Zahn's Theory of Dynamical Tides and Its Application to Stars

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Abstract—Zahn's theory of dynamical tides is analyzed critically. We compare the results of this theory with our numerical calculations for stars with a convective core and a radiative envelope and with masses of one and a half and two solar masses. We show that for a binary system consisting of stars of one and a half or two solar masses and a point object with a mass equal to the solar mass and with an orbital period of one day under the assumption of a dense spectrum and moderately rapid dissipation, the evolution time scales of the semimajor axis will be shorter than those in Zahn's theory by several orders of magnitude.

INTRODUCTION

Zahn's asymptotic theory of dynamical tides is one of the first theories of dynamical tides developed in the 1970s for stars with a convective core and a radiative envelope (Zahn 1970). This theory has both advantages and disadvantages. One advantage of this theory is that although it is rather cumbersome, its corollaries are very simple. It turns out that the intensity of tidal interactions in this theory is determined by the dependence of the so-called overlap integral Q (or the resonance coefficient defined in Zahn's papers, 1970, and proportional to Q; for their definition see below) on the stellar normal mode frequency, with this dependence being a power law $Q \sim \omega^{17/6}$. According to this theory, the power-law dependence does not evolve with time and remains valid for any stars satisfying some conditions to be discussed below. In reality, as our numerical calculations show, it is quite difficult to obtain this dependence (Chernov et al. 2013), and many of the assumptions that were made in deriving this relation hold only for some very short interval of the stellar lifetime. Nevertheless, owing to its simple result, this theory is used in a large number of papers to explain various observed phenomena (Zahn 1977; Claret and Cunha 1997; Goodman and Dicksun 1998).

For close binary systems those stellar eigenfrequencies that resonantly interact with the external gravitational potential will play a major role. For a point object revolving around a star with an orbital period of the order of a few days, and precisely such orbital periods are believed to be able to lead to significant changes in the orbital parameters of the binary system, the resonant eigenfrequencies will fall within the frequency range $0.1 < \sigma < 1$, where the dimensionless frequency $\sigma = \omega/\sqrt{GM/R^3}$. This frequency range refers to the so-called g-modes (Cowling 1941). It is the g-modes that play a crucial role in the theory of dynamical tides. The g-mode spectrum is assumed to be sufficiently dense. This means that the distance between neighboring frequencies is much smaller than the frequency itself, $\omega_j - \omega_{j+1} \ll \omega_j$. For main-sequence stars this condition may be deemed to be fulfilled. As a result of the interaction of the star with the external potential, its energy and angular momentum are transferred from the orbital motion to the stellar eigenmodes, which are then damped either through viscosity or as a consequence of nonlinear effects.

In Zahn's theory of dynamical tides the boundary between the convective core and the radiative envelope plays a major role. For an orbital period longer than the dynamical time of the star the main interaction between the g-modes and the tidal potential will occur at this boundary (Zahn 1970; Goodman and Dicksun 1998). The resonant gravitational g-modes are assumed to be damped through moderately rapid dissipation in a thin layer beneath the stellar surface (Zahn 1975). Moderately rapid dissipation implies that the characteristic rate of viscous dissipation at the resonant frequency is much greater than the difference between the resonant and neighboring frequencies (Ivanov et al. 2013). This will occur when the mode decay time due to viscosity is much shorter than the travel time of a wave packet with

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frequencies of the order of the resonant one (i.e., of the order of the reciprocal distance between normal mode eigenfrequencies; Ivanov et al. 2013; Goodman and Dicksun 1998).

A huge number of papers devoted to dynamical tides have been published in the last several years (Bolmont and Mathis 2016; Weinberg et al. 2012; Ivanov and Papaloizou 2004, 2007, 2010; Essick and Weinberg 2016; Lanza and Mathis 2016; Ogilvie 2014; Papaloizou and Ivanov 2005, 2010). Tassoul (1980) and Smeyers and Tassoul (1987) developed an asymptotic theory of adiabatic free oscillations for the p- and g-modes. These works generalize the work by Zahn (1970). The rotation effects were considered by Rocca (1987). A theory of dynamical tides for stars with a radiative core and a convective envelope was developed by Ivanov et al. (2013). In contrast to Zahn's theory, in this paper two stellar regions contribute to the overlap integral: the convective envelope and the radiative region near the base of the convective envelope. Using Zahn's formalism, Claret and Cunha (1997) and Kushnir et al. (2016) calculated the orbital evolution time scales as a function of problem parameters.

