ISSN 1063-7761, Journal of Experimental and Theoretical Physics, 2015, Vol. 120, No. 6, pp. 1055–1063. © Pleiades Publishing, Inc., 2015.

ELECTRONIC PROPERTIES OF SOLID

Attractive Hubbard Model with Disorder and the Generalized Anderson Theorem[1]

E. Z. Kuchinskii[a,*], N. A. Kuleeva[a,], and M. V. Sadovskii[a,b,***]**

[a] *Institute for Electrophysics, Ural Branch, Russian Academy of Sciences, Yekaterinburg, 620016 Russia*

[b] *Institute for Metal Physics, Ural Branch, Russian Academy of Sciences, Yekaterinburg, 620290 Russia*

**e-mail: kuchinsk@iep.uran.ru*

***e-mail: strigina@iep.uran.ru*

****e-mail: sadovski@iep.uran.ru*

Received November 6, 2014

Abstract—Using the generalized DMFT+Σ approach, we study the influence of disorder on single-particle properties of the normal phase and the superconducting transition temperature in the attractive Hubbard model. A wide range of attractive potentials U is studied, from the weak coupling region, where both the instability of the normal phase and superconductivity are well described by the BCS model, to the strong-coupling region, where the superconducting transition is due to Bose–Einstein condensation (BEC) of compact Cooper pairs, formed at temperatures much higher than the superconducting transition temperature. We study two typical models of the conduction band with semi-elliptic and flat densities of states, respectively appropriate for three-dimensional and two-dimensional systems. For the semi-elliptic density of states, the disorder influence on all single-particle properties (e.g., density of states) is universal for an arbitrary strength of electronic correlations and disorder and is due to only the general disorder widening of the conduction band. In the case of a flat density of states, universality is absent in the general case, but still the disorder influence is mainly due to band widening, and the universal behavior is restored for large enough disorder. Using the combination of DMFT+Σ and Nozieres–Schmitt-Rink approximations, we study the disorder influence on the superconducting transition temperature T_c for a range of characteristic values of U and disorder, including the BCS–BEC crossover region and the limit of strong-coupling. Disorder can either suppress T_c (in the weak-coupling region) or significantly increase T_c (in the strong-coupling region). However, in all cases, the generalized Anderson theorem is valid and all changes of the superconducting critical temperature are essentially due to only the general disorder widening of the conduction band.

DOI: 10.1134/S1063776115050143

1. INTRODUCTION

The problem of strong-coupling superconductivity has been studied for a long time, starting with the pioneering papers by Eagles and Leggett [1, 2]. Significant progress here was achieved by Nozieres and Schmitt-Rink [3], who suggested an effective method to study the transition temperature crossover from weak-coupling BCS-like behavior to the Bose–Einstein condensation (BEC) scenario in the strong-coupling region. Recent progress in experimental studies of quantum gases in magnetic and optical dipole traps, as well as in optical lattices, with controllable parameters, such as the density and interaction strength (see reviews [4, 5]), has increased the interest in superconductivity (superfluidity of fermions) with strong pairing interaction, including the region of the BCS–BEC crossover. One of the simplest models allowing the study of the BCS–BEC crossover is the Hubbard model with an attractive on-site interaction. The most successive approach to the solution of the Hubbard model, both in the case of repulsive interaction and for the studies of BCS–BEC crossover in the case of attraction, is the dynamical mean field theory (DMFT) [6–8]. The attractive Hubbard model was studied within the DMFT in a number of recent papers [9–13]. However, up to now there have been only a few studies of the disorder influence on the properties of normal and superconducting phases in this model, especially in the region of the BCS–BEC crossover. Disorder effects in this region were analyzed qualitatively in [14], where it was argued that the Anderson theorem remains valid in the BCS–BEC crossover region in the case of s-wave pairing. A diagrammatic approach to (weak) disorder effects on the superconducting transition temperature and the properties of the normal phase in the crossover region was developed recently in [15].

In recent years, we have developed a generalized DMFT+Σ approach to the Hubbard model [16–19], which is very convenient for the studies of different external interactions with respect to those taken into account in the DMFT, such as pseudogap fluctuations

[1] The article is published in the original.

[16–19], disorder [20, 21], electron–phonon interaction [22], etc. This approach is also well suited to the analysis of two-particle properties, such as optical (dynamic) conductivity [20, 23]. In [13], we used this approximation to calculate single-particle properties of the normal phase and optical conductivity in the attractive Hubbard model. In a recent paper [24], the DMFT+Σ approach was used by us to study the disorder influence on the superconducting transition temperature, which was calculated in the Nozieres–Schmitt-Rink approximation. In that paper, for the semi-elliptic density of states of the "bare" conduction band, which is adequate for three-dimensional systems, we numerically demonstrated the validity of the generalized Anderson theorem according to which all changes in the critical temperature are controlled only by the general widening of the conduction band by disorder.

In this paper, we present an analytic proof of such universal influence of disorder (in the DMFT+Σ approximation) on single-particle characteristics and the superconducting transition temperature for the semi-elliptic density of states and also investigate disorder effects in the case of the "bare" band with a flat density of states, qualitatively appropriate for two-dimensional systems. We show that for the flat band model, the universal dependence of single-particle properties and the superconducting transition temperature is also realized for the case of sufficiently strong disorder.

