ISSN 1063-7737, Astronomy Letters, 2015, Vol. 41, No. 12, pp. 693–703. © Pleiades Publishing, Inc., 2015.
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Energy Density of Standing Sound Waves at the Radiation-Dominated Phase of the Universe Expansion (Hydrodynamic Derivation)

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Received January 29, 2012

Abstract—In the early Universe up to hydrogen recombination in the Universe, the radiation pressure was much greater than the pressure of baryons and electrons. Moreover, the energy density of cosmic microwave background (CMB) photons was greater than or close to the energy density contained in the rest mass of baryonic matter, i.e., the primordial plasma was a radiated-dominated one and the adiabatic index was close to 4/3. The small density perturbations from which the observed galaxies have grown grew as long as the characteristic perturbation scales exceeded the horizon of the Universe ct at that time. On smaller scales, the density perturbations were standing sound waves. Radiative viscosity and heat conduction must have led to the damping of sound waves on very small scales. After the discovery of the cosmic microwave background, J. Silk calculated the scales of this damping, which is now called Silk damping, knowing the CMB temperature and assuming the density of baryons and electrons. Observations with the South Pole Telescope, the Atacama Cosmology Telescope, and the Planck satellite have revealed the predicted damping of acoustic peaks in the CMB power spectrum and confirmed one important prediction of the theory. In 1970, R.A. Sunyaev and Ya.B. Zeldovich showed that such energy release in the early Universe should lead to characteristic deviations of the CMB spectrum from the Planck one. The development of the technology of cryogenic detectors of submillimeter and millimeter wavelength radiation has made it possible to measure the CMB spectral distortions at 10^{-8} of its total intensity (PIXIE). This has sharply increased the interest of theoretical cosmologists in the problem of energy release when small-scale sound waves are damped. We have derived a relativistic formula for the energy of a standing sound wave in a photon—baryon—electron plasma from simple hydrodynamic and thermodynamic relations. This formula is applicable for an arbitrary relation between the energy density of photons and the rest energy density of baryons and their thermal energy density. It continuously describes the transition between the two extreme cases. We obtain the expression for a radiation-dominated plasma in one limit and return to the expression for a gas of classical massive particles in the other limit. We have derived the relations that relate the amplitudes of velocity, baryon number density, and temperature perturbations in a radiation-dominated plasma of photons, baryons, and electrons.

DOI: 10.1134/S106377371512004X

Keywords: *radiation-dominated Universe, Silk damping, standing sound waves, Cosmic Microwave Background Radiation, energy release in early Universe.*

INTRODUCTION

The matter density perturbations at the radiation-dominated expansion phase of the Universe on sub-horizon scales are standing sound waves (Lifshitz 1946). At redshifts $z > 10^3$, not only the radiation pressure $e_r/3$ exceeded the matter pressure $\approx 2NT$ but also the radiation energy density e_r was greater than Nmc^2, the energy density of baryonic matter. Recall that the number density of CMB photons exceeds the mean number density of protons or electrons in the Universe by a factor of 10^9. The main photon—electron interaction mechanism was Thomson scattering. The photon mean free path

$$l = 1/(\sigma_T N_e) = 10^{31}/(1+z)^3 = 10^{13}(10^6/z)^3 \text{ [cm]}$$

was much smaller than the wavelength of sound waves λ on the scales of interest for observational

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cosmology:

$$\lambda = 5 \times 10^{17}(M_\mathrm{b}/(10^7 M_\odot)^{1/3}(10^6/z)\ [\mathrm{cm}],$$

where σ_T is the Thomson cross section, and M_b in the mass of the baryons in the volume λ^3. In this problem, the pressure in the wave and its kinetic and internal energies are completely determined by radiation. The presence of electrons (and the Coulomb attraction of protons and helium nuclei associated with them) allows the photon gas to be treated as a collisional gas where sound waves can exist. The presence of dark matter interacting with the baryon and photon gases only via gravity has virtually no effect whatsoever on the properties of sound waves in the photon gas.

Lifshitz (1946) pointed out that the sound waves in the early Universe should be damped due to radiative viscosity. Lifshitz and Khalatnikov (1963) drew attention to the role of radiative heat conduction. Silk (1968) was the first to calculate the redshift dependence of the damping scale using the CMB temperature measured by that time. Sunyaev and Zeldovich (1970a, 1970b) noted that constraints on the spectrum of primordial density perturbations could be obtained by measuring the CMB spectral distortions resulting from the dissipation of small-scale sound waves and the release of their energy (and primarily the presence of μ-distortions) (see also Hu and White 1996); the μ-distortions are associated with the deviation of the Bose–Einstein spectrum from a blackbody one.

In recent years, the development of the technology of cryogenic submillimetric and millimetric detectors has made it possible to measure the spectral distortions at 10^{-8} of the total CMB energy density (see Fixsen and Mather (2002), and Kogut et al. (2010) for a description of the proposed PIXIE space project). This has increased sharply the interest of theoretical cosmologists in the problem of energy release when small-scale sound waves are damped at redshifts $z \sim 10^5$–2×10^6 (Chluba et al. 2012; Khatri et al. 2012a, 2012b; Pajer and Zaldarriaga 2013). Unfortunately, we cannot obtain information about the earlier times due to the existence of a blackbody photosphere of the Universe at $z \sim 2 \times 10^6$ (Sunyaev and Zeldovich 1970b; Danese and Zotti 1982; Khatri and Sunyaev 2013). Before this time, the production of photons by double Compton scattering and bremsstrahlung allows Comptonization to form a blackbody spectrum and causes all spectral distortions to be nulled.

For the subsequent consideration, it is important that the sound waves in the early Universe were standing ones. This fact was pointed out by Sunyaev and Zeldovich (1970c). It follows directly from the solution of Lifshitz (1946) for the amplitude of the growing mode of density perturbations, which transform into sound waves when crossing the horizon, i.e., when the perturbation wavelength becomes less than the horizon.

Standing waves with different wavelengths arrive with different phases by the time of hydrogen recombination in the Universe. As a result, a characteristic dependence of the amplitude on the angular scale, the so-called acoustic peaks (predicted by Sunyaev and Zeldovich (1970c) and Peebles and Yu (1970) and detected by the instruments on the high-altitude BOOMERanG and Maxima balloons and the WMAP and Planck satellites), is observed in the power spectrum of angular CMB fluctuations. This fact is the most important observational consequence of the theory of evolution of the matter density fluctuations in the Universe. It confirms the existence of standing sound waves in the Universe at the radiation-dominated phase.

A standing sound wave has a very elegant property. At two phases of its period in time, the wave, as it were, is damped—the matter velocities in the wave are zero in the entire space. This means that the entire wave energy at these instants of time is concentrated in the internal energy. This remarkable fact allows the calculation of the energy density of standing sound waves in the photon gas in the early Universe to be simplified. Below, we calculate the energy density using, in particular, this fact. Note that this problem important for cosmology was solved by Chluba et al. (2012) (see also Khatri et al. 2012a, 2012b; Pajer and Zaldarriaga 2013) using the Boltzmann equation. These authors showed that the formula for the energy release during the dissipation of sound waves used by Sunyaev and Zeldovich (1970a), Hu and White (1996), and subsequent authors underestimates the energy release by a factor of 2.25. For a complete picture, we provide a solution to the problem of the energy of sound waves in an ordinary plasma, where $2NT \gg e_r/3$ and $Nmc^2 \gg e_r$. Under these conditions, the solution coincides with the classical result presented, for example, in the book by Landau and Lifshitz (1986). Below, we also provide a general formula describing the energy density in a standing sound wave for an arbitrary relation between the contributions of the radiation, the baryon rest energy, and the thermal energy. We also derive a general formula for the sound speed.

It is well known that the formulas for the growth of density perturbations in the Universe in the case where the perturbation scale is much less than the horizon can be derived by using the Newtonian approximation and by disregarding the general relativity effects. We will work in the same approximation.

FORMULATION OF THE PROBLEM AND RESULTS

For a gas of massive nonrelativistic particles, the local instantaneous energy density (erg cm^{-3}) of a standing wave is

$$\frac{mN_0}{2}u^2 + \frac{mN_0c_s^2}{2}\frac{n^2}{N_0^2}, \quad (1)$$

where m is the mass of the particles, N_0 is the unperturbed particle number density, u is the particle fluid velocity in the standing-wave oscillations, n is the number density perturbation, and c_s is the sound speed. The sound speed in Eq. (1) is taken in the unperturbed state $c_s|_0$; the index zero refers to the unperturbed state. Formula (1) gives a small correction to the energy density of any gas element that moves with a low ($|u| \ll c_s$) velocity u and in which the number density is changed by n compared to the equilibrium number density N_0; $|n| \ll N_0$. The derivation of Eq. (1) for a nonrelativistic medium with an arbitrary equation of thermodynamic state can be found in the textbook of Landau and Lifshitz (1986). In the case of a standing wave, the kinetic energy (the first term in (1)) and internal energy (the second term in (1)) of matter averaged over the spatial and temporal periods $\langle\, \rangle_{xt}$ are equal to each other. Therefore, the total energy of the standing wave E_w is

$$E_w = mN_0\langle u^2\rangle_{xt} = mN_0c_s^2\frac{\langle n^2\rangle_{xt}}{N_0^2}. \quad (2)$$

The previous estimates of the wave energy in a photon gas were based on the hydrodynamic expressions (2). The following substitutions were made: the mass density $mN_0 \to \rho_r = e_r/c^2$, $e_r = aT^4$, and the sound speed $c_s \to c/\sqrt{3}$. In addition, the density perturbation was substituted as follows: $mn \to \delta\rho_r$. As a result, it was found that

$$E_w \simeq \frac{e_r}{c^2}\langle u^2\rangle_{xt} = \frac{1}{3}e_r\frac{\langle n^2\rangle_{xt}}{N_0^2}, \quad (3)$$

$$E_w \simeq \frac{1}{3}e_r\langle\delta^2\rangle_{xt},$$

where

$$\delta \equiv \frac{\delta\rho_r}{\rho_r}.$$

In new papers (Khatri et al. 2012a, 2012b; Chluba et al. 2012), the Boltzmann equation for CMB photons is solved. In particular, the coefficient in Eq. (3) was calculated using this equation:

$$E_w = \frac{3}{4}e_r\langle\delta^2\rangle_{xt}, \quad (4)$$

which differs from the estimate (3) by a factor of 9/4. In this paper, we neglect the primordial abundance of helium by assuming this correction to be small at the radiation-dominated phase of the Universe expansion considered here.

The calculations presented here confirm the coefficient 3/4 in Eq. (4). Our calculations are based on the thermodynamics of a photon–electron–baryon gas and the hydrodynamic equations valid for a small (compared to λ_w) photon mean free path; λ_w is the wavelength of the standing wave. The following universal formula was derived:

$$E_w = \frac{3e_r}{2^3}\frac{2^{11} + 5\times 3\times 2^9\epsilon + 29\times 3^2\times 2^5\epsilon^2 + 17\times 2^3\times 3^3\epsilon^3 + 5\times 3^4\epsilon^4}{(2^2+3\epsilon)^3}\frac{\langle t^2\rangle_{xt}}{T_0^2}, \quad (5)$$

which is applicable for any relation between the contributions of photons and massive particles to the wave energy. Here, $e_r = aT_0^4$, $\epsilon = p_{pl}/e_r$, $p_{pl} = 2N_0T_0$ is the thermal pressure of the fully ionized hydrogen plasma, N is the baryon number density equal to the electron number density (electrical neutrality), T is the temperature of the single-temperature photon–electron–baryon plasma, T_0 is the unperturbed, spatially uniform temperature, and t is the temperature perturbation, $T = T_0 + t$. The universal formula (5) is reduced to Eq. (4) in the radiation-dominated limit, when $\epsilon \to 0$. Indeed, in this limit we have: $E_w \to 12e_r\langle t^2\rangle_{xt}/T_0^2$; $\rho_r = e_r/c^2$, $\delta\rho_r = 4(e_r/c^2)t/T_0$, $t/T_0 = (1/4)\delta$, $\delta = \delta\rho_r/\rho_r$; therefore, $E_w = (3/4)e_r\langle\delta^2\rangle_{xt}$.

THE WAVE EQUATION AND A WAVE

The problem is considered in the approximation of tight coupling between CMB photons and electrons, because in the early Universe the Compton mean free path of a photon is small compared to the acoustic perturbation wavelength. Since ions are coupled with electrons by Coulomb forces, a mixture of photons and plasma forms an elastic medium that oscillates with a single fluid velocity $\vec{u}(\vec{r},\tau)$. We assume that the damping effects are negligible on time scales of the order of several wave periods.

Let, for simplicity, the fully ionized plasma consist of protons and electrons and N be the proton number density (note that the cosmological abundance of helium is easy to take into account). At the evolutionary phase of the Universe under consideration, the number of particles is conserved. The particle number consideration equation in the case of one-dimensional motion is

$$\frac{\partial(\Gamma N)}{\partial t} + \frac{\partial(\Gamma N u)}{\partial x} = 0, \quad (6)$$
$$\Gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}.$$

Equation (6) is written in the laboratory frame of coordinates x and t, in which the nodes and antinodes of the standing wave are at rest. The local instantaneous fluid velocity of particles in the wave in the laboratory frame is denoted by $u(x,t)$. In (6) N is the proton number density in the rest frame, and Γ is the Lorentz factor. In the rest frame, the fluid velocity of protons is zero. In the laboratory frame, the number density is ΓN. The number density is the number of protons per unit volume. The factor Γ that appears when passing from the rest frame to the laboratory frame, $N \to \Gamma N$, is related to the *Lorentz contraction of a unit volume* in the laboratory frame compared to the rest frame.

We consider small-amplitude waves:

$$N = N_0 + n, \quad T = T_0 + t, \quad (7)$$
$$|n| \ll N_0, \quad |t| \ll T_0.$$

The notation (7) is adopted to remove notation like $\delta, \delta n, \Delta, n^{(1)}, \ldots$ from the formulas. In unperturbed equilibrium, the thermodynamic parameters are N_0 and T_0; the velocity is zero.

Let us rewrite Eq. (6) for the number density in a form linear in wave amplitude. Substituting (7) into (6), using the spatial uniformity of the unperturbed number density N_0 and the smallness of the fluid velocity, $u \ll c$, we will obtain the linearized equation

$$n_t + N_0 u_x = 0, \quad n_t \equiv \partial n/\partial t, \quad (8)$$
$$u_x \equiv \partial u/\partial x.$$

The factor Γ in this equation is equal to one in the linear (in u) approximation.

The energy–momentum tensor $T^{\alpha\beta} = T^{\beta\alpha}$ for the photon–baryon–electron plasma in the laboratory frame is

$$\begin{matrix} \Gamma^2 h - p & \Gamma^2 hu/c & 0 & 0 \\ \Gamma^2 hu/c & \Gamma^2 hu^2/c^2 + p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{matrix}$$

$$h = e + p = mc^2 N + 5NT + \frac{4}{3}aT^4, \quad (9)$$

where h is the relativistic enthalpy per unit volume in the rest frame, e and p are the volume energy density and pressure in the rest frame; the indices α and β run 0, 1, 2, 3. The matrix $T^{\alpha\beta}$ in the laboratory frame in reduced form is obtained from the diagonal matrix $T^{\alpha\beta}_{\text{rest}}$ (diagonal e, p, p, p) in the rest frame using the Fourier transformation.

The relativistic momentum conservation law is given by the equation

$$\frac{1}{c}\frac{\partial[\Gamma^2(e+p)u/c]}{\partial t} \quad (10)$$
$$+ \frac{\partial[\Gamma^2(e+p)u^2/c^2 + p]}{\partial x} = 0.$$

This equation corresponds to the second row from top in the matrix $T^{\alpha\beta}$. Equation (10) states: the change in the volume momentum density is equal to the divergence of the momentum flux. In the case of an adiabatic small-amplitude acoustic wave, Eq. (10) after linearization takes the form

$$h_0 u_t + c^2 p_x = 0, \quad (11)$$
$$u_t + c_s^2 (n_x/N_0) = 0, \quad h_0 = e_0 + p_0,$$

where e_0 and p_0 are the unperturbed quantities, and c_s is the sound speed:

$$c_s^2 = c^2 p_N|_s \frac{N_0}{h_0} = \frac{p_N|_s}{\tilde{m}}, \quad (12)$$
$$p_N|_s \equiv \frac{dp}{dN}|_s.$$

Since the wave is adiabatic, the entropy s of the photon–baryon–electron plasma is conserved at compression and rarefaction in the wave. The symbol $|_s$ reminds of this. Let us explain the meaning of the important mass $\tilde{m}$ in Eq. (12) for the sound speed.

For a nonrelativistic gas of massive particles (of one type and *without photons*), the adiabatic sound speed is

$$c_s^2 = \frac{dp}{m dN}|_s, \quad (13)$$

where m is the mass of the massive particle. In our case (provided that the radiation contribution is negligible), this is the proton rest mass; we neglect the electron mass. In the photon–baryon–electron plasma, the following quantity acts as the mass in Eq. (12) for the sound speed:

$$\tilde{m} = (h_0/c^2)N_0^{-1} = m + \frac{5T_0}{c^2} + \frac{4}{3}\frac{aT_0^4}{c^2}N_0^{-1}, \quad (14)$$

where N_0^{-1} is the volume per proton,

$$h_0 = mc^2N_0 + 5N_0T_0 + \frac{4}{3}aT_0^4$$

is the unperturbed (background) relativistic enthalpy per unit volume in the laboratory frame. For an unperturbed background at rest, the laboratory and rest frames coincide. The quantity $\tilde{m}$ in Eq. (14) is the proton mass together with the mass of the photons accounted for by this proton plus the mass related to the enthalpy $5NT$ of the plasma proper. It can be seen that when this mass is introduced, the form of Eqs. (12) and (13) for the sound speed becomes the same in the relativistic case and in the nonrelativistic limit.

The linearized particle (8) and momentum (11) conservation laws can be easily reduced to the wave equation

$$n_{tt} - c_s^2 n_{xx} = 0, \quad u_{tt} - c_s^2 u_{xx} = 0. \quad (15)$$

The harmonic standing wave

$$n = n_m \cos kc_s t \sin kx, \quad (16)$$
$$u = -u_m \sin kc_s t \cos kx$$

is a solution of the wave equation (15). The expression

$$n_m/N_0 = u_m/c_s \quad (17)$$

relates the amplitudes of the standing wave (16). Equation (17) follows from Eqs. (8) or (11) after the substitution of solutions (16) into them.

The origin $x = 0$, $t = 0$ in (16) is chosen in such a way that, first, the center $x = 0$ is half way between the fixed walls $u(x = \pm\pi/2k, t) \equiv 0$ and, second, the phases $t = 0$ and $\pi/2kc_s$ correspond to the instants the motion stops ($u \equiv 0$) and the conversion of the entire wave energy into kinetic energy (the time of the greatest matter acceleration). At $t = \pi/2kc_s$, the profile of the particle number density $N_0 + n$ levels off, $n \equiv 0$, where n is the deviation of the number density from its equilibrium value N_0 (see (7)). Thus, the phases $t = 0$ and $\pi/2kc_s$ refer to the maxima of the potential and kinetic energies (see the figure).

THE SOUND SPEED, A UNIVERSAL FORMULA

Let us calculate the sound speed (12). The fluid velocities of plasma particles in a standing wave are small compared to the speed of light, $u \ll c$. We will restrict ourselves to the case of $T < m_ec^2$, where m_e is the electron mass. Accordingly, the thermodynamic characteristics of the plasma are described by the formulas for a nonrelativistic ideal gas of massive particles. The volume energy density, pressure, and entropy are

$$e = 3NT + aT^4, \quad (18)$$

$$p = 2NT + aT^4/3, \quad (19)$$

$$s = 2\ln\frac{T^{3/2}}{N} + \frac{4}{3}\frac{aT^3}{N}, \quad (20)$$

where m is the proton mass, a is the radiation constant, and $e_r = aT^4$ is the blackbody radiation energy density. Formula (20) for the entropy is written per proton, because the entropy of a variable volume with a fixed number of protons is conserved in an adiabatic flow.

Let us expand the isentropic condition

$$s(N,T) \equiv s_0 = s(N_0, T_0) \quad (21)$$

near the unperturbed solution (7). We will obtain

$$s_N n + s_T t = 0, \quad t = -(s_N/s_T)n, \quad (22)$$

where the derivatives $s_N = \partial s/\partial N|_T$ and $s_T = \partial s/\partial T|_N$ are taken at point N_0, T_0. Condition (21) specifies a curve emerging from point N_0, T_0 on the N, T plane. This is the curve of the isentrope. Equation (22) defines the tangent to this curve at point N_0, T_0.

Let us expand the pressure $p(N,T) = p_0 + p_N n + p_T t$ at point N_0, T_0; here, $p_N = \partial p/\partial N|_T$ and $p_T = \partial p/\partial T|_N$. Let us eliminate the temperature increment t (7) using Eq. (22). Thus, we find

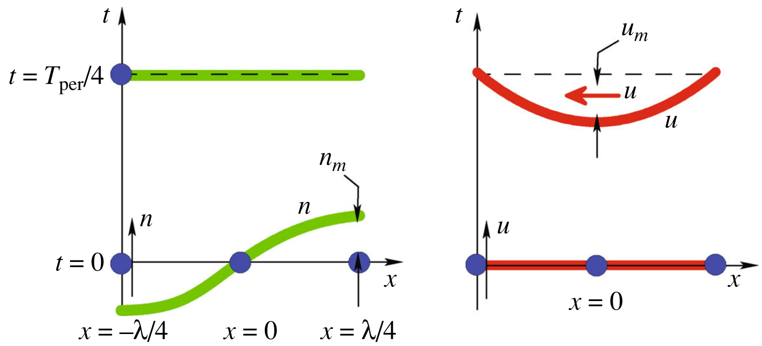
$$p_N|_s = \frac{p(N,T) - p_0}{n} = p_N - p_T\frac{s_N}{s_T}.$$

Calculating the partial derivatives in accordance with the formulas for pressure (19) and entropy (20) and using Eq. (12), we arrive at the formula for the sound speed

$$\frac{c_s^2}{c^2} = \frac{2}{3}\frac{32e_r^2 + 120e_r p_{pl} + 45p_{pl}^2}{(8e_r + 6mc^2N_0 + 15p_{pl})(8e_r + 3p_{pl})}, \quad (23)$$

$$e_r = aT_0^4, \quad p_{pl} = 2N_0T_0,$$

$$\frac{c_s^2}{c^2} = \frac{2}{3}\frac{32 + 120\epsilon + 45\epsilon^2}{(8 + 3\epsilon)(8 + 15\epsilon + 6mc^2N_0/e_r)}, \quad \epsilon = \frac{p_{pl}}{e_r}.$$



Alternation of the phases of stopping $t = 0$ and fastest motion $t = T_{\text{per}}/4$ in the standing wave whose profiles are specified by Eqs. (16); the wave period is $T_{\text{per}} = 2\pi/kc_s = \lambda/c_s$; n_m and u_m are the amplitudes of the proton number density and velocity perturbations. They correspond to the greatest perturbations in the one-dimensional plane wave (16).

Into (23) we substituted the unperturbed relativistic enthalpy

$$h_0 = e_0 + p_0 = mc^2 N_0 + 5N_0T_0 \quad (24)$$
$$+ \frac{4}{3}aT_0^4 = \frac{e_r}{6}\left(8 + 15\epsilon + 6\frac{mc^2N_0}{e_r}\right),$$

which enters into Eqs. (11), (12), and (14). The derived expression takes into account all of the contributions associated with the radiation and the rest energy of baryons together with the thermal contribution of the plasma. This allows both the radiation-dominated and nonrelativistic cases to be described in a unified way.

Hence at $p_{pl} \ll e_r$ and $p_{pl} \ll mc^2N$ we obtain the well-known formula for the radiative sound speed with baryon loading

$$\frac{c_s^2}{c^2} = \frac{1/3}{1 + (3/4)mc^2N/e_r}. \quad (25)$$

In the intermediate region $e_r \sim p_{pl} \ll mc^2N$, from (23) we obtain a formula,

$$c_s^2 = (2/3)(32e_r^2 + 120e_rp_{pl} + 45p_{pl}^2)/[(8e_r + 3p_{pl})6mN],$$

similar to that given in problem 2 to Section 64 in the textbook of Landau and Lifshitz (1986). For a nonrelativistic plasma with a low photon pressure, $e_r \ll p_{pl} \ll mc^2N$, the speed of light c cancels out, and Eq. (23) tends to the formula for the hydrodynamic sound speed

$$c_s^2 \to \frac{10}{3}\frac{T}{m}.$$

It can be seen that Eq. (23) is universal in the sense that it encompasses all limiting cases.

THE POTENTIAL AND TOTAL ENERGIES OF A STANDING WAVE

Whereas we expanded the adiabat $s(N_0 + n, T_0 + t)$ (21) up to terms of the first order of n, t in deviation from the equilibrium point (7) when calculating the sound speed (23), (25), the curvature of adiabat (21) at the equilibrium point N_0, T_0 should be estimated when calculating the standing-wave energy. Indeed, the average of the linear deviation is zero. That is why the curvature needs to be estimated, i.e., the expansion should be brought to terms of the second order inclusive. The expansion of curve (21) at point (7) is

$$s(N,T) - s(N_0,T_0) = s_N n + s_T t \quad (26)$$
$$+ \frac{1}{2}s_{NN}n^2 + s_{NT}nt + \frac{1}{2}s_{TT}t^2 = 0,$$

where the first-order derivatives were determined above, and the second-order derivatives are

$$s_{NN} = \frac{\partial^2 s}{\partial N^2}, \quad s_{NT} = \frac{\partial^2 s}{\partial N \partial T}, \quad s_{TT} = \frac{\partial^2 s}{\partial T^2}.$$

Consider expansion (26) together with the isentropicity condition $s - s_0 = 0$ as an equation for the deviation t. Solving this equation, we find

$$t|_s = -\frac{s_N}{s_T}n + \left(-\frac{s_{NN}}{2s_T}\right. \quad (27)$$
$$\left. + \frac{s_N s_{NT}}{s_T^2} - \frac{s_N^2 s_{TT}}{2s_T^3}\right)n^2.$$

An additional (compared to tangent (22)) term that defines the curvature appears in (27).

If Eq. (26) is solved for n, then we will obtain

$$n|_s = -\frac{s_T}{s_N}t - \frac{s_{NN}s_T^2 - 2s_N s_{NT} s_T + s_N^2 s_{TT}}{2s_N^3}t^2. \quad (28)$$

Let us expand Eq. (18) for the energy density per unit volume up to the second-order perturbations near the equilibrium state:

$$e = e_0 + e_N n + e_T t + \frac{1}{2}e_{NN}n^2 + e_{NT}nt + \frac{1}{2}e_{TT}t^2; \quad (29)$$

here again the subscripts N and T at energy e denote differentiation. Let us related the increments in number density n and temperature t by requiring that the shift on the N, T plane described by them remain on the adiabat. To find this relation between the differentials n and t, we will substitute parabola (27) into expansion (29) of the energy e. As a result, we obtain a formula that expresses the adiabatic energy perturbation in terms of the density perturbation:

$$(e - e_0)|_s = \frac{8e_r + 15p_{pl}}{6}\frac{n}{N_0} + \frac{1}{18}\frac{32e_r^2 + 120e_r p_{pl} + 45p_{pl}^2}{8e_r + 3p_{pl}}\frac{n^2}{N_0^2}. \quad (30)$$

The calculations of the derivatives, the power expansions, and the calculations of the coefficients were performed with the Mathematica program of symbolic transformations. The adopted notation e_r and p_{pl} is presented above in (23).

The second-order correction is of interest, because the first-order correction becomes zero when averaged over the harmonic oscillations (16). Let us express the second-order correction $(e - e_0)_2|_s$ in Eq. (30) in terms of the ratio of the thermal and radiation energies $\epsilon = p_{pl}/e_r$. We have

$$(e - e_0)_2|_s = \frac{32 + 120\epsilon + 45\epsilon^2}{18(8 + 3\epsilon)}\frac{n^2}{N_0^2}e_r, \quad (31)$$

where n and N_0 refer to the proton number density, with n being the correction to the unperturbed number density N_0. The correction n results from the wave perturbation of the homogeneous background. In the limit $\epsilon \ll 1$ (the dominance of photons), from (31) we obtain

$$(e - e_0)_2|_s \to \frac{2}{9}e_r\left(\frac{n}{N_0}\right)^2.$$

The factors in the formulas for the sound speed (23) and for the correction $(e - e_0)_2|_s$ to the energy due to the presence of wave perturbations (30), (31) are similar. Factoring out the sound speed, we bring the formula for the perturbation $(e - e_0)_2|_s$ to the form

$$(e - e_0)_2|_s = \frac{c_s^2}{c^2}\frac{8 + (6mc^2N_0/e_r) + 15\epsilon}{12}e_r\frac{n^2}{N_0^2} = \tilde{m}\frac{c_s^2}{2}\frac{n^2}{N_0}. \quad (32)$$

The effective mass $\tilde{m}$ (14) was substituted for the relativistic enthalpy in (32).

For a nonrelativistic plasma dominated by the thermal pressure of particles, $e_r \ll p_{pl} \ll mc^2 N$, Eqs. (31) and (32) take the form

$$(e - e_0)_2|_s = m\frac{c_s^2}{2}\frac{n^2}{N_0}. \quad (33)$$

It coincides with Eq. (65.1) from the book of Landau and Lifshitz (1986) for the second-order correction $(n/N_0)^2$ to the internal energy of gas compression (rarefaction) at a low degree of compression (rarefaction), $|n|/N_0 \ll 1$. The answer (33) is also given on page 21 in the book of Zel'dovich and Raizer (1966). We will emphasize that Eqs. (32) for the general situation and (33) for the special one are identical in appearance if the mass and the sound speed are used in the formula for the compression energy.

The total energy E_w of the standing wave (16) can be easily found from the above formulas. At the instant the motion stops $t = 0$ (see Eqs. (16) and the figure), the entire energy of the standing wave is converted into potential energy. Substituting the number density distribution (16) at $t = 0$ into Eq. (31) or (32) for n, integrating Eq. (31) over the period $\int_0^\lambda dx$, and dividing the integral by the wavelength $\int_0^\lambda dx/\lambda$, we arrive at the formula for the total energy of the standing wave

$$E_w = \frac{32 + 120\epsilon + 45\epsilon^2}{36(8 + 3\epsilon)}\frac{n_m^2}{N_0^2}e_r = \tilde{m}\frac{c_s^2}{4}\frac{n_m^2}{N_0}. \quad (34)$$

These formulas are derived from (31) or (32) when $n_m^2/2$ is substituted for n^2, where n_m is the amplitude of baryon number density oscillations in the plane wave (16) (see the figure). The coefficient 1/2 is equal to the mean value of the sine squared, $\langle(\sin kx)^2\rangle$. Note that the parameters e_r, N_0, ϵ, and c_s in the above formulas refer to the unperturbed uniform values and do not depend on time and coordinate. In the limit $\epsilon \ll 1$ (the radiation-dominated case), we have

$$E_w \to \frac{1}{9} e_r \frac{n_m^2}{N_0^2}. \quad (35)$$

We will emphasize that Eq. (35) is applicable at $mc^2 N \sim e_r$. The term with the baryon rest energy (baryon loading) remaining in the sound speed (25) drops out of (35) even in the case of $mc^2 N > e_r$.

TEMPERATURE OSCILLATIONS AND THE POTENTIAL ENERGY OF A WAVE

Let us calculate the energy perturbation via the correction t/T_0 to the temperature. For this purpose, we will substitute parabola (28) into expansion (29) of the energy. As a result, we find the first two terms of the expansion of the perturbation of the internal energy (18) in powers of t/T_0 along the curve of constant entropy (20):

$$(e - e_0)|_s = \frac{e_r}{4} \frac{(8 + 3\epsilon)(8 + 15\epsilon)}{4 + 3\epsilon} \frac{t}{T_0} \quad (36)$$
$$+ \frac{3e_r}{16} \frac{2048 + 7680\epsilon + 8352\epsilon^2 + 3672\epsilon^3 + 405\epsilon^4}{(4 + 3\epsilon)^3} \frac{t^2}{T_0^2},$$

where, as before, $e_r = aT_0^4$ and $\epsilon = p_{pl}/e_r$. If the parabola $t|_s = t(n) = an + bn^2$ (27) is substituted into the expansion $(e - e_0)|_s = At + Bt^2$ (36) for the perturbation t, then expansion (36) transforms back to the formula $(e - e_0)|_s = Cn + Dn^2$ (30). We will emphasize that, in this case, the coefficient B transforms to the coefficient D *not according to the rule* $D = Ba^2$! The correct expression is $D = Ba^2 + Ab$. The additional term Ab originates from the linear term At in Eq. (36).

From (36) at $e_r \ll p_{pl} = 2NT \ll mc^2 N$ (a nonrelativistic plasma without radiation) we obtain

$$(e - e_0)|_s = \frac{15p_{pl}}{4} \frac{t}{T_0} + \frac{45p_{pl}}{16} \frac{t^2}{T_0^2}. \quad (37)$$

We arrive at the same expression by expanding the energy of an ideal gas $e|_s = e_0(T/T_0)^{\gamma/(\gamma-1)}$ with a constant adiabatic index γ at $\gamma = 5/3$ on the isentrope $s \equiv s_0$. Indeed,

$$e|_s = e_0 \left(\frac{T_0 + t}{T_0}\right)^{\gamma/(\gamma-1)} = e_0 \quad (38)$$
$$\times \left[1 + \frac{\gamma}{\gamma - 1}\left(\frac{t}{T_0}\right) + \frac{\gamma}{2(\gamma - 1)^2}\left(\frac{t}{T_0}\right)^2 + \ldots\right].$$

At $\gamma = 5/3$ for a fully ionized hydrogen plasma, we have $e_0 = 3N_0T_0$. Substituting γ and e_0 into (38) and taking into account the fact that $e_{pl} = 3p_{pl}/2$, we obtain Eq. (37).

Let us compare the quadratic terms in temperature $Bt^2 = (45/8)(N_0/T_0)t^2$ in (33) and in number density $Dn^2 = (5/3)(T_0/N_0)n^2$ in (33). We will relate the increments n and t by the entropy constancy condition. The following relation holds on the adiabat of a classical gas: $(T_0 + t)/T_0 = [(N_0 + n)/N_0]^{\gamma-1}$. Consequently,

$$\frac{t}{T_0} = \left[(\gamma - 1)\frac{n}{N_0} + \frac{1}{2}(\gamma - 1)(\gamma - 2)\left(\frac{n}{N_0}\right)^2\right]\Bigg|_{\gamma=5/3} = \frac{2}{3}\frac{n}{N_0} - \frac{1}{9}\left(\frac{n}{N_0}\right)^2.$$

The incorrect value of $D' = Ba^2$ mentioned above is equal to $D' = (5/2)(T_0/N_0)$, while the correct value of D consistent with (33) is a factor of 1.5 smaller: $D = Ba^2 + Ab = (5/3)(T_0/N_0)$.

In the case of radiation dominance, $\epsilon \ll 1$ (i.e., $e_r \gg p_{pl}$), Eq. (36) for the second-order correction (at an *arbitrary ratio* between e_r and mc^2N) takes the form

$$(e - e_0)_2|_s = 6e_r(t/T_0)^2. \quad (39)$$

From (36) and (39) we obtain the general and asymptotic ($\epsilon \ll 1$) expressions for the total energy of the standing wave E_w

$$E_w = \frac{3e_r}{32} \quad (40)$$
$$\times \frac{2048 + 7680\epsilon + 8352\epsilon^2 + 3672\epsilon^3 + 405\epsilon^4}{(4 + 3\epsilon)^3} \frac{t_m^2}{T_0^2},$$

$$E_w = 3e_r(t_m/T_0)^2, \quad (41)$$

because $2048 = 2^{11}$. The quantity t_m in Eqs. (40) and (41) is the amplitude of harmonic temperature oscillations in the standing wave (16), $e_r = aT_0^4$, and $\epsilon = p_{pl}/e_r = 2N/aT_0^3$.

THE KINETIC ENERGY OF A WAVE

Consider the kinetic energy of a standing sound wave in the photon–baryon–electron plasma. The energy is given by the first component

$$T^{00} = \Gamma^2 h - p \quad (42)$$

of the energy–momentum tensor $T^{\alpha\beta}$; here, $\Gamma^2 = (1 - u^2/c^2)^{-1}$, $h = e + p$. The wave energy at a space-time point x, t in the laboratory frame is

$$T^{00}(n, u) - T^{00}(0, 0),$$

where n and u are the perturbations of the proton number density and proton fluid velocity at this point.

Consider the instant of time $t_4 = (1/4)\lambda/c_s$ (see the figure) when the profile of the standing wave (16) is homogeneous in number density, $n(x, t_4) \equiv 0$. At instant t_4 the entire energy of the wave is concentrated in the kinetic energy. In this case, the kinetic energy at a point with coordinates x and t in the laboratory frame is

$$T^{00}(0, u) - T^{00}(0, 0). \tag{43}$$

The quantities h and p in the energy (42) are taken in the rest frame. They do not depend on the velocity u and change only when the number density $N_0 + n$ changes. Since $n(x, t_4) \equiv 0$ at instant t_4, the values of h and p at this instant are equal to the unperturbed ones:

$$h|_{t=t4} = h_0, \quad p|_{t=t4} = p_0. \tag{44}$$

In view of condition (44), the difference (43) giving the kinetic energy at instant t_4 takes the form

$$(\Gamma^2 - 1)h_0 = \Gamma^2 h_0 - h_0 = \Gamma(\Gamma h_0) - h_0, \tag{45}$$

because $T^{00}(0, u) = \Gamma^2 h_0 - p_0$ and $T^{00}(0, 0) = h_0 - p_0$.

The wave amplitude is small; accordingly, it is appropriate to pass to the nonrelativistic limit. The quantities e, p, and h have the dimensions of energy per unit volume. When passing from the rest frame to the laboratory frame, the Lorentz dilation of a unit volume takes place. Therefore, the energy, pressure, and enthalpy in the rest frame are e, p, and h, while these quantities in the laboratory frame are $e_{\text{lab}} = \Gamma e, p_{\text{lab}} = \Gamma p$, and $h_{\text{lab}} = \Gamma h$. In the nonrelativistic limit, instead of two frames, the laboratory and rest ones, we have one frame, the laboratory one. Accordingly, the enthalpy in the laboratory frame at instant t_4 is Γh_0.

Let us expand (45) in small u^2/c^2. We have

$$h_{\text{lab}}|_{t4} = \Gamma h_0 \approx h_0, \tag{46}$$

$$(\Gamma^2 - 1)h_0 = \Gamma(\Gamma h_0) - h_0 \tag{47}$$
$$\approx \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right) h_0 - h_0 = \frac{h_0/c^2}{2} u^2.$$

Hence it follows that the quantity h_0/c^2 acts as the mass density for a small-amplitude standing wave in the photon–baryon–electron plasma. Accordingly, the kinetic energy takes the form

$$\frac{\tilde{m} N_0}{2} u^2, \tag{48}$$

where the effective mass $\tilde{m}$ was defined above by Eq. (14). We see that this formula for the mass enters into the sound speed (12), the internal energy (32), and the kinetic energy (48) and everywhere makes the formulas for the photon–baryon–electron plasma and a gas of classical massive particles similar in appearance.

COMPARISON OF THE KINETIC AND POTENTIAL ENERGIES, THE TOTAL WAVE ENERGY

Let us compare the kinetic and potential (internal) energies. The total energy of the wave E_w is equal to its total kinetic energy at instant t_4. Integrating (according to the rule $E_w = \int_0^\lambda dx/\lambda$) Eq. (48) for the local kinetic energy at point x, t_4, we will obtain

$$E_w = \frac{\tilde{m} N_0}{4} u_m^2, \tag{49}$$

where u_m is the amplitude of the wave (16); the harmonic wave form (16) is used in this integration.

Let us compare (49) with the total wave energy at the instant the motion stops $t = 0$. At this instant, the entire wave energy is stored in the total internal energy (34) of the photon–baryon–electron plasma. Let us substitute the amplitude of the change in proton number density n_m for the velocity amplitude u_m in Eq. (49). This substitution is made according to Eq. (17), which relates the amplitudes in the standing wave: $u_m = (n_m/N_0)c_s$. It can be seen that after substitution (17) Eq. (49) transforms to Eq. (34). Thus, the total kinetic energy of the standing wave at instant t_4 is equal to the total potential energy at instant $t = 0$.

At an arbitrary instant of time, the total energy E_w is the sum of the potential and kinetic contributions. The first of these contributions is

$$E_{\text{pot}} = (1/4)(\tilde{m} c_s^2)(n_m/N_0)^2 \cos^2(k c_s t)$$
$$= (1/4)\tilde{m} u_m^2 \cos^2(k c_s t).$$

This expression is obtained by the integration $\int_0^\lambda dx/\lambda$ of Eq. (32), into which Eq. (16) for the field of number densities n in the standing wave was substituted for the local instantaneous value of $n(x, t)$. The second contribution (kinetic energy) is

$$E_{\text{kin}} = (1/4)\tilde{m} u_m^2 \sin^2(k c_s t).$$

The sum of the first and second contributions is equal to E_w (the energy conservation law).

In Eqs. (34) and (49) for the total energy E_w, we may write the means $\langle n^2 \rangle_{x,t}$ and $\langle u^2 \rangle_{x,t}$ instead of the amplitudes squared n_m^2 and u_m^2. The averaging should be performed in a rectangle (see the figure) over the spatial, λ, and temporal, λ/c_s, wave periods.

The subscript x, t in $\langle\rangle_{x,t}$ denote a spatiotemporal averaging. According to this definition, we have

$$\langle n^2\rangle_{x,t} = \frac{c_s}{\lambda^2}\int\limits_0^\lambda dx \tag{50}$$

$$\times \int\limits_0^{\lambda/cs} [n(x,t)]^2 dt = \frac{n_m^2}{4}.$$

Similarly, $\langle u^2\rangle_{x,t} = u_m^2/4$. When integrating (50), we used Eq. (16) for a plane standing wave. Thus, the mean square is equal to a quarter of the amplitude squared. One factor (1/2) is formed when averaging $\cos^2(kc_s t)$ over the time; the other factor is formed when averaging $\sin^2(kx)$ over the coordinate x. The total energy of the standing wave

$$E_w = \frac{\tilde{m}c_s^2\langle n^2\rangle_{x,t}}{2N_0} + \frac{\tilde{m}N_0\langle u^2\rangle_{x,t}}{2} \tag{51}$$

$$= \tilde{m}N_0\langle u^2\rangle_{x,t} = \frac{\tilde{m}c_s^2\langle n^2\rangle_{x,t}}{N_0}$$

is the sum of the averaged energies (32) and (48). The terms in this sum are *equal* to each other. Whereas the local instantaneous potential energy is equal to the local instantaneous kinetic energy in the case of a traveling wave (see below), this equality of the potential and kinetic energies holds only for the means $\langle\ \rangle_{x,t}$ in the case of a standing wave.

THE TOTAL ENERGY AND TEMPERATURE OSCILLATIONS

Formula (51) expresses the total energy E_w in terms of the corrections to the proton number density n and the velocity u. Consider the coefficient with which the energy E_w is expressed in terms of the correction t to the temperature. We will neglect the thermal contribution of the plasma $\sim N_0 T_0$. The energy E_w is then given by Eq. (41) at an *arbitrary ratio* between the rest energy of massive particles $mc^2 N_0$ and the radiation energy $e_r = aT_0^4$. Let us substitute the mean $\langle t^2\rangle_{x,t}$ for the amplitude squared t_m^2 in (41) just as was done in Eqs. (51).

The variability of the correction to the temperature $t(x,t)$ obeys Eq. (16) for a standing wave, because the variability of $t(x,t)$ is proportional to the variability of the number density $n(x,t)$. Consequently, the following relations hold:

$$t = t_m \cos kc_s t \sin kx, \tag{52}$$

$$t_m^2 = 4\langle t^2\rangle_{x,t}. \tag{53}$$

Let us write

$$\rho_r = \frac{aT^4}{c^2}, \quad (\delta\rho_r) = 4\frac{aT_0^3}{c^2}t, \tag{54}$$

$$\frac{\langle(\delta\rho_r)^2\rangle_{x,t}}{(\rho_r|_0)^2} = 16\frac{\langle t^2\rangle_{x,t}}{T_0^2},$$

where $\rho_r|_0 = aT_0^4/c^2$, the subscript r denotes radiation. Substituting Eqs. (53) and (54) into (41) and transforming, we will obtain

$$E_w = \frac{3}{4}e_r\langle\delta^2\rangle_{x,t}, \quad \delta = \frac{(\delta\rho_r)}{\rho_r|_0}. \tag{55}$$

Formula (55) remains valid at an arbitrary ratio mc^2N_0/e_r.

The answer (55) is easy to obtain. Let the radiation energy e_r dominate the rest energy mc^2N and the thermal energy $3NT$, i.e., the massive particles are insignificant dynamically and thermally. The electron–baryon component is involved only in the Compton restriction of the photon mean free path $l_{\rm ph}$: $l_{\rm ph} \ll \lambda$, where λ is the acoustic perturbation wavelength; as a result, the problem becomes hydrodynamic. In such a situation, the energy of the photon–electron–baryon system with perturbation (7) is

$$aT^4 = a(T_0 + t)^4 \tag{56}$$

$$= e_r\left(1 + 4\frac{t}{T_0} + 6\frac{t^2}{T_0^2}\right), \quad e_r = aT_0^4.$$

Compare (56) with (36) at $\epsilon = 0$. In this case, the perturbation in (56) may be even not isentropic.

Substituting the wave field of the standing wave (52) into (56), we return to the formula $E_w/e_r = 3(t_m/T_0)^2$ (41). Recall that this requires the integration $\int_0^\lambda dx/\lambda$ of the correction to the energy (56) at the instant the motion stops $t = 0$. Finally, from the formula $E_w/e_r = 3(t_m/T_0)^2$ we easily arrive at Eq. (55) using the simple substitutions (53) and (54). Formula (55) was derived in Chluba et al. (2012) in this way. See also the description of previous attempts to calculate the numerical coefficient in Eq. (55) in the above paper, which, for example, led to a factor of 9/4 smaller coefficient.

It is important to note that Chluba et al. (2012) showed that 2/3 of the energy released through the dissipation of standing sound waves in the early Universe goes into increasing the mean CMB blackbody temperature and only 1/3 of this energy goes into the distortions of the CMB spectrum.

The approach applied here allows one, first, to establish the role of baryon loading (the ratio mc^2N/e_r) and, second, to describe the continuous transitions in sound speed and wave energy from the radiation-dominated case to a classical plasma of massive particles. This transition occurs when varying the ratio $\epsilon = p_{pl}/e_r$ from zero (the photons dominate) to infinity (the massive particles dominate).

ACKNOWLEDGMENTS

We are grateful to Jens Chluba and Rishi Khatri for numerous fruitful discussions. R.A. Sunyaev Thanks the Russian Science Foundation for partial support of this work (RSCF project no. 14-22-00271).

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Translated by V. Astakhov