

On the Influence of Angular Momentum and Dynamical Friction on Structure Formation. II. Turn-Around and Structure Mass

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Abstract—In this paper, we use the solution of the collapse model presented in a previous paper by Del Popolo to obtain more precise estimates on the turn-around radius (TAR) (namely the radius at which the expansion is divided into the region of recollapse from the material continuing its motion following the Hubble flow), R_0 (note that in this paper TAR or R_0 represent the turn-around radius, as specified in the previous footnote), and structure mass, M , of the Local Group, M81, NGC 253, IC 342, Cen A/M83, and to the Virgo clusters. To this aim, similarly to Peirani and Pacheco, we found a relationship between the velocity, v , and radius, R , depending on the Hubble parameter, and structure mass, M , and fitted them to data of the Local Group, M81, NGC 253, IC 342, Cen A/M83, and to the Virgo clusters obtained by Peirani and Pacheco. In this way, we obtained optimized values of the mass, the Hubble constant, and R_0 , of the objects studied. Similarly to the quoted Del Popolo’s paper, the Lemaitre–Tolman model took into account the cosmological constant, angular momentum and dynamical friction. The fit gives values of the masses 30–40% larger than the v – R relationship obtained from the standard Lemaitre–Tolman (LT) model (not taking account nor cosmological constant, angular momentum, or dynamical friction). Differently from mass, the Hubble parameter becomes smaller with respect to the LT model, when angular momentum, and dynamical friction are introduced. This is in agreement with Peirani and Pacheco, who improved the standard Lemaitre–Tolman model taking into account the cosmological constant. After determining, the optimized values of the TAR, R_0 , and mass, M , of the studied objects, we put constraints to the dark energy equation of state parameter, w .

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1. INTRODUCTION

From several observations, it is clear that universe contain a larger content of mass-energy than that predicted by General Relativity (GR) [1–6]. Two are the dominant matter components of the Universe: dark matter [3], and a second component, dubbed “dark energy” (DE). In its simplest form, and in the Λ CDM model, DE is represented by the cosmological constant Λ . Λ CDM model describes correctly the Universe [1, 7, 8], but discrepancies with observations are observed both at large and small scales [3, 9–22].

Nevertheless the success of the Λ CDM model [1, 7, 8], precision data are revealing drawbacks, and tensions both at large scales [23–27], and at small ones [3, 9–14, 20–22, 28–30].

As observed in [31], calculation of the mass-to-light (M/L) ratios of group of galaxies in the past [32] is in disagreement with newer measurements based on the virial theorem [33]. The new measurements give smaller (M/L) ratios. Lynden–Bell [34] and Sandage [35] proposed an alternative approach to the virial the-

orem based on the Lemaitre–Tolman (LT) model [36, 37]. The LT model was applied to the local group [35] and to the Virgo cluster [38–40].

The model was modified taking into account the cosmological constant, by Peirani and de Freitas Pacheco [41, 42] applying it to the Virgo cluster, the pair M31–MW, M81, the Centaurus A/M83 group, the IC 342/Maffei-I group, and the NGC 253 group. In [31], we extended their model to take account of the effect of angular momentum, and dynamical friction. The introduction of angular momentum, and dynamical friction modifies the mass, M , and TAR, R_0 , relation.

The TAR has been proposed as a promising way to test cosmological models [43], dark energy, and disentanglement between Λ CDM model, dark energy, and modified gravity models [27, 43–48]. Several papers (e.g., [43–48]) claimed that TAR is independent on baryon physics.

Contrarily to [44], we already showed in [2, 31, 49–54] that TAR depends from baryon physics, through

its dependence from shear, and vorticity, and dynamical friction [50, 54–61].

The effects of shear, rotation, and dynamical friction (slowing down the collapse [58, 59, 62, 63], changing turnaround epoch and collapse time) were investigated in [49, 50, 64–68] for smooth DE models, [69] in clustering DE cosmologies, and [70] in Chaplygin cosmologies. Moreover, shear, vorticity, and dynamical friction change the typical parameters of the spherical collapse model (SCM) [2, 16, 19, 55, 71, 72], the mass function [2, 49, 51, 52, 55–57], the two-point correlation function [1, 15, 16, 56, 73], and the weak lensing peaks [54] (see also [19, 74–79]).

In order to get estimates of R_0 , and the structure mass, [41, 42], differently from [35], and similarly to [41, 42], we build up a velocity–distance relationship, v – R , describing the kinematic status of the systems studied. Knowing the values of v , and R for the members of the groups studied, the mass of the group, M , and the Hubble parameter can be obtained by means of a non-linear fit of the v – R relation to the data. The value of R_0 is obtained by the integration of equation of motion, and using the value of the Hubble constant, and mass, obtained by the quoted fit.

In the following, we apply the results of [31] to build up a velocity–distance relationship, v – R , similar to that of [41, 42]. Differently from [41, 42], we take account of angular momentum, and dynamical friction. The quoted relation will be applied to the Virgo cluster, the pair M31–MW, M81, the Centaurus A/M83 group, the IC342/Maffei-I group, and the NGC 253 group, in order to get R_0 , the Hubble parameter, and M .

The paper is organized as follows. In Section 2, we show how to obtain the velocity–radius, v – R , relation. In Section 3, we use the quoted relation to fit the data of the Virgo cluster, the pair M31–MW, M81, the Centaurus A/M83 group, the IC 342/Maffei-I group, and the NGC 253 group, obtaining R_0 , the Hubble parameter, and M . In Section 4, we use the obtained values of R_0 , and mass M , to put constraints to the equation of state (EoS) parameter of DE. Section 5 is devoted to conclusions.

2. THE RELATION BETWEEN RADIUS AND VELOCITY

In [31] the equations of Lemaitre–Tolman model with cosmological constant, angular momentum, and dynamical friction were solved and it was shown how to use them to obtain an estimation of the mass, and the TAR of a given structure. The goal of the present paper is to extend the previous study following [41, 42] to obtain a relationship between the radius, R and velocity, $V(R)$. The quoted relationship will be fitted to the data of a series of groups of galaxies to obtain a value of the mass, TAR, and Hubble constant for each group.

Going a step back, the Lemaitre–Tolman model is fundamentally a spherical collapse model. The simplest form of the spherical collapse model (SCM) was introduced by [80]. It is a simple and popular method to study analytically the non-linear evolution of perturbations of dark matter (DM) and dark energy (DE). The model describes the evolution of a spherical symmetric over-density which initially expands with the Hubble flow, then detaches from it, when the density overcomes a critical value, reaches a maximum radius, dubbed turn-around radius (TAR), and finally collapse and virialize. Gunn and Gott model [80] SCM is