2. DISORDERED HUBBARD MODEL WITHIN THE DMFT+Σ APPROACH

We consider the disordered nonmagnetic Hubbard model with attractive interaction with the Hamiltonian

$$H = -t\sum_{\langle ij\rangle\sigma} a^{\dagger}_{i\sigma}a_{j\sigma} + \sum_{i\sigma}\epsilon_i n_{i\sigma} - U\sum_i n_{i\uparrow}n_{i\downarrow}, \qquad (1)$$

where $t > 0$ is the transfer integral between nearest neighbors on the lattice, U represents Hubbard-like on site attraction, $a_{i\sigma}(a^{\dagger}_{i\sigma})$ is the annihilation (creation) operator of an electron with spin σ, $n_{i\sigma} = a^{\dagger}_{i\sigma}a_{i\sigma}$ is the particle number operator on a lattice site i, while local on-site energies are assumed to be random variables (independent on the lattice sites). For the standard "impurity" diagram technique to be valid, we take the Gaussian distribution of energy levels ϵ_i:

$$\mathcal{P}(\epsilon_i) = \frac{1}{\sqrt{2\pi}\Delta}\exp\left(-\frac{\epsilon_i^2}{2\Delta^2}\right). \qquad (2)$$

The parameter Δ is a measure of the disorder strength, while the Gaussian random field of random on-site energy levels, which are independent on different sites ("white noise" correlation) induces "impurity" scattering, which is analyzed using the standard formalism of averaged Green's functions [25].

The generalized DMFT+Σ approach [16–19] extends the standard dynamical mean field theory (DMFT) [6–8] taking into account an additional "external" self-energy part $\Sigma_{\mathbf{p}}(\varepsilon)$ (in the general case, momentum dependent), which is due to some additional interaction outside the DMFT, and gives an effective method to calculate both single-particle and two-particle properties [20, 23]. The success of this generalized approach is based on the choice of the single-particle Green's function in the form

$$G(\varepsilon, \mathbf{p}) = \frac{1}{\varepsilon + \mu - \varepsilon(\mathbf{p}) - \Sigma(\varepsilon) - \Sigma_{\mathbf{p}}(\varepsilon)}, \qquad (3)$$

where $\varepsilon(\mathbf{p})$ is the "bare" electron dispersion, while the complete self-energy is assumed to be an additive sum of the local DMFT self-energy and some "external" self-energy $\Sigma_{\mathbf{p}}(\varepsilon)$, due to the neglect of the interference of Hubbard and "external" interactions. This allows the conservation of the standard form of self-consistent equations of the standard DMFT [6–8]. At the same time, at each step of DMFT iterations, we consistently recalculate the "external" self-energy $\Sigma_{\mathbf{p}}(\varepsilon)$ using an appropriate approximate scheme, corresponding to the form of the additional interaction, while the local Green's function is also "dressed" by $\Sigma_{\mathbf{p}}(\varepsilon)$ at each step of the standard DMFT procedure.

For the "external" self-energy entering the DMFT+Σ cycle for the problem of random scattering by disorder, we use the simplest self-consistent Born approximation, neglecting diagrams with crossing "impurity" lines, which gives

$$\Sigma_{\mathbf{p}}(\varepsilon) \longrightarrow \tilde{\Sigma}(\varepsilon) = \Delta^2\sum_{\mathbf{p}} G(\varepsilon, \mathbf{p}), \qquad (4)$$

where $G(\varepsilon, \mathbf{p})$ is the single-electron Green's function (3) and Δ is the amplitude of site disorder.

To solve the effective single-Anderson-impurity problem of DMFT, we use the numerical renormalization group approach (NRG) [26].

In what follows, we consider two models of the "bare" conduction band. The first is the band with a semi-elliptic density of states (per unit cell and single spin projection)

$$N_0(\varepsilon) = \frac{2}{\pi D^2}\sqrt{D^2 - \varepsilon^2}, \qquad (5)$$

where D is the band half-width. This model is appropriate for a three-dimensional system. The second model is the one with the flat density of states, appropriate for the two-dimensional case:

$$N_0(\varepsilon) = \begin{cases} \dfrac{1}{2D} & |\varepsilon| \le D, \\ 0 & |\varepsilon| > D. \end{cases} \qquad (6)$$

In principle, for two-dimensional systems, we should take the presence of the weak (logarithmic) Van Hove singularity in the density of states into account. However, this singularity is already effectively suppressed by rather small disorder, and hence the simple model in Eq. (6) is quite sufficient for our aims.

All calculations in this paper are done for a quarter-filled band (the number of electrons per lattice site is $n = 0.5$).

The superconducting transition temperature in the attractive model was analyzed in a number of papers [9, 10, 12], both from the condition of instability of the normal phase [9] (divergence of the Cooper susceptibility) and from the condition of the superconducting order parameter going to zero [10, 12]. In recent paper [13], we determined the critical temperature from the condition of instability of the normal phase, reflected in the instability of the DMFT iteration procedure. The results obtained in this way in fact coincide with those in [9, 10, 12]. Also, to calculate T_c in [13], we used the approach due to Nozieres and Schmitt-Rink [3], which allows the correct (though approximate) description of T_c in the BCS–BEC crossover region. In a later paper [24], we used the combination of Nozieres and Schmitt-Rink and DMFT+Σ approximations for detailed numerical studies of the disorder dependence of T_c and the number of local pairs in the model with the semi-elliptic density of states.

3. DISORDER INFLUENCE ON SINGLE-PARTICLE PROPERTIES FOR THE SEMI-ELLIPTIC DENSITY OF STATES

In this section, we analytically demonstrate that in the DMFT+Σ approximation, the disorder influence on single-particle properties of the disordered Hubbard model (both attractive or repulsive) with a semi-elliptic "bare" conduction band is completely described by effects of general band widening by disorder scattering.

