)a_2 + k_{x3}ga_1 - f_0a_3] \\ &+z_3a_1[\omega(\mathbf{k}_3)k_{y3}a_3 - 2k_{y1}k_{y2}ga_1 + 2ik_{y2}f_0a_2 + \beta a_2] \\ &-2i\omega(\mathbf{k}_2)a_1(z_4a_4 + z_5a_5). \end{aligned} \tag{3.30}$$

In this expression, the product of the form $a_ia_j = [a_i(\mathbf{k}_1)a_j(\mathbf{k}_2) + a_i(\mathbf{k}_2)a_j(\mathbf{k}_1)]/2$.

Thus, we have obtained the system of interacting amplitudes of three Rossby waves in magnetic fluid dynamics in the shallow-water approximation in an external vertical magnetic field. For convenience of further analysis, we write the resultant system in the form

$$s_{v1}\phi = f_{v1}\psi^*\chi, \tag{3.31}$$

$$s_{v2}\psi = f_{v2}\phi^*\chi, \tag{3.32}$$

$$s_{v3}\chi = f_{v3}\phi\psi. \tag{3.33}$$

We have obtained a set of three equations for amplitudes ϕ, ψ, χ of the interacting waves. The system (3.31)–(3.33) describes the three-wave interaction of Rossby waves, which satisfy synchronism condition (2.35). In Eqs. (3.31)–(3.33), coefficients f_{v1}, f_{v2}, and f_{v3} are defined by expressions (3.18), (3.24), and (3.30), respectively, and operators s_{v1}, s_{v2}, and s_{v3} are defined by expressions (3.14), (3.20), and (3.26).

We can use the system (3.31)–(3.33) for analyzing parametric instabilities of Rossby waves qualitatively [34]. Let us consider the case when the amplitude of one of the three interacting waves at the initial instant is much larger than the amplitudes of the other two waves ($\phi \gg \psi, \chi$). Then we can approximately assume that the amplitude of the first wave is constant ($\phi = \phi_0$); in this case, we can disregard the reverse effect of waves with amplitudes ψ and χ on the pump wave with amplitude ϕ. Then system (3.31)–(3.33) assumes the form

$$s_{v2}\psi = f_{v2}\phi_0^*\chi, \tag{3.34}$$

$$s_{v3}\chi = f_{v3}\phi_0\psi. \tag{3.35}$$

We seek the solution of the resultant linear system of equations in the form

$$\begin{pmatrix}\psi\\ \chi\end{pmatrix} = \begin{pmatrix}\psi'\\ \chi'\end{pmatrix}\exp(\Gamma T_1). \tag{3.36}$$

This gives the instability increment

$$\Gamma = \sqrt{\frac{|f_{v2}f_{v3}|}{|r_{v2}r_{v3}|}}|\phi_0| > 0, \tag{3.37}$$

where f_{v2} and f_{v3} are defined by expressions (3.24) and (3.30), and r_{v2} and r_{v3} are defined by (3.21) and (3.27).

Therefore, one of the magnetic Rossby waves with wavevector $\mathbf{k}_1$ and frequency $\omega_1 = \omega(\mathbf{k}_1)$ splits into two magnetic Rossby waves with wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$ and frequencies ω_2 and ω_3 with increment (3.37).

The instability can evolve in two ways when the amplitudes of growing waves become comparable with the amplitude of the pump wave: explosive growth and instability saturation. The approximation considered above holds as long as amplitude ϕ_0 of the pump wave is much larger than the amplitudes ψ, χ of the other two magnetic Rossby waves. However, at a certain stage of the process, the amplitudes of the growing waves become comparable with amplitude ϕ_0. Therefore, we must include Eq. (3.31) into our analysis. For $f_{v1} < 0$, amplitude ϕ of the pump wave, as well as the growth rates of amplitudes ψ, χ, decrease, which leads to parametric instability saturation.