In this paper the approximations used in the original paper of Zahn (1970) are analyzed critically. The evolution time scales of the semimajor axis are calculated for a binary system containing a point object with a mass equal to the solar mass and a star with masses of one and a half and two solar masses. Below we show that for these stars Zahn's theory provides reasonable agreement with our numerical calculations only for sufficiently long orbital periods. In the regime of moderately rapid dissipation the tidal orbital evolution can be faster than that in Zahn's theory by several orders of magnitude. In addition, we show how the quantities appearing in Zahn's theory and determined for polytropic stars can be calculated in the case of stars with a realistic structure.

DEFINITION OF THE BRUNT−VÄSÄLÄ FREQUENCY

There exist three types of eigenfrequencies for a nonrotating star without a magnetic field: the p-, f-, and g-modes (Christensen-Dalsgaard 1998). Zahn's asymptotic theory of dynamical tides was considered in the low-frequency limit for the g-modes. The properties of the g-modes are determined by the Brunt−Väsälä (BV) frequency. It is specified by the following relation (Christensen-Dalsgaard 1998):

$$N^2 = g \left(\frac{1}{\Gamma p} \frac{\partial p}{\partial r} - \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right), \quad (1)$$

where p is the pressure, ρ is the density, g is the gravitational acceleration, r is the radial coordinate, and

$\Gamma = (\partial \ln p / \partial \ln \rho)_{ad}$ is the adiabatic index. However, this definition of the BV frequency is difficult to apply to many stars. This is related to the numerical errors, to the calculation of the derivative of the density, and to the fact that there exist regions in stars where both terms in parentheses can be of the same order of magnitude. Therefore, a different definition of the BV frequency is used (Brassard et al. 1991):

$$N^2 = \frac{g^2 \rho}{P} \frac{\chi_T}{\chi_\rho} \left[\nabla_{ad} - \nabla - \sum_{i=1}^{N-1} \frac{\chi_{X_i}}{\chi_T} \frac{d \ln X_i}{d \ln p} \right], \quad (2)$$

where

$$\nabla_{ad} = \left(\frac{\partial \ln T}{\partial \ln P} \right)_{ad, X_i}, \quad \chi_T = \left(\frac{\partial \ln p}{\partial \ln T} \right)_{\rho, X_i}, \quad (3)$$

$$\chi_\rho = \left(\frac{\partial \ln p}{\partial \ln \rho} \right)_{T, X_i},$$

$$\chi_{X_i} = \left(\frac{\partial \ln p}{\partial \ln X_i} \right)_{\rho, T, X_{i \neq j}},$$

the temperature gradient

$$\nabla = \frac{\partial \ln T}{\partial \ln P}, \quad (4)$$

X_i is the mass fraction of atoms of species i, and

$$\sum_{i=1}^{N-1} X_i + X_N = 1. \quad (5)$$

It is convenient to divide the BV frequency into two parts (Chernov 2017): the structure term

$$N_{st}^2 = \frac{g^2 \rho}{P} \frac{\chi_T}{\chi_\rho} \left[\nabla_{ad} - \nabla \right] \quad (6)$$

and the composition term

$$N_{com}^2 = -\frac{g^2 \rho}{P} \frac{1}{\chi_\rho} \sum_{i=1}^{N-1} \chi_{X_i} \frac{d \ln X_i}{d \ln p}. \quad (7)$$

The composition term is related to the change in stellar chemical composition due to nuclear reactions, and it is of great importance at the boundary of the convective and radiative regions.