In the system of self-consistent DMFT+Σ equations [17, 19, 20], information on the "bare" band and disorder scattering enter only at the stage of calculations of the local Green's function

$$G_{ii} = \sum_{\mathbf{p}} G(\varepsilon, \mathbf{p}), \tag{7}$$

where the full Green's function $G(\varepsilon, \mathbf{p})$ is determined by Eq. (3), while the self-energy due to disorder, in the self-consistent Born approximation, is defined by Eq. (4). Then the local Green's function takes the form

$$\begin{aligned} G_{ii} &= \int_{-D}^{D} d\varepsilon' \frac{N_0(\varepsilon')}{\varepsilon + \mu - \varepsilon' - \Sigma(\varepsilon) - \Delta^2 G_{ii}} \\ &= \int_{-D}^{D} d\varepsilon' \frac{N_0(\varepsilon')}{E_t - \varepsilon'}, \end{aligned} \tag{8}$$

where we introduce the notation $E_t = \varepsilon + \mu - \Sigma(\varepsilon) - \Delta^2 G_{ii}$. In the case of semi-elliptic density of states (5), this integral is easily calculated in analytic form, and hence the local Green's function is written as

$$G_{ii} = 2\frac{E_t - \sqrt{E_t^2 - D^2}}{D^2}. \tag{9}$$

It can be easily seen that Eq. (9) represents one of the roots of the quadratic equation

$$G_{ii}^{-1} = E_t - \frac{D^2}{4} G_{ii}, \tag{10}$$

corresponding to the correct limit of $G_{ii} \longrightarrow E_t^{-1}$ for an infinitely narrow band ($D \longrightarrow 0$). Then

$$\begin{aligned} G_{ii}^{-1} &= \varepsilon + \mu - \Sigma(\varepsilon) - \Delta^2 G_{ii} - \frac{D^2}{4} G_{ii} \\ &= \varepsilon + \mu - \Sigma(\varepsilon) - \frac{D_{\text{eff}}^2}{4} G_{ii}, \end{aligned} \tag{11}$$

where we introduce D_{eff} as the effective half-width of the band (in the absence of electronic correlations, i.e., for $U = 0$) widened by disorder scattering:

$$D_{\text{eff}} = D\sqrt{1 + 4\frac{\Delta^2}{D^2}}. \tag{12}$$

Equation (10) was obtained from (8), and hence comparing (11) and (10), we obtain:

$$G_{ii} = \int_{-D_{\text{eff}}}^{D_{\text{eff}}} d\varepsilon' \frac{\tilde{N}_0(\varepsilon')}{\varepsilon + \mu - \varepsilon' - \Sigma(\varepsilon)}, \tag{13}$$

where

$$\tilde{N}_0(\varepsilon) = \frac{2}{\pi D_{\text{eff}}^2}\sqrt{D_{\text{eff}}^2 - \varepsilon^2} \tag{14}$$

represents the density of states in the absence of the interaction U "dressed" by disorder. This density of states remains semi-elliptic in the presence of disorder, and therefore all effects of disorder scattering on single-particle properties of the disordered Hubbard model in the DMFT+Σ approximation reduce to only disorder widening of the conduction band, i.e., to the replacement $D \longrightarrow D_{\text{eff}}$.

4. DISORDER INFLUENCE ON THE SUPERCONDUCTING TRANSITION TEMPERATURE

The superconducting transition temperature T_c is not a single-particle characteristic of the system. The Cooper instability determining T_c is related to the divergence of a two-particle loop in the Cooper channel. In the weak-coupling limit, when superconductivity is due to the appearance of Cooper pairs at T_c, disorder only slightly influences superconductivity with the *s*-wave pairing [27, 28]. The so-called Anderson theorem is valid and changes of T_c are connected only with the relatively small changes of the density of states by disorder. The standard derivation of the Anderson theorem [27, 28] uses the formalism of exact eigenstates of an electron in the random field of impurities. Here, we present another derivation of the Anderson theorem, using the exact Ward identity, which allows us to derive the equation for T_c, which is then used to calculate T_c in the Nozieres–Schmitt-Rink approximation in a disordered system.

In general, the Nozieres–Schmitt-Rink approach [3] assumes that corrections due to strong pairing attraction significantly change the chemical potential of the system, while possible corrections due to this interaction to the Cooper instability condition can be neglected, and we can therefore always use the weak-coupling (ladder) approximation. In that approximation, the Cooper instability condition in the disordered Hubbard model takes the form

$$1 = U\chi_0(q = 0, \omega_m = 0), \tag{15}$$

where

$$\chi_0(q = 0, \omega_m = 0) = T\sum_n \sum_{\mathbf{pp'}} \Phi_{\mathbf{pp'}}(\varepsilon_n) \tag{16}$$

represents the two-particle loop (susceptibility) in the Cooper channel "dressed" only by disorder scattering, and $\Phi_{\mathbf{pp'}}(\varepsilon_n)$ is the averaged two-particle Green's function in the Cooper channel ($\omega_m = 2\pi m T$ and $\varepsilon_n = \pi T(2n + 1)$ are the usual boson and fermion Matsubara frequencies).

To obtain $\sum_{\mathbf{pp'}} \Phi_{\mathbf{pp'}}(\varepsilon_n)$, we use the exact Ward identity, derived by us in [23]:

$$G(\varepsilon_n, \mathbf{p}) - G(-\varepsilon_n, -\mathbf{p}) = -\sum_{\mathbf{p'}} \Phi_{\mathbf{pp'}}(\varepsilon_n) \times (G_0^{-1}(\varepsilon_n, \mathbf{p'}) - G_0^{-1}(-\varepsilon_n, -\mathbf{p'})). \tag{17}$$