In the case of linear damping, we can write system (3.34), (3.35) in the form [27]

$$s_{v2}\psi + \eta_2\psi = f_{v2}\phi_0^*\chi, \tag{3.38}$$

$$s_{v3}\chi + \eta_3\chi = f_{v3}\phi_0\psi, \tag{3.39}$$

where terms $\eta_2\psi$ and $\eta_3\chi$ determine damping. In this case, the exponentially increasing solutions of type (3.36) exist only provided that

$$\phi_0 > \sqrt{\eta_2\eta_3|r_{v2}r_{v3}|/|f_{v2}f_{v3}|}.$$

Thus, there exists a threshold value $\phi_0^{\rm cr}$ of the pump wave amplitude,

$$\phi_0^{\rm cr} = \sqrt{\frac{\eta_2\eta_3}{|f_{v2}f_{v3}|}|r_{v2}r_{v3}|}, \tag{3.40}$$

beginning with which instability evolves with the increment

$$\Gamma = \sqrt{\frac{|f_{v2}f_{v3}|}{|r_{v2}r_{v3}|}}\phi_0^{\rm cr}. \tag{3.41}$$

Let us now consider the case when the amplitude of one of the interacting waves is much smaller than the other two amplitudes ($\phi \ll \psi, \chi$) so that we can assume that amplitudes ψ, χ are constant, $\psi = \psi_0$ and $\chi = \chi_0$. Equation (3.31) for amplitude ϕ then assumes the form

$$s_{v1}\phi = f_{v1}\psi_0^*\chi_0. \tag{3.42}$$

We seek the solutions to this equation in the form

$$\phi = \phi'\exp(\Gamma T_1). \tag{3.43}$$

Substituting the solution in this form into Eq. (3.42), we obtain the following expression for the gain,

$$\Gamma = \frac{|f_{v1}|}{|r_{v1}|}|\psi_0\chi_0| > 0, \tag{3.44}$$

where quantity f_{v1} is defined in (3.18) and r_{v1} in (3.15). In the given case of parametric amplification, two initial Rossby waves with wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$ and frequencies $\omega_1 = \omega(\mathbf{k}_1)$ and $\omega(\mathbf{k}_2)$ amplify the magnetic Rossby wave with wavevector $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$ and frequency $\omega_3 = \omega_1 + \omega_2$ with gain (3.44).

The approximation considered above is valid as long as amplitudes ψ_0 and χ_0 of the pump waves are much larger than amplitude ϕ of the third Rossby wave. However, at a certain stage of the process, the amplitude of the growing wave becomes comparable with amplitudes ψ_0, χ_0. Therefore, we must include Eqs. (3.32) and (3.33) in our analysis. Amplitudes ψ_0 and χ_0 of the pump waves, as well as the growth rate of amplitude ϕ, decrease, which leads to the parametric instability saturation.

In the case of damping, Eq. (3.42) for the amplitude of the wave can be written in the form

$$s_{v1}\phi + \eta_1\phi = f_{v1}\psi_0^*\chi_0, \tag{3.45}$$

where η_1 is the linear damping coefficient for amplitude ϕ in Eq. (3.45). Thus, the necessary condition for the evolution of instability is

$$\eta_1 < |\psi_0^*\chi_0||f_{v1}|/|r_{v1}|.$$

This conditions determines the threshold value for the product of amplitudes of waves $(\psi_0\chi_0)^{cr}$, at which the solution increases exponentially,

$$(\psi_0^*\chi_0)^{\rm cr} = \eta_1\frac{|r_{v1}|}{|f_{v1}|}, \tag{3.46}$$

with the increment

$$\Gamma = \frac{|f_{v1}|}{|r_{v1}|}(\psi_0^*\chi_0)^{\rm cr}. \tag{3.47}$$

4. THREE-WAVE INTERACTIONS AND PARAMETRIC INSTABILITIES OF ROSSBY WAVES IN A HORIZONTAL MAGNETIC FIELD

To obtain a qualitative description of the interaction of Rossby waves in a horizontal magnetic field, we perform, analogously to the case of the vertical field, the substitution of variables, passing from t, x, y to "fast" (T_0, X_0, Y_0) and "slow" variables (T_1, X_1, Y_1) arguments in accordance with expressions (3.2). The solution to the system (2.7)–(2.9) for $B_0 = 0$ can be written in the form

$$\mathbf{u} = \varepsilon\mathbf{u}_1(T_1, X_1, Y_1)\exp[i(k_xX_0 + k_yY_0 - \omega T_0)] + \ldots. \tag{4.1}$$

The evolution equation for a slowly varying amplitude can be obtained from the compatibility condition. For the initial system of equations (2.7)–(2.9) for $B_0 = 0$, we write the terms proportional to ε^2:

$$\begin{pmatrix} \dfrac{\partial h_2}{\partial T_0} + H(\nabla_0\cdot\mathbf{v}_2) \\ H\dfrac{\partial^2 v_{x2}}{\partial T_0\partial Y_0} - H(\mathbf{B}^0\cdot\nabla_0)\dfrac{\partial B_{x2}}{\partial Y_0} + gH\dfrac{\partial^2 h_2}{\partial X_0\partial Y_0} - f_0H\dfrac{\partial v_{y2}}{\partial Y_0} - \beta H v_{y2} \\ H\dfrac{\partial v_{y2}}{\partial T_0} - H(\mathbf{B}^0,\nabla_0)B_{y2} + gH\dfrac{\partial h_2}{\partial Y_0} + f_0H\partial v_{x2} \\ H\dfrac{\partial B_{x2}}{\partial T_0} - H(\mathbf{B}^0\cdot\nabla_0)v_{x2} \\ H\dfrac{\partial B_{y2}}{\partial T_0} - H(\mathbf{B}^0\cdot\nabla_0)v_{y2} \end{pmatrix}$$

$$= -\begin{pmatrix} \dfrac{\partial h_1}{\partial T_1} + H(\nabla_1\cdot\mathbf{v}_1) \\ H\left(\dfrac{\partial^2}{\partial T_1\partial Y_0} + \dfrac{\partial^2}{\partial T_0\partial Y_1}\right)v_{x1} - H(\mathbf{B}^0\cdot\nabla_0)\dfrac{\partial B_{x1}}{\partial Y_1} - H(\mathbf{B}^0\cdot\nabla_1)\dfrac{\partial B_{x1}}{\partial Y_0} + gH\left(\dfrac{\partial^2}{\partial X_0\partial Y_1} + \dfrac{\partial^2}{\partial X_1\partial Y_0}\right)h_1 - f_0H\dfrac{\partial v_{y1}}{\partial Y_1} \\ H\dfrac{\partial v_{y1}}{\partial T_1} - H(\mathbf{B}^0\cdot\nabla_1)\partial B_{y1} + gH\dfrac{\partial h_1}{\partial Y_1} \\ H\dfrac{\partial B_{x1}}{\partial T_1} - H(\mathbf{B}^0\cdot\nabla_1)v_{x1} \\ H\dfrac{\partial B_{y1}}{\partial T_1} - H(\mathbf{B}^0\cdot\nabla_1)v_{y1} \end{pmatrix} \tag{4.2}$$

$$
-\begin{pmatrix}
\dfrac{h_1 v_{x1}}{\partial X_0}+\dfrac{h_1 v_{y1}}{\partial Y_0}\\
\dfrac{\partial^2 h_1 v_{x1}}{\partial T_0\partial Y_0}+H\dfrac{\partial^2(v_{x1}^2-B_{x1}^2)}{\partial X_0\partial Y_0}+H\dfrac{\partial^2(v_{x1}v_{y1}-B_{x1}B_{y1})}{\partial^2 Y_0}+g\dfrac{\partial h_1}{\partial X_0}\dfrac{\partial h_1}{\partial Y_0}+gh_1\dfrac{\partial^2 h_1}{\partial X_0}\partial Y_0-f_0\dfrac{\partial h_1 v_{y1}}{\partial Y_0}-\beta h_1 v_{y1}\\
\dfrac{\partial h_1 v_{y1}}{\partial T_0}+H\dfrac{\partial(v_{x1}v_{y1}-B_{x1}B_{y1})}{\partial X_0}+H\dfrac{\partial(v_{y1}^2-B_{y1}^2)}{\partial Y_0}+gh_1\dfrac{\partial h_1}{\partial Y_0}+f_0 h_1 v_{x1}\\
\dfrac{\partial h_1 B_{x1}}{\partial T_0}+H\dfrac{\partial(v_{y1}B_{x1}-v_{x1}B_{y1})}{\partial Y_0}\\
\dfrac{\partial h_1 B_{y1}}{\partial T_0}+H\dfrac{\partial(v_{x1}B_{y1}-v_{y1}B_{x1})}{\partial X_0}
\end{pmatrix}
$$

$$
-\begin{pmatrix}
0\\
(\mathbf{B}^0\cdot\nabla_0)\dfrac{\partial h_1 B_{x1}}{\partial Y_0}\\
(\mathbf{B}^0\cdot\nabla_0)h_1 B_{y1}\\
(\mathbf{B}^0\cdot\nabla_0)h_1 v_{x1}\\
(\mathbf{B}^0\cdot\nabla_0)h_1 v_{y1}
\end{pmatrix}.
$$

As in the previous case, for eliminating the resonance terms on the right-hand side, we will use the condition of orthogonality of the right-hand side of expression (4.2) to the kernel of operator A_h (2.32). To this end, we denote the eigenvector of operator A_h^* by $\mathbf{z}_h = (z_{h1}, z_{h2}, z_{h3}, z_{h4}, z_{h5})^{\mathrm{T}}$ and determine the eigenvectors of operator A_h^*:

$$
\det\begin{pmatrix}
-i\omega & ik_xH & ik_yH & 0 & 0\\
igk_x & -i\omega & -f_0 & -i(\mathbf{k}\cdot\mathbf{B}^0) & 0\\
-gk_y^2 & ik_yf_0=\beta & \omega k_y & 0 & k_y(\mathbf{k}\cdot\mathbf{B}^0)\\
0 & -i(\mathbf{k}\cdot\mathbf{B}^0) & 0 & -i\omega & 0\\
0 & 0 & -i(\mathbf{k}\cdot\mathbf{B}^0) & 0 & -i\omega
\end{pmatrix}=0. \tag{4.3}
$$

This leads to expression (2.33). Eigenvector $\mathbf{z}_h = (z_{h1}, z_{h2}, z_{h3}, z_{h4}, z_{h5})^{\mathrm{T}}$ satisfies the system $A_h^*\mathbf{z}_h = 0$, which gives

$$-i\omega z_{h1}+ik_xHz_{h2}+ik_yHz_{h3}=0, \tag{4.4}$$

$$igk_xz_1-i\omega z_{h2}-f_0z_{h3}-i(\mathbf{k}\cdot\mathbf{B}^0)z_{h4}=0, \tag{4.5}$$

$$-gk_y^2z_{h1}+(ik_yf_0+\beta H)z_{h2}+\omega k_yz_{h3}+k_y(\mathbf{k}\cdot\mathbf{B}^0)z_{h5}=0, \tag{4.6}$$

$$-i(\mathbf{k}\cdot\mathbf{B}^0)z_{h2}-i\omega z_{h4}=0, \tag{4.7}$$

$$-i(\mathbf{k}\cdot\mathbf{B}^0)z_{h3}-i\omega z_{h5}=0. \tag{4.8}$$

Using this system of equations, we can define the eigenvector of operator A_h^* to within a constant:

$$
\mathbf{z}_h=\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\end{pmatrix}. \tag{4.9}
$$

Let us again represent the solution in form (3.11), where $\mathbf{a}$ is now the eigenvector of operator A_h (i.e., in the form of three magnetic Rossby waves satisfying synchronism condition (2.35)). We can use the right-hand side of system of equations (4.2), which is proportional to $\exp(i\theta_1)$, for obtaining the first equation of three-wave interactions for wave amplitude ϕ. We choose from the right-hand side of system (4.2) the terms proportional to $\exp(i\theta_1)$ and $\exp(i\theta_3)\exp(-i\theta_2)$. As before, multiplying this part by eigenvector $\mathbf{z}$ of the conjugate operator and using compatibility condition $A_h^*\mathbf{z} = 0$, we obtain the following equation for the amplitude of the first of the interacting magnetic Rossby waves:

$$s_{h1}\phi = f_{h1}\psi^*\chi, \tag{4.10}$$

where operator s_{h1} can be expressed as

$$s_{h1} = r_{h1}\frac{\partial}{\partial T_1} + p_{h1}\frac{\partial}{\partial X_1} + q_{h1}\frac{\partial}{\partial Y_1}, \tag{4.11}$$

$$r_{h1} = z_1a_1 + z_2a_2H + ik_{y1}z_3a_3H + z_4a_4H + z_5a_5H, \tag{4.12}$$

$$\begin{gathered} p_{h1} = z_1a_2H \\ + z_2(a_1gH - a_4B_x^0H) - z_4(a_2 + a_3)B_x^0H, \end{gathered} \tag{4.13}$$

$$\begin{gathered} q_{h1} = z_1a_3H + z_2a_4B_y^0H + i\omega(\mathbf{k}_1)z_3a_3H \\ -2ik_{y1}z_3a_5B_y^0H + 2ik_{y1}z_3a_1gH \\ +z_3a_2f_0H - z_4a_2B_y^0H - z_5a_3B_y^0H. \end{gathered} \tag{4.14}$$

In these expressions, eigenvector $\mathbf{a} = \mathbf{a}(\mathbf{k}_1)$. Coefficient f_{h1} in Eq. (4.10) has the form

$$\begin{gathered} f_{h1} = f_{h1}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, f_0, \beta, g, H, \mathbf{B}^0) = 2ik_{x1}z_1a_1a_2 \\ +2ik_{y1}z_1a_1a_3 - 2i\omega(\mathbf{k}_1)z_2a_1a_2 + 2ik_{x1}z_2a_2^2 - 2ik_{x1}z_2a_4^2 \\ +2ik_{y1}z_2a_2a_3 - 2ik_{y1}z_2a_4a_5 + 2ik_{x1}z_2a_1^2g \\ +2(\mathbf{B}_0\cdot\mathbf{k}_1)z_2a_1a_4 - 2f_0z_2a_1a_3 + 2\omega_1k_{y1}z_3a_1a_3 \\ +2k_{x1}k_{y1}z_3(a_2a_3 - a_4a_5)H + 2k_{y1}^2z_3(a_3^2 - a_5^2)H \\ +2k_{y1}^2z_3a_1^2g + (k_{y2}^2 + k_{y3}^2)z_3a_1^2g - 2ik_{y1}(\mathbf{B}^0\cdot\mathbf{k}_1)z_3a_1a_5 \\ +2ik_{y1}z_3a_1a_2f_0 + z_3a_1a_2\beta + 2i\omega(\mathbf{k}_1)z_4a_1a_4 \\ +2ik_{y1}z_4(a_3a_4 - a_2a_5)H - 2(\mathbf{B}^0\cdot\mathbf{k}_1)z_4a_1a_2 \\ +2i\omega(\mathbf{k}_1)z_4a_1a_5 + 2ik_{x1}z_4(a_2a_5 - a_3a_4)H \\ -2(\mathbf{B}^0\cdot\mathbf{k}_1)z_4a_1a_3. \end{gathered} \tag{4.15}$$

In this expression, the product of the form $a_ia_j = [a_i^*(\mathbf{k}_2)a_j(\mathbf{k}_3) + a_i(\mathbf{k}_3)a_j^*(\mathbf{k}_2)]/2$.

Using an analogous procedure for the terms proportional to $\exp(i\theta_2)$, we obtain from the right-hand side of system (4.2) multiplied by eigenvector $\mathbf{z}$ the equation for amplitude β of the second of the interacting magnetic Rossby waves,

$$s_{h2}\psi = f_{h2}\phi^*\chi, \tag{4.16}$$

where

$$s_{h2} = r_{h2}\frac{\partial}{\partial T_1} + p_{h2}\frac{\partial}{\partial X_1} + q_{h2}\frac{\partial}{\partial Y_1}, \tag{4.17}$$

$$r_{h2} = z_1a_1 + z_2a_2H + ik_{y2}z_3a_3H + z_4a_4H + z_5a_5H, \tag{4.18}$$

$$\begin{gathered} p_{h2} = z_1a_2H + z_2(a_1gH - a_4B_x^0H) \\ -z_4(a_2 + a_3)B_x^0H, \end{gathered} \tag{4.19}$$

$$\begin{gathered} q_{h2} = z_1a_3H + z_2a_4B_y^0H + i\omega(\mathbf{k}_1)z_3a_3H \\ -2ik_{y1}z_3a_5B_y^0H + 2ik_{y1}z_3a_1gH \\ +z_3a_2f_0H - z_4a_2B_y^0H - z_5a_3B_y^0H. \end{gathered} \tag{4.20}$$

In these expressions, eigenvector $\mathbf{a} = \mathbf{a}(\mathbf{k}_2)$. The expression for coefficient f_{h2} in Eq. (4.16) has the form

$$\begin{gathered} f_{h2} = f_{h2}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, f_0, \beta, g, H, \mathbf{B}^0) = 2ik_{x2}z_1a_1a_2 \\ +2ik_{y2}z_1a_1a_3 - 2i\omega(\mathbf{k}_2)z_2a_1a_2 + 2ik_{x2}z_2a_2^2 - 2ik_{x2}z_2a_4^2 \\ +2ik_{y2}z_2a_2a_3 - 2ik_{y2}z_2a_4a_5 + 2ik_{x2}z_2a_1^2g \\ +2(\mathbf{B}_0\cdot\mathbf{k}_2)z_2a_1a_4 - 2f_0z_2a_1a_3 + 2\omega_2k_{y2}z_3a_1a_3 \\ +2k_{x2}k_{y2}z_3(a_2a_3 - a_4a_5)H + 2k_{y2}^2z_3(a_3^2 - a_5^2)H \\ +2k_{y2}^2z_3a_1^2g + (k_{y1}^2 + k_{y3}^2)z_3a_1^2g \\ -2ik_{y2}(\mathbf{B}^0\cdot\mathbf{k}_2)z_3a_1a_5 \\ +2ik_{y2}z_3a_1a_2f_0 + z_3a_1a_2\beta + 2i\omega(\mathbf{k}_2)z_4a_1a_4 \\ +2ik_{y2}z_4(a_3a_4 - a_2a_5)H - 2(\mathbf{B}^0\cdot\mathbf{k}_2)z_4a_1a_2 \\ +2i\omega(\mathbf{k}_2)z_4a_1a_5 + 2ik_{x2}z_4(a_2a_5 - a_3a_4)H \\ -2(\mathbf{B}^0\cdot\mathbf{k}_2)z_4a_1a_3. \end{gathered} \tag{4.21}$$

In this expression for f_{h2}, the product of the form $a_ia_j = [a_i^*(\mathbf{k}_1)a_j(\mathbf{k}_3) + a_i(\mathbf{k}_3)a_j^*(\mathbf{k}_2)]/2$.

Finally, writing only the terms proportional to $\exp(i\theta_3)$ on the right-hand side of system (4.2), we obtain the following equation for amplitude χ of the third of the interacting waves:

$$s_{h3}\chi = f_{h3}\phi\psi, \tag{4.22}$$

where

$$s_{h3} = r_{h3}\frac{\partial}{\partial T_1} + p_{h3}\frac{\partial}{\partial X_1} + q_{h3}\frac{\partial}{\partial Y_1}, \tag{4.23}$$

$$\begin{gathered} r_{h3} = z_1a_1 + z_2a_2H \\ +ik_{y3}z_3a_3H + z_4a_4H + z_5a_5H, \end{gathered} \tag{4.24}$$

$$\begin{gathered} p_{h3} = z_1a_2H \\ +z_2(a_1gH - a_4B_x^0H) - z_4(a_2 + a_3)B_x^0H, \end{gathered} \tag{4.25}$$

$$\begin{gathered} q_{h3} = z_1a_3H + z_2a_4B_y^0H + i\omega(\mathbf{k}_1)z_3a_3H \\ -2ik_{y1}z_3a_5B_y^0H + 2ik_{y1}z_3a_1gH + z_3a_2f_0H \\ -z_4a_2B_y^0H - z_5a_3B_y^0H. \end{gathered} \tag{4.26}$$

In these expressions, eigenvector $\mathbf{a} = \mathbf{a}(\mathbf{k}_3)$. The interaction coefficient for Eq. (4.22) has the form

$$\begin{gathered} f_{h3} = f_{h3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, f_0, \beta, g, H, \mathbf{B}^0) = 2ik_{x3}z_1a_1a_2 \\ +2ik_{y3}z_1a_1a_3 - 2i\omega(\mathbf{k}_3)z_2a_1a_2 + 2ik_{x3}z_2a_2^2 - 2ik_{x3}z_2a_4^2 \\ +2ik_{y3}z_2a_2a_3 - 2ik_{y3}z_2a_4a_5 + 2ik_{x3}z_2a_1^2g \\ +2(\mathbf{B}_0\cdot\mathbf{k}_3)z_2a_1a_4 - 2f_0z_2a_1a_3 + 2\omega_1k_{y3}z_3a_1a_3 \end{gathered}$$

$$+2k_{x3}k_{y3}z_3(a_2a_3 - a_4a_5)H + 2k_{y3}^2z_3(a_3^2 - a_5^2)H$$
$$+2k_{y3}^2z_3a_1^2g + (k_{y1}^2 + k_{y2}^2)z_3a_1^2g \quad (4.27)$$
$$-2ik_{y3}(\mathbf{B}^0 \cdot \mathbf{k}_3)z_3a_1a_5$$
$$+2ik_{y3}z_3a_1a_2f_0 + z_3a_1a_2\beta + 2i\omega(\mathbf{k}_3)z_4a_1a_4$$
$$+2ik_{y3}z_4(a_3a_4 - a_2a_5)H - 2(\mathbf{B}^0 \cdot \mathbf{k}_3)z_4a_1a_2$$
$$+2i\omega(\mathbf{k}_3)z_4a_1a_5 + 2ik_{x3}z_4(a_2a_5 - a_3a_4)H$$
$$-2(\mathbf{B}^0 \cdot \mathbf{k}_3)z_4a_1a_3.$$

In the expression for interaction coefficient f_{h3}, the product of the form $a_ia_j = a_i(\mathbf{k}_1)a_j(\mathbf{k}_2) + a_i(\mathbf{k}_1)a_j(\mathbf{k}_2)$.

Thus, we have obtained the system of interacting amplitudes of three Rossby waves in magnetic fluid dynamics in the shallow-water approximation in a horizontal magnetic field. For convenience of further analysis, we write the resultant system in the following form:

$$s_{h1}\phi = f_{h1}\psi^*\chi, \quad (4.28)$$

$$s_{h2}\psi = f_{h2}\phi^*\chi, \quad (4.29)$$

$$s_{h3}\chi = f_{h3}\phi\psi. \quad (4.30)$$

Writing separately the terms proportional to each of the interacting waves from Eq. (4.2), we have obtained system (4.28)–(4.30) of three equations for amplitudes ϕ, ψ, χ of the interacting waves. This system describes the interaction of Rossby waves satisfying synchronism conditions (2.35). Coefficients f_{h1}, f_{h2}, f_{h3} and operators s_{h1}, s_{h2}, s_{h3} in system (4.28)–(4.30) depend only on the initial parameters of the problem and are uniquely determined by the compatibility condition $A_h^*\mathbf{z} = 0$ analogously to the case with the vertical external magnetic field (see Eqs. (4.15), (4.21), (4.27), and (4.11), (4.17), (4.23)). We will use the system (4.28)–(4.30) for qualitative analysis of parametric instabilities of Rossby waves.

Let us consider the case when the amplitude of one of the interacting waves at the initial instant is much larger than the amplitudes of the other two waves ($\phi \gg \psi, \chi$). We can assume that the amplitude of the first wave is constant ($\phi = \phi_0$) and disregard the reciprocal influence of waves with amplitudes ψ, χ on the wave with amplitude ϕ. In this case, the system (4.28)–(4.30) assumes the form

$$s_{h2}\psi = f_{h2}\phi_0^*\chi, \quad (4.31)$$

$$s_{h3}\chi = f_{h3}\phi_0\psi. \quad (4.32)$$

We seek the solution to this linear system of equations in the form

$$\begin{pmatrix}\psi\\ \chi\end{pmatrix} = \begin{pmatrix}\psi'\\ \chi'\end{pmatrix}e^{i\varphi}. \quad (4.33)$$

This gives the instability increment

$$\Gamma = \sqrt{\frac{|f_{h2}f_{h3}|}{|r_{h2}r_{h3}|}}|\phi_0| > 0, \quad (4.34)$$

where f_{h2} and f_{h3} are defined in expressions (4.21) and (4.27). Therefore, one of the magnetic Rossby waves with wavevector $\mathbf{k}_1$ and frequency $\omega_1 = \omega(\mathbf{k}_1)$ splits into two magnetic Rossby waves with wavevectors $\mathbf{k}_2$ and $\mathbf{k}_3$, frequencies ω_2 and ω_3, and with increment (4.34).

The approximation considered above holds as long as amplitude ϕ_0 of the pump wave is much larger than amplitudes ψ, χ of the other two magnetic Rossby waves. However, the amplitudes of the growing waves become comparable with amplitude ϕ_0 at a certain stage of the process. Therefore, Eq. (4.28) must be included into our analysis. For $f_{h1} < 0$, amplitude ϕ of the pump wave, as well as the rates of growth of amplitudes ψ and χ, decreases leading to parametric instability saturation.

In the case of linear damping, system (4.31), (4.32) can be written in the form

$$s_{h2}\psi + \eta_2\psi = f_{h2}\phi_0^*\chi, \quad (4.35)$$

$$s_{h3}\chi + \eta_3\chi = f_{h3}\phi_0\psi, \quad (4.36)$$

where terms $\eta_2\psi$ and $\eta_3\chi$ determine damping. In this case, the exponentially increasing solutions of the form (4.33) exist only when

$$\phi_0 > \sqrt{\eta_2\eta_3|r_{h2}r_{h3}|/|f_{h2}f_{h3}|}.$$

Therefore, there exists a threshold value ϕ_0^{cr} of the pump wave amplitude,

$$\phi_0^{\mathrm{cr}} = \sqrt{\frac{\eta_2\eta_3|r_{h2}r_{h3}|}{|f_{h2}f_{h3}|}}, \quad (4.37)$$

beginning with which instability evolves with increment

$$\Gamma = \sqrt{\frac{|f_{h2}f_{h3}|}{|r_{h2}r_{h3}|}}\phi_0^{\mathrm{cr}}. \quad (4.38)$$

Let us now consider the case when the amplitude of one of the interacting waves is much smaller than the amplitudes of the other two waves, $\phi \ll \psi, \chi$, so that we can treat amplitudes ψ and χ as constant ($\psi = \psi_0$ and $\chi = \chi_0$). The equation for amplitude ϕ then assumes the form

$$s_{h1}\phi = f_{h1}\psi_0^*\chi_0. \quad (4.39)$$

We seek its solution in the form

$$\phi = \phi'\exp(\Gamma T_1). \quad (4.40)$$

Substituting this solution into Eq. (4.39), we obtain the following expression for the gain:

$$\Gamma = \frac{|f_{h1}|}{|r_{h1}|}|\psi_0^*\chi_0| > 0, \quad (4.41)$$

where quantity f_{h1} is defined in (4.15). In the given case of parametric amplification, the two initial magnetic

Rossby waves with wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$ and frequencies $\omega_1 = \omega(\mathbf{k}_1)$ and $\omega_2 = \omega(\mathbf{k}_2)$ amplify the magnetic Rossby wave with wavevector $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$ and frequency $\omega_3 = \omega_1 + \omega_2$ with the gain (4.41).

The approximation considered above holds as long as amplitudes ψ_0 and χ_0 of the pump waves are much larger than amplitude ϕ of the first magnetic Rossby wave. However, the amplitude of the growing wave becomes comparable with amplitudes ψ_0 and χ_0 at a certain stage of the process. Therefore, we must include Eqs. (4.29) and (4.30) in our analysis. Amplitudes ψ_0 and χ_0 of the pump waves, as well as the growth rate of amplitude ϕ, decrease, leading to saturation of the parametric instability.

In the case of damping, expression (4.39) for the wave amplitude can be written in the form

$$s_{h1}\phi + \eta_1\phi = f_{h1}\psi_0^*\chi_0, \quad (4.42)$$

where η_1 is the linear damping coefficient for amplitude ϕ in Eq. (4.42). Therefore, the necessary condition for instability evolution is

$$\eta_1 < |\psi_0^*\chi_0||f_{h1}|/|r_{h1}|.$$

This condition determines the threshold value $(\psi_0^*\chi_0)^{\rm cr}$ for the product of the wave amplitudes, for which the solution increases exponentially,

$$(\psi_0^*\chi_0)^{\rm cr} = \eta_1\frac{|r_{h1}|}{|f_{h1}|}, \quad (4.43)$$

with increment

$$\Gamma = \frac{|f_{h1}|}{|r_{h1}|}(\psi_0^*\chi_0)^{\rm cr}. \quad (4.44)$$

In the particular case of a toroidal magnetic field ($B_x = B_x^0$, $B_y = 0$), the expressions for interaction coefficients f_{h1}, f_{h2}, and f_{h3} and differential operators s_{h1}, s_{h2}, and s_{h3} include factor $k_x v_{Ax}$ instead of $\mathbf{k}\cdot\mathbf{v}_A$.

In the case of a poloidal magnetic field ($B_x = 0$, $B_y = B_y^0$), the expressions for interaction coefficients f_{h1}, f_{h2}, and f_{h3} and differential operators s_{h1}, s_{h2}, and s_{h3} include factor $k_y v_{Ay}$ instead of $\mathbf{k}\cdot\mathbf{v}_A$.

5. CONCLUSIONS

We have considered the shallow-water model for describing large-scale processes in astrophysical plasmas. It is shown that the magnetic field structure in an external vertical magnetic field in the shallow-water magnetohydrodynamic flow differs substantially from the magnetic field structure in zero external field. The required magnetic field nondivergence condition necessitates the introduction of the vertical magnetic field component in the system, which makes the magnetic field structure of Rossby waves essentially three-dimensional. It is well known that in zero external vertical magnetic field, the magnetic field is two-dimensional in the shallow-water approximation. It has been concluded that the Rossby waves in magnetic fluid dynamics in an external vertical field cannot be analyzed using 2D equations of magnetic fluid dynamics. Magnetohydrodynamic equations averaged over depth must be used as initial equations. It is noted that the proposed system of equations is transformed into the traditional system in the particular case of zero vertical magnetic field.

The shallow-water approximation has been used for the development of the weakly nonlinear theory of Rossby waves in an external vertical magnetic field and in the absence of a magnetic field also for stationary states in the presence of a horizontal field (poloidal, toroidal, and their sum). Qualitative analysis of the dispersion curves for the Rossby waves in magnetic fluid dynamics revealed the possibility of three-wave interactions in the weak nonlinearity approximation.

The weakly nonlinear theory of Rossby waves has been developed using the method of multiscale asymptotic expansions, and three-wave equations for slowly varying amplitudes have been derived. Approximate analysis of the resultant systems of equations has revealed that two types of parametric instability can evolve in the system: parametric decay and parametric amplification of Rossby waves. The increments of these instabilities have been obtained.

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