When considering Zahn's theory of dynamical tides (Zahn 1970), we imposed significant constraints on the BV frequency. The BV frequency N^2 was assumed to have a first-order pole on the stellar surface (see Eq. (16a) in Zahn (1970)). This behavior of the BV frequency is typical for polytropic stars. In stars with a realistic structure there is an atmosphere on the stellar surface that significantly distorts the BV frequency N^2. Therefore, the BV frequency on the stellar surface actually has a more complex structure.
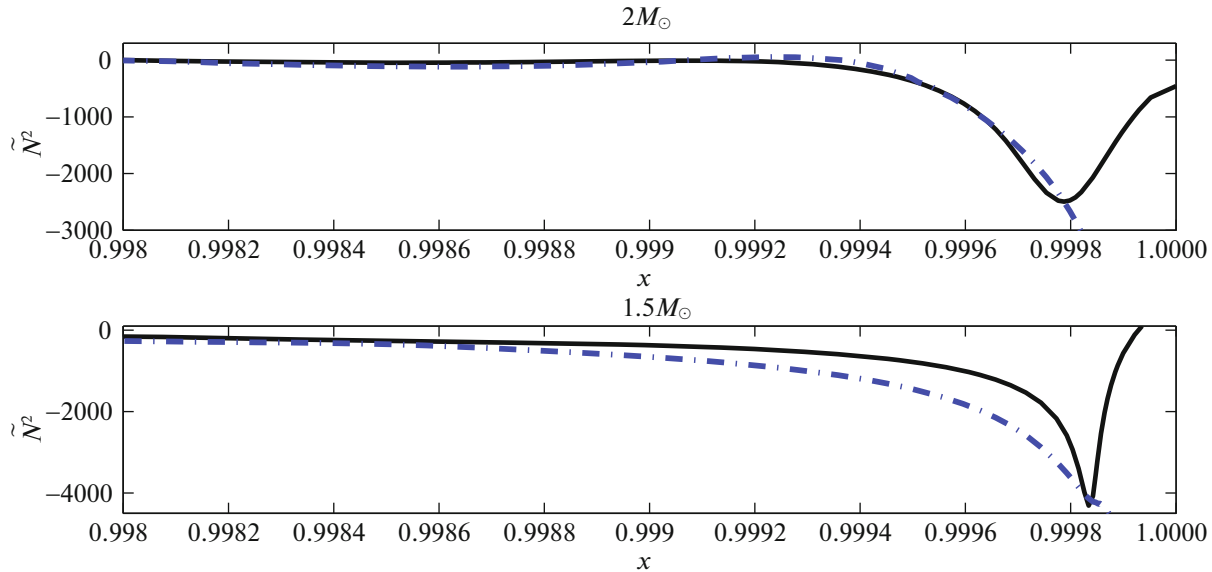
Fig. 1. The BV frequency near the stellar surface.

In Fig. 1 the BV frequency is plotted against the radius near the surface for a star with a mass of one and a half and two solar masses and an age $t = 7.82 \times 10^8$ and 1.24×10^8 yr, respectively (see Chernov 2017). The solid line indicates a realistic curve of the BV frequency for the star of two solar masses on the upper panel and the star of one and a half solar masses on the lower panel. The dash−dotted line indicates the curve calculated using Eq. (16a) from Zahn (1970). The effective polytropic index ν (Zahn 1970), which is defined as

$$\frac{\nu + 1}{\nu} = \lim_{x \to 1} \frac{d \log P}{d \log \rho}, \qquad (8)$$

is $\nu = 2.92$ for the star of two solar masses at $x = 1$, where $x = r/R$ is the dimensionless radial coordinate and R is the stellar radius, and $\nu = 0.95$ for the star of one and a half solar masses at $x = 0.999$. It can be seen from Fig. 1 that in realistic stars the BV frequency near the stellar surface has a sag, and Zahn's assumption describes this sag only partly. This assumption works more poorly as one approaches the stellar surface. It is worth noting that the calculations of the effective polytropic index are also complicated due to the presence of an atmosphere and can lead to significant errors. For example, for the star of one and a half solar masses the parameter ν was chosen at radius $x = 0.999$. This approximation turned out to describe best the BV frequency precisely for this radius. This approximation does not work for the parameter ν at $x = 1$. The BV frequency can also take zero values (near the stellar surface), which was disregarded by Zahn (1970).

The next assumption is that the BV frequency is zero exactly at the boundary of the convective and radiative regions (below the subscript f in a particular quantity A_f will mean that the quantity is taken at this boundary) and is a linear function of radius in the radiative region near the boundary with the convective region. In reality, only the structure term of the BV frequency is zero exactly at the boundary of this region; the composition term of the BV frequency has a spike related to hydrogen burning. Thus, the approximation used by Zahn works only for stars in which the thermonuclear burning is sufficiently weak and the composition term of the BV frequency is approximately zero, $N_{\text{com}}^2 \approx 0$. In Fig. 2 the BV frequency is plotted against the radius near the boundary of the convective and radiative regions. The upper panel is for the star of two solar masses and an age $t = 1.24 \times 10^8$ yr; the lower panel is for the star of one and a half solar masses and an age $t = 7.82 \times 10^8$ yr. The solid curve indicates the BV frequency, while the dash−dotted curve indicates the structure term of the BV frequency. It follows from Fig. 2 that only the structure term becomes zero at the boundary of the convective and radiative regions $N_{\text{st}}^2 \sim (x - x_f)$, while the BV frequency itself at the boundary of the convective and radiative regions is not zero but has a spike.

The approximations used by Zahn in developing the theory of dynamical tides are not quite correct for stars with a realistic structure. This is because the method of solving second−order ordinary differential equations proposed by Olver (1974), which is a mathematically rigorous method, is used in Zahn's theory. The essence of Olver's method is briefly as follows.
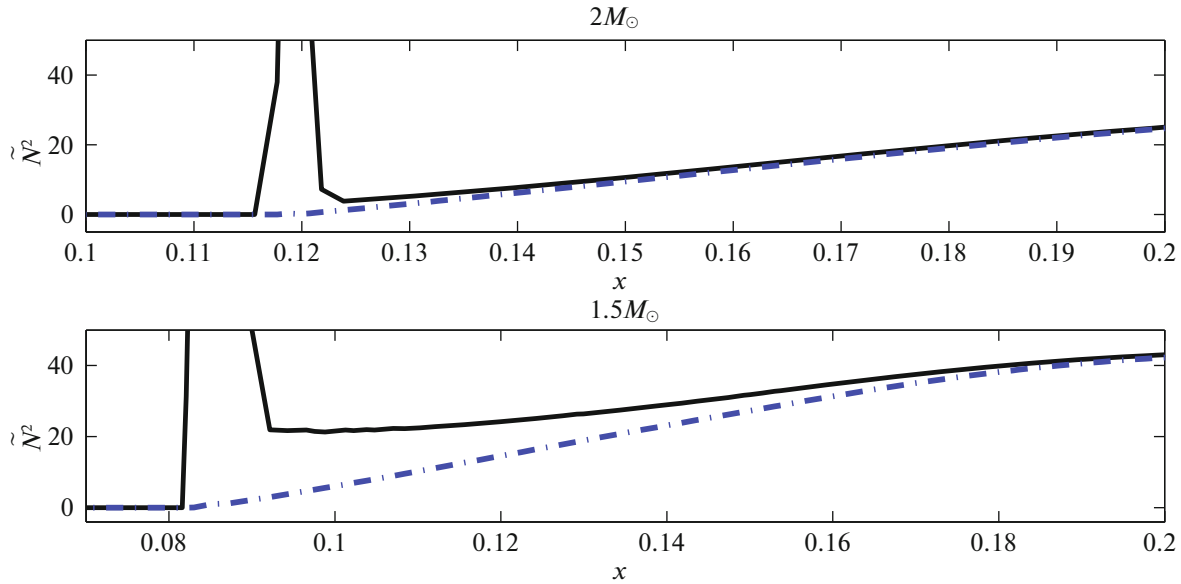
$$2M_\odot$$

$$1.5M_\odot$$

Fig. 2. The BV frequency near the boundary of the convective and radiative regions.

An equation of the form

$$\frac{d^2 w}{dz^2} = \{u p(z) + q(z)\} w \qquad (9)$$

is considered for a large value of the parameter u. Olver (1954) considered three cases: when the function $p(z)$ has no singularities, when the function $p(z)$ has one singularity in the form of a simple zero at some point z_0, and when the function $p(z)$ has one singularity in the form of a second-order pole. In the latter case, the function $q(z)$ can also have a first- or second-order pole at the same point z_0. Olver (1956) considered the fourth case where the function $p(z)$ has one singularity in the form of a first-order pole, while the function $q(z)$ has a second-order pole at the same point. Thus, different forms of solutions are possible, depending on which singularities the BV frequency has. Zahn's above assumptions about the singularities in the BV frequency are important for this method, and not quite correct approximations can affect significantly the solution of our problem.

OVERLAP INTEGRALS

The overlap integrals play an important role in the theory of dynamical tides. These are defined as follows (Press and Teukolsky 1977):

$$Q = \frac{R^2 \int \mathrm{grad}\left[\left(\frac{r}{R}\right)^n Y_n^m\right] \delta\vec{r}_i \, dm}{\sqrt{\int (\delta\vec{r}_i \cdot \delta\vec{r}_i) dm}}, \qquad (10)$$

where $\delta\vec{r}_i$ is the Lagrangian displacement vector and Y_n^m is a spherical harmonic. The overlap integral serves as a measure of the excitation efficiency of

normal modes in a star by tidal forces (Press and Teukolsky 1977) and is of great importance in the theory of dynamical tides. Below we will present our results for the quadrupole case $n = 2$, because it is this part that makes the largest contribution to the overlap integral (Press and Teukolsky 1977). A different quantity, the resonance coefficient, which differs from the overlap integral by the normalization factor, was calculated in Zahn's theory. The overlap integral is easy to express via the resonance coefficient. Using Eqs. (32) and (34) from Zahn (1970), we obtain an expression for the overlap integral in Zahn's theory of dynamical tides:

$$Q = \tilde{Q}\sqrt{M}R, \qquad (11)$$

$$\tilde{Q} = (-1)^i \frac{\sqrt{2\pi} H_n}{3^{1/6}\Gamma\left(\frac{2}{3}\right)} \sqrt{\frac{R^3}{M}\rho_f} \left(\frac{\omega^2 R^3}{GM}\right)^{\frac{17}{12}}$$

$$\times \frac{\left[\left(\frac{-R^3 N^2}{GMx^2}\right)'_{x_f}\right]^{-\frac{1}{6}}}{(n(n+1))^{\frac{11}{12}}\sqrt{\int_{x_f}^{1}\sqrt{-\frac{R^3 N^2}{GMx^2}}dx}},$$

where M is the stellar mass, R is the stellar radius, G is the gravitational constant, and H_n is a constant coefficient determined by the convective stellar core (see Eq. (36) in Zahn (1970)). In contrast to the resonance coefficient, the overlap integral is a quantity with dimensions $\sqrt{M}R$. Thus, a universal frequency dependence of the overlap integral $Q \sim \omega^{17/6}$, where the exponent does not depend on the stellar structure, is obtained in Zahn's theory. The proportionality factor of the overlap integral is determined entirely

Stellar parameters and characteristic quantities from Zahn's theory calculated for our models. The mass and radius are measured in solar masses and radii, respectively; the age is in years

	2b	2c	2d	1.5b	1.5c
Mass	2	2	2	1.5	1.5
Radius	1.63	2.25	2.91	1.46	1.82
Age	2.93×10^7	5.93×10^8	8.44×10^8	5.96×10^7	1.58×10^9
$\sqrt{\int_{x_f}^{1} \sqrt{\dfrac{\tilde{N}^2}{x^2}}dx}$	2.87	3.81	5.07	3.12	3.94
$\dfrac{R^3}{M}\rho_f$	16.27	42.93	117.1	23.15	49.95
$\left(\dfrac{\tilde{N}^2}{x^2}\right)'_{|x_f}$	1.5×10^4	5×10^4	1×10^5	1.8×10^4	1.1×10^5
H_2	8.63×10^{-5}	1.99×10^{-5}	3.73×10^{-6}	2.09×10^{-5}	1.20×10^{-5}

by the stellar density and the BV frequency at the boundary of the convective and radiative regions. In contrast to the resonance coefficient, the overlap integral does not depend on the stellar density and the BV frequency at the stellar surface. The evolution of the overlap integral is related to the change in stellar density and BV frequency. This is responsible for the change of the overlap integral with time.

Difficulties in calculating the contribution from the convective core to the overlap integral also arise in Zahn's theory (Kushnir et al. 2016). The difficulties stem from the fact that the prefactor consists of two terms that depend strongly on stellar parameters. The first and second terms take fairly large and fairly small values, respectively. As a result of their multiplication, the error can be compensated (Kushnir et al. 2016). Kushnir et al. (2016) showed that the prefactor could be calculated quite accurately from a polytrope stellar model, and an alternative form of the angular momentum transfer through the quantities that are determined at the boundary of the convective and radiative regions was derived.

COMPARISON OF ZAHN's THEORY WITH OUR NUMERICAL CALCULATIONS

In this section we will compare the results of Zahn's analytical theory of dynamical tides (11) with our numerical calculations for a realistic stellar structure. We considered stars of one and a half and two solar masses. We chose stellar models similar to the stellar models 2b, 2c, 2d and 1.5b, 1.5c from

Chernov et al. (2013). The table presents the stellar parameters and the quantities that were calculated for each stellar model using Zahn's theory. To calculate the integral in the fourth row of the table, we used the full BV frequency. To calculate the derivative at point x_f, we used only the structure term of the BV frequency, because this term behaves smoothly at this point. All these stars have a convective core and a radiative envelope. The stars were modeled with the MESA software package (Paxton 2011, 2013, 2015); the data obtained were then interpolated by the Steffen (1990) method to two million points. Using these data, we solved the eigenfrequency and eigenfunction problem and calculated the overlap integrals in the adiabatic approximation. The methods of finding the eigenfrequencies and eigenfunctions are described in Christensen-Dalsgaard (1998). The fourth-order Runge–Kutta method was used to solve the differential equations.

The overlap integrals are shown in Figs. 3–7. The solid line indicates our numerical calculations; the dash–dotted line indicates the analytical curve obtained from Zahn's formula (11). There is a close asymptotic correspondence of the theory to the numerical calculations only for model 2d (Fig. 5a). There is a discrepancy by one order of magnitude or more for other models. This discrepancy may be related to the rough approximations that were used in Zahn's theory to describe the BV frequency.

For all values of the overlap integral there is a successive change of sign in the dependence on the number of nodes i of the eigenfunctions, which is
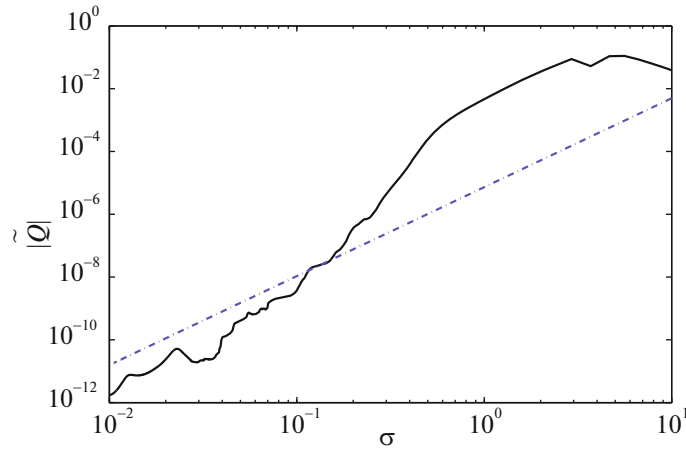
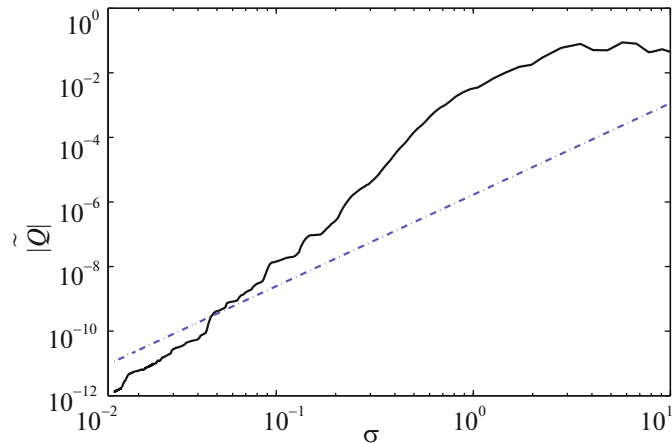**Fig. 3.** Overlap integral versus frequency for model 2b.

Fig. 4. Overlap integral versus frequency for model 2c.

consistent with Eq. (11) (the factor $(-1)^i$), except for some cases. At frequencies $\omega \sim 0.4$ there is a glitch, when the overlap integral does not change its sign for two neighboring frequencies. This may stem from the fact that the two solutions obtained near the stellar surface and at the boundary of the convective and radiative regions are joined in the theory when finding the eigenfrequencies of stellar oscillations in the radiative region. However, since the joining occurs at moderately low frequencies, the next orders of smallness in frequency can contribute to the oscillation phase. As a result, the prefactor of the overlap integral decreases sharply, which is reflected in the sharp oscillations in Figs. 3–6. At lower frequencies the contribution to the oscillation phase will decrease, and the overlap integrals are asymptotically smoothed out.

Some structure in the from of sags can also be seen in Figs. 4–7 for the overlap integral at frequencies $\sigma \sim 4$. This may be related to the interaction of the g- and p-modes (Chernov et al. 2013).

In Figs. 8 and 9 the evolution time scales of the semimajor axis are plotted against the orbital period of a point object around a star of one and a half and two solar masses, respectively (see Eq. (131) from Ivanov et al. (2013)). The stars are assumed to be nonrotating ones; the mass of the point object is equal to the solar mass. The orbit of the point object is assumed to be nearly circular; the eccentricity is approximately zero, $e \to 0$. The solid line indicates our numerical calculations; the dashed line indicates our calculations using Zahn's overlap integral (11). For an orbital period of the order of one day ($P_{\rm orb} \approx 1$) the orbital evolution time derived from
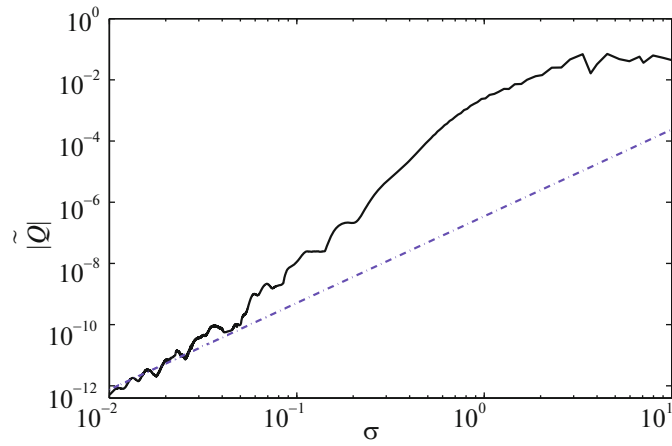
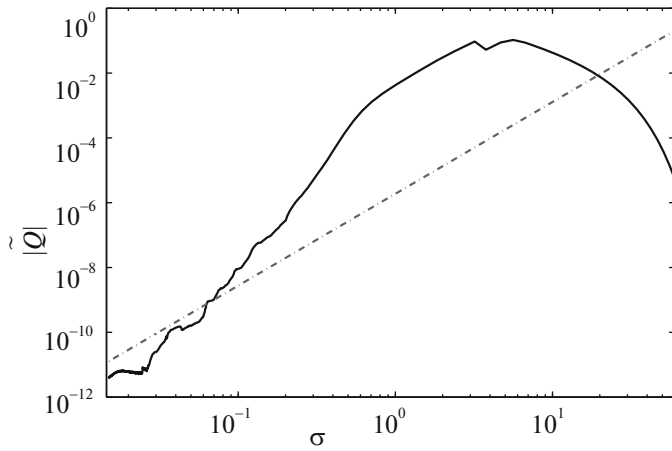**Fig. 5.** Overlap integral versus frequency for model 2d.

Fig. 6. Overlap integral versus frequency for model 1.5b.
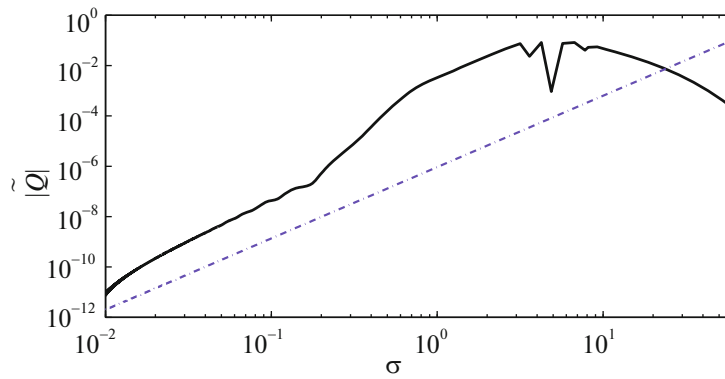
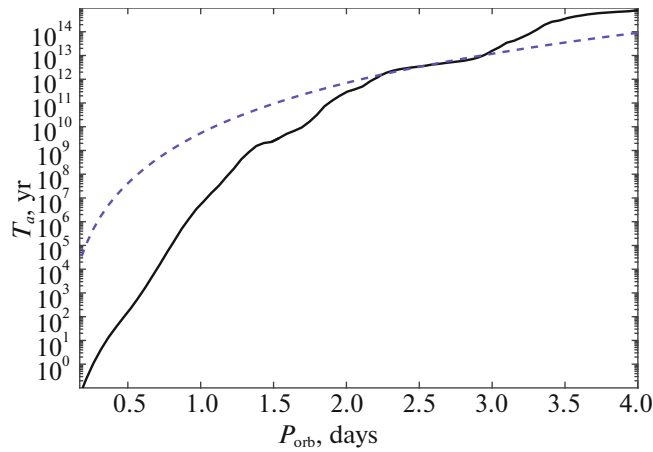**Fig. 7.** Overlap integral versus frequency for model 1.5c.

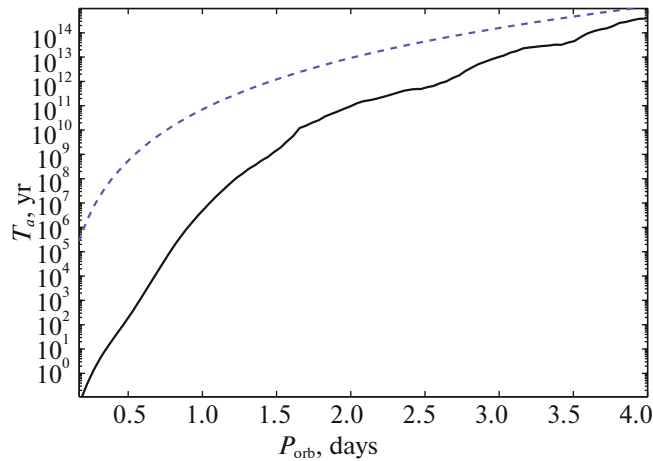**Fig. 8.** Evolution time scale of the semimajor axis for model 2b.

Fig. 9. Evolution time scale of the semimajor axis for model 1.5b.

Zahn's formula is longer than that in our numerical calculations by several orders of magnitude. This is easy to explain, because for an orbital period of one day the characteristic eigenfrequencies lie within the range $\sigma \sim 0.3-0.8$. Zahn's asymptotic theory at such frequencies gives overlap integrals than are smaller than those in our numerical calculations by several orders of magnitude.

CONCLUSIONS

We showed that for the models of stars with a convective core and a radiative envelope with masses of one and a half and two solar masses in the approximation of moderately rapid dissipation and a dense spectrum, the tidal evolution is much faster than follows from Zahn's theory for an orbital period of the order of one day. Such significant discrepancies

probably stem from the fact that in the models for the tidally excited resonant terms under consideration, the asymptotic expression for the overlap integrals derived by Zahn is smaller than the numerical one by several orders of magnitude (see Figs. 3–7).

The fact that our approach gives much shorter tidal evolution time scales of the semimajor axis may turn out to be very important for questions related to the tidal evolution of both binary stars and systems of exoplanets, because in both cases the existing theoretical approaches constructed "from first principles" often give time scales that are too long to explain the observations. Note, however, that since the resonant frequencies and, accordingly, the distances between them are relatively large in our case, the question arises as to whether the approximation of moderately rapid dissipation is valid.

However, despite all its shortcomings, Zahn's theory of dynamical tides is one of the few theories that qualitatively describes the dynamical tides in stars with a convective core and a radiative envelope and explains many observations. Therefore, this theory deserves rapt attention, but it should be applied with caution to specific stars with a realistic structure.

ACKNOWLEDGMENTS

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