Here, $G(\varepsilon_n, \mathbf{p})$ is the impurity-averaged single-particle Green's function (not containing Hubbard interaction corrections!). Using the obvious symmetry $\varepsilon(\mathbf{p}) = \varepsilon(-\mathbf{p})$ and $G(\varepsilon_n, -\mathbf{p}) = G(\varepsilon_n, \mathbf{p})$, we obtain from the Ward identity (17) that

$$\sum_{\mathbf{pp'}} \Phi_{\mathbf{pp'}}(\varepsilon_n) = -\frac{\sum_{\mathbf{p}} G(\varepsilon_n, \mathbf{p}) - \sum_{\mathbf{p}} G(-\varepsilon_n, \mathbf{p})}{2i\varepsilon_n}, \tag{18}$$

and hence for Cooper susceptibility (16) we have

$$\chi_0(q = 0, \omega_m = 0) = -T\sum_n \frac{\sum_{\mathbf{p}} G(\varepsilon_n, \mathbf{p}) - \sum_{\mathbf{p}} G(-\varepsilon_n, \mathbf{p})}{2i\varepsilon_n} = -T\sum_n \frac{\sum_{\mathbf{p}} G(\varepsilon_n, \mathbf{p})}{i\varepsilon_n}. \tag{19}$$

Performing the standard summation over Matsubara frequencies [25], we obtain

$$\chi_0(q = 0, \omega_m = 0) = -\frac{1}{4\pi i}\int_{-\infty}^{\infty} d\varepsilon \times \frac{\sum_{\mathbf{p}} G^R(\varepsilon, \mathbf{p}) - \sum_{\mathbf{p}} G^A(\varepsilon, \mathbf{p})}{\varepsilon} \tanh\frac{\varepsilon}{2T} = \int_{-\infty}^{\infty} d\varepsilon \frac{\tilde{N}(\varepsilon)}{2\varepsilon}\tanh\frac{\varepsilon}{2T}, \tag{20}$$

where $\tilde{N}(\varepsilon)$ is the density of states ($U = 0$) "dressed" by disorder scattering. In Eq. (20), the energy ε is referenced to the chemical potential, and if we reference it to the center of the conduction band, we have to replace $\varepsilon \longrightarrow \varepsilon - \mu$, such that Cooper instability condition (15) leads to the following equation for T_c:

$$1 = \frac{U}{2}\int_{-\infty}^{\infty} d\varepsilon \tilde{N}_0(\varepsilon)\frac{\tanh((\varepsilon - \mu)/2T_c)}{\varepsilon - \mu}, \tag{21}$$

where $\tilde{N}_0(\varepsilon)$ is again the density of states (calculated at $U = 0$) "dressed" by disorder scattering. At the same time, the chemical potential of the system at different values of U and Δ should be determined from DMFT+Σ calculations, i.e., from the standard equation for the number of electrons (band filling) determined by the Green's function in Eq. (3), which allows us to find T_c for the wide range of model parameters, including the BCS–BEC crossover and strong-coupling regions, as well as for different levels of disorder. This reflects the physical meaning of the Nozieres–Schmitt-Rink approximation: in the weak-coupling region, the transition temperature is controlled by Cooper instability equation (21), while in the limit of strong-coupling, it is determined as the BEC temperature controlled by the chemical potential. Thus, the joint solution of Eq. (21) and the equa-

tion for the chemical potential guarantees the correct interpolation for T_c through the BCS–BEC crossover region. This approach gives the results for the critical temperature that are quantitatively close to the exact results obtained by direct numerical DMFT calculations [13], but demands much less numerical effort.

We stress that we have used the exact Ward identity, which also allows using Eq. (21) in the region of strong disorder, when the effects of Anderson localization may become relevant. Equation (21) demonstrates that the critical temperature depends on disorder only through the disorder dependence of the density of states $\tilde{N}(\varepsilon)$, which is the main statement of the Anderson theorem. In the framework of the Nozieres–Schmitt-Rink approach, Eq. (21) is also preserved in the strong-coupling region, when the critical temperature is determined by the BEC condition for compact Cooper pairs. In this case, the chemical potential μ entering Eq. (21) may significantly depend on disorder. However, in the DMFT+Σ approximation, this dependence of the chemical potential (as well as of any other single-particle characteristic) in the model with a semi-elliptic density of states is only due to disorder widening of the conduction band. Thus, in both the BCS–BEC crossover and strong-coupling regions, the generalized Anderson theorem actually remains valid. Accordingly, in the model of a semi-elliptic band, Eq. (21) leads to a universal dependence of T_c on disorder, due to the change $D \longrightarrow D_{\text{eff}}$. Such universality is fully confirmed by numerical calculations of T_c in this model, performed in [24] (cf. also the results presented below).

5. MAIN RESULTS

We now discuss the main results of our numerical calculations, explicitly demonstrating the universal behavior of single-particle properties and the superconducting transition temperature with disorder. We see below that all disorder effects are effectively controlled, in fact, only by the growth of the half-width of conduction band, which for a semi-elliptic density of states is given by Eq. (12). In the case of the band with a flat density of states, the growth of disorder changes the shape of the density of states, making it semi-elliptic in the limit of sufficiently strong disorder, while the effective half-width of the band is given by (cf. Appendix A)

$$\frac{D_{\text{eff}}}{D} = \sqrt{1+\frac{\Delta^2}{D^2}} + \frac{1}{2}\frac{\Delta^2}{D^2}\ln\left(\frac{\sqrt{1+\Delta^2/D^2}+1}{\sqrt{1+\Delta^2/D^2}-1}\right). \quad (22)$$

As an example of the most important single-particle property, we take the density of states. In Fig. 1, we show the evolution of the density of states with disorder in the model of a semi-elliptic band [13]. We can see that the growth of disorder smears the density of states and widens the band. This smearing somehow

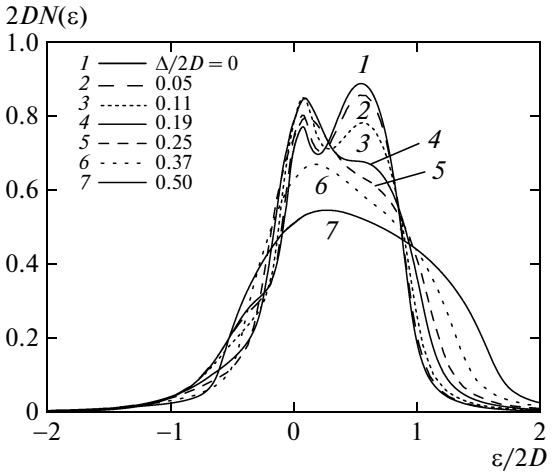

Fig. 1. Dependence of the density of states on disorder in the model with a semi-elliptic band, $|U|/2D = 0.8$, $I/2D = 0.05$.

masks the peculiarities of the density of states due to correlation effects. In particular, both the quasiparticle peak and the lower and upper Hubbard bands, observed in Fig. 1 in the absence of disorder, are completely destroyed in the limit of strong enough disorder. However, we can easily convince ourselves that this evolution is only due to the general widening of the band due to disorder (cf. Eqs. (12) and (22)), because all the data for the density of states belong to the same universal curve replotted in appropriate new variables, with all energies (and temperature) normalized by the effective bandwidth by replacing $D \longrightarrow D_{\text{eff}}$, as shown in Fig. 2a, in complete agreement with the general results obtained above. For the conduction band with a flat density of states, there is no complete universality, as can be seen from Fig. 2b for low enough values of disorder. However, for large enough disorder, the dashed curve shown in Fig. 2b practically coincides with the universal curve for the density of states shown in Fig. 2a. This reflects the simple fact that at large disorder, the flat density of states effectively transforms into a semi-elliptic one (cf. Appendix A).

Going now to the analysis of the superconducting transition temperature, in Fig. 3 we present the dependence of T_c (normalized by the critical temperature in the absence of disorder, $T_{c0} = T_c(\Delta = 0)$) on disorder for different values of the pairing interaction U for both models of the initial "bare" density of states, semi-elliptic (Fig. 3a) and flat (Fig. 3b). Qualitatively, the evolution of T_c with disorder is the same for both models. We can see that in the weak-coupling limit ($U/2D \ll 1$), disorder slightly suppresses T_c (curves *1*). At intermediate couplings ($U/2D \sim 1$), weak disorder increases T_c, while the further increase in disorder suppresses the critical temperature (curves *3*). In the strong-coupling region ($U/2D \gg 1$), the growth of disorder leads to a significant increase in the critical tem-

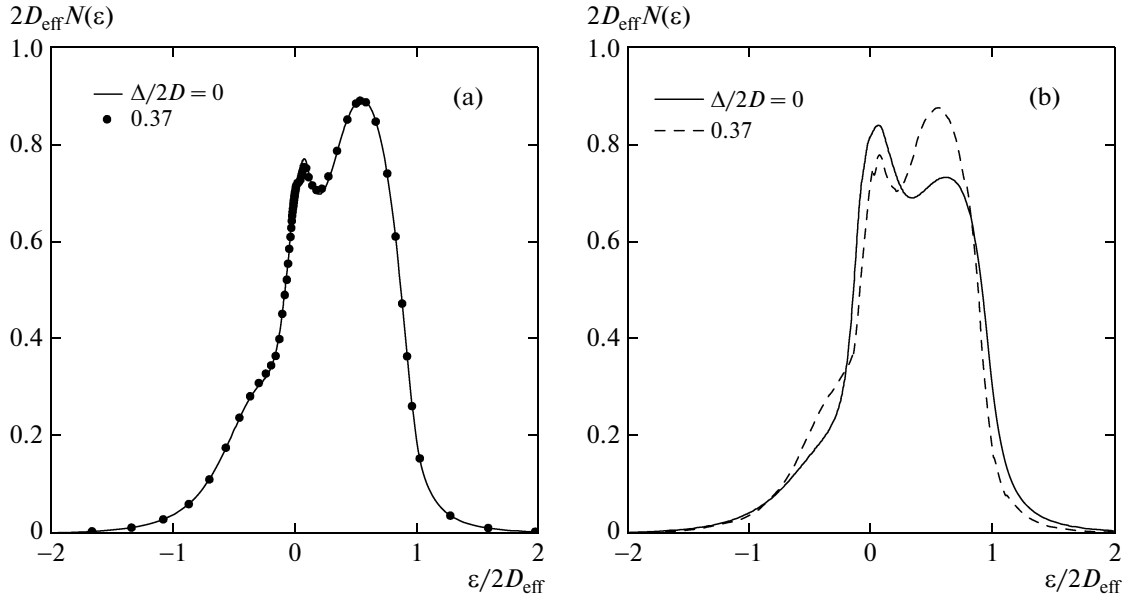

Fig. 2. Universal dependence of the density of states on disorder: (a) the model of a semi-elliptic "bare" density of states; (b) the model of a flat "bare" density of states.

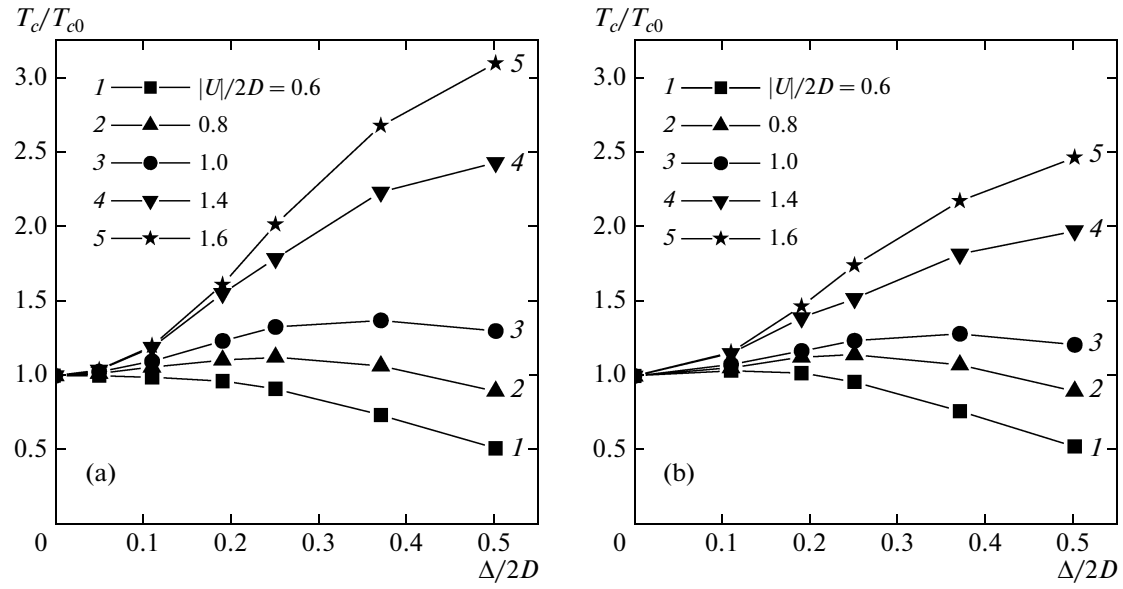

Fig. 3. Dependence of the superconducting transition temperature on disorder for different values of the Hubbard attraction U: (a) semi-elliptic band; (b) flat band.

perature (curves *5*). However, we can easily see that such a complicated dependence of T_c on disorder is completely determined by the disorder widening of the "bare" ($U = 0$) conduction band, demonstrating the validity of the generalized Anderson theorem for all values of U. In Fig. 4, the curve with octagons shows the dependence of the critical temperature $T_c/2D$ on the coupling strength $U/2D$ in the absence of disorder ($\Delta = 0$) for both models of "bare" conduction bands, semi-elliptic (Fig. 4a) and flat (Fig. 4b). We can see that in both models, in the weak-coupling region, the superconducting transition temperature is well described by the BCS model (in Fig. 4a), the dashed curve represents the result of the BCS model, with T_c defined by Eq. (21), with the chemical potential independent of U and determined by the quarter-filling of the "bare" band), while in the strong-coupling region, the critical temperature is determined by the BEC

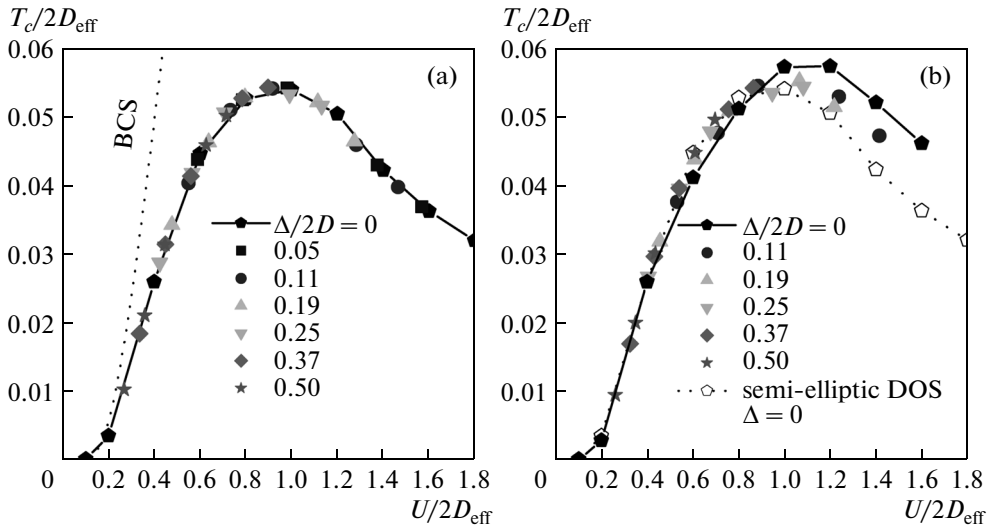

Fig. 4. Universal dependence of the superconducting critical temperature on the Hubbard attraction U for different disorder levels: (a) semi-elliptic band; the dotted curve represents the BCS dependence in the absence of disorder; (b) flat band; the dotted line represents a similar dependence for the semi-elliptic band for $\Delta = 0$.

condition for Cooper pairs and decreases as t^2/U as U increases (inversely proportional to the effective mass of the pair), passing through the maximum at $U/2D_{\text{eff}} \sim 1$. The other symbols in Fig. 4a show the results for T_c obtained by a combination of the DMFT+Σ and Nozieres–Schmitt-Rink approximations for a semi-elliptic "bare" band. We can see that all data (expressed in normalized units of $U/2D_{\text{eff}}$ and $T_c/2D_{\text{eff}}$) ideally fit the universal curve obtained in the absence of disorder. For a flat "bare" band, results of our calculations are shown in Fig. 4b and we do not observe the complete universality: data points, corresponding to different degrees of disorder, somehow deviate from the curve obtained in the absence of disorder. However, with the increase in disorder, the form of the band becomes close to semi-elliptic and our data points move towards the universal curve obtained for the semi-elliptic case and shown by the dashed curve in Fig. 4b, thus confirming the validity of the generalized Anderson theorem.

6. CONCLUSION

In this paper, in the framework of the DMFT+Σ generalization of dynamical mean field theory, we have studied the disorder influence on single-particle properties (e.g., the density of states) and the superconducting transition temperature in the attractive Hubbard model. Calculations were done for a wide range of attractive interactions U, from the weak-coupling region $U/2D_{\text{eff}} \ll 1$, where both instability of the normal phase and superconductivity are well described by the BCS model, to the strong-coupling limit $U/2D_{\text{eff}} \gg 1$, where the superconducting transition is determined by Bose–Einstein condensation of compact Cooper pairs forming at temperatures much higher than the superconducting transition temperature. We have shown analytically that for the conduction band with a semi-elliptic density of states, which is a good approximation in the three-dimensional case, disorder influences all single-particle properties in a universal way: all changes of these properties are only due to the disorder widening of the band. In the model of the conduction band with a flat density of states, which is appropriate for two-dimensional systems, there is no universality in the region of weak disorder. However, the main effects are again due to the general widening of the band and complete universality is restored for high enough disorder, when the density of states effectively becomes semi-elliptic.

To study the superconducting transition temperature, we have used the combination of the DMFT+Σ approach and the Nozieres–Schmitt-Rink approximation. For both models of the conduction band, disordering the density of states may either suppress the critical temperature T_c (in the region of weak coupling) or significantly increase it (in the strong-coupling region). However, in all these cases, we have actually proved the validity of the generalized Anderson theorem, and hence all changes of the transition temperature are in fact controlled only by the effects of general disorder widening of the conduction band. In the case of the initial semi-elliptic band, the disorder influence on T_c is completely universal, while in the case of the initial flat band, such universality is absent at weak disorder, but is completely restored for high enough disorder levels.

Finally, we present some additional comments on the methods and approximations used. Both the DMFT+Σ and Nozieres–Schmitt-Rink approaches represent certain approximate interpolation schemes, strictly valid only in the corresponding limit cases

(e.g., small disorder or small (large) U). However, both schemes demonstrate their effectiveness also in the case of intermediate values of U and intermediate (or even strong) disorder. Actually, the effectiveness of the Nozieres–Schmitt-Rink approximation (neglecting U corrections in the Cooper channel) was verified by comparison with the direct DMFT calculations [13]. The use of DMFT+Σ to analyze the disorder effects in the repulsive Hubbard model was shown to produce reasonable results for the phase diagram, compared to exact numerical simulations of disorder in DMFT, including the region of large disorder (the Anderson localized phase) [19–21]. However, the role of the approximations made in DMFT+Σ, such as the neglect of the interference of disorder scattering and correlation effects, deserves further studies.

ACKNOWLEDGMENTS

This paper is supported by the Russian Science Foundation, grant no. 14-12-00502.

APPENDIX A

For the band with a flat density of states (at $U = 0$ and $\Delta = 0$), disorder leads both to widening of the band and to a change of the form of the density of states. Taking the density of states in the form given by Eqs. (6), we calculate the local Green's function as

$$G_{ii} = \frac{1}{2D}\int_{-D}^{D} d\varepsilon' \frac{1}{\varepsilon - \varepsilon' - \Delta^2 G_{ii}} = \frac{1}{2D}\ln\left(\frac{\varepsilon - \Delta^2 G_{ii} + D}{\varepsilon - \Delta^2 G_{ii} - D}\right), \quad \text{(A1)}$$

where the energy ε is referenced to the middle of the "bare" band. We introduce the auxiliary notation $G_{ii} = R - iI$. At the band edges, $I \longrightarrow 0$, and therefore expanding the r.h.s. of Eq. (A.1) up to linear terms in I, we obtain

$$R - iI \approx \frac{1}{2D}\ln\left(\frac{\varepsilon - \Delta^2 R + D}{\varepsilon - \Delta^2 R - D}\right) - iI\frac{\Delta^2}{(\varepsilon - \Delta^2 R)^2 - D^2}. \quad \text{(A2)}$$

Equating the real parts in (A.2), we obtain

$$R = \frac{1}{2D}\ln\left(\frac{\varepsilon - \Delta^2 R + D}{\varepsilon - \Delta^2 R - D}\right).$$

Similarly, equating the imaginary parts at the band edges, we obtain $\varepsilon - \Delta^2 R = \pm\sqrt{D^2 + \Delta^2}$, and substituting this expression into the logarithm in the preceding expression, we find R and the band edge positions at

$$\varepsilon = \pm\left(\sqrt{D^2 + \Delta^2} + \frac{\Delta^2}{2D}\ln\left(\frac{\sqrt{D^2 + \Delta^2} + D}{\sqrt{D^2 + \Delta^2} - D}\right)\right). \quad \text{(A3)}$$

Thus, the half-width of the band D_{eff} widened by disorder in this model is determined by Eq. (22) used above.

We note that although the Born approximation for disorder scattering that we use is formally valid only for small disorder $\Delta \ll D$, the effects of Anderson localization at large disorder $\Delta \sim D$ do not qualitatively change the density of states [27], and hence the Born approximation gives qualitatively correct results also in the region of large disorder. Actually, this approximation neglects only the appearance of exponentially small "tails" in the density of states, outside the "mean field" band edges [27] and gives more or less correct results inside such a band.

At large enough disorder, almost any "bare" band width $2D$ and an arbitrary density of states $N_0(\varepsilon)$ acquires a semi-elliptic density of states. In the limit of very large disorder $\Delta \gg D$, almost in the whole band, widened by disorder, we have $|\varepsilon - \Delta^2 R| \gg D$ and in the expression for the local Green's function we can neglect the ε'-dependence in the denominator of the integrand:

$$R - iI = G_{ii} = \int_{-\infty}^{\infty} d\varepsilon' \frac{N_0(\varepsilon')}{\varepsilon - \varepsilon' - \Delta^2 G_{ii}} \approx \frac{1}{\varepsilon - \Delta^2 R + i\Delta^2 I}. \quad \text{(A4)}$$

Then we immediately obtain

$$\varepsilon - \Delta^2 R = \frac{\varepsilon}{2}, \quad I = \frac{1}{2\Delta^2}\sqrt{4\Delta^2 - \varepsilon^2} \quad \text{(A5)}$$

whence the density of states "dressed" by disorder

$$N(\varepsilon) = -\frac{1}{\pi}\text{Im}G_{ii} = \frac{I}{\pi} = \frac{2}{\pi(2\Delta)^2}\sqrt{(2\Delta)^2 - \varepsilon^2} \quad \text{(A6)}$$

becomes semi-elliptic, Eq. (5), with the half-width $D_{\text{eff}} = 2\Delta$. Thus, at strong enough disorder, any "bare" band becomes semi-elliptic, restoring the universal dependence of single-particle properties on disorder discussed above. In this sense, the model of the "bare" band with a semi-elliptic density of states is most appropriate for the studies of the effects of strong disorder.

REFERENCES

1. D. M. Eagles, Phys. Rev. **186**, 456 (1969).
2. A. J. Leggett, in: *Modern Trends in the Theory of Condensed Matter*, Ed. by A. Pekalski and J. Przystawa (Springer-Verlag, Berlin, 1980).
3. P. Nozieres and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985).

4. I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
5. L. P. Pitaevskii, Phys.—Usp. **49** (4), 333 (2006).
6. Th. Pruschke, M. Jarrell, and J. K. Freericks, Adv. Phys. **44**, 187 (1995).
7. A. Georges, G. Kotliar, W. Krauth, and M. J. Rozenberg, Rev. Mod. Phys. **68**, 13 (1996).
8. D. Vollhardt, in *Lectures on the Physics of Strongly Correlated Systems XIV*, Ed. by A. Avella and F. Mancini (American Institute of Physics, Melville, New York, 2010), AIP Conf. Proc. **1297**, 339; D. Vollhardt, arXiv:1004.5069.
9. M. Keller, W. Metzner, and U. Schollwock, Phys. Rev. Lett. **86**, 4612 (2001); M. Keller, W. Metzner, and U. Schollwock, arXiv:cond-mat/0101047.
10. A. Toschi, P. Barone, M. Capone, and C. Castellani, New J. Phys. **7**, 7 (2005); A. Toschi, P. Barone, M. Capone, and C. Castellani, arXiv:cond-mat/0411637v1.
11. J. Bauer, A. C. Hewson, and N. Dupis, Phys. Rev. B: Condens. Matter **79**, 214518 (2009); J. Bauer, A. C. Hewson, and N. Dupis, arXiv:0901.1760v2.
12. A. Koga and P. Werner, Phys. Rev. A: At., Mol., Opt. Phys. **84**, 023638 (2011); A. Koga and P. Werner, arXiv:1106.4559v1.
13. N. A. Kuleeva, E. Z. Kuchinskii, and M. V. Sadovskii, J. Exp. Theor. Phys. **119** (2), 264 (2014); N. A. Kuleeva, E. Z. Kuchinskii, and M. V. Sadovskii, arXiv:1401.2295.
14. A. I. Posazhennikova and M. V. Sadovskii, JETP Lett. **65** (3), 270 (1997).
15. F. Palestini and G. C. Strinati, Phys. Rev. B: Condens. Matter **88**, 174504 (2013); F. Palestini and G. C. Strinati, arXiv:1311.2761.
16. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, JETP Lett. **82** (4), 198 (2005); E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, arXiv:cond-mat/0506215.
17. M. V. Sadovskii, I. A. Nekrasov, E. Z. Kuchinskii, Th. Prushke, and V. I. Anisimov, Phys. Rev. B: Condens. Matter **72**, 155105 (2005); M. V. Sadovskii, I. A. Nekrasov, E. Z. Kuchinskii, Th. Prushke, and V. I. Anisimov, arXiv:cond-mat/0508585.
18. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, Low Temp. Phys. **32** (4), 398 (2006); E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, arXiv:cond-mat/0510376.
19. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, Phys.—Usp. **55** (4), 325 (2012); E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, arXiv:1109.2305.
20. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, J. Exp. Theor. Phys. **106** (3), 581 (2008); E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, arXiv:0706.2618.
21. E. Z. Kuchinskii, N. A. Kuleeva, I. A. Nekrasov, and M. V. Sadovskii, J. Exp. Theor. Phys. **110** (2), 325 (2010); E. Z. Kuchinskii, N. A. Kuleeva, I. A. Nekrasov, and M. V. Sadovskii, arXiv:0908.3747.
22. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, Phys. Rev. B: Condens. Matter **80**, 115124 (2009); E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, arXiv:0906.3865.
23. E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, Phys. Rev. B: Condens. Matter **75**, 115102 (2007); E. Z. Kuchinskii, I. A. Nekrasov, and M. V. Sadovskii, arXiv:cond-mat/0609404.
24. E. Z. Kuchinskii, N. A. Kuleeva, and M. V. Sadovskii, JETP Lett. **100** (3), 192 (2014); E. Z. Kuchinskii, N. A. Kuleeva, and M. V. Sadovskii, arXiv:1406.5603.
25. A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskii, *Quantum Field Theoretical Methods in Statistical Physics* (Pergamon, Oxford, 1965); M. V. Sadovskii, *Diagrammatics* (World Scientific, Singapore, 2006).
26. R. Bulla, T. A. Costi, and T. Pruschke, Rev. Mod. Phys. **60**, 395 (2008).
27. M. V. Sadovskii, *Superconductivity and Localization* (World Scientific, Singapore, 2000).
28. P. G. De Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, New York, 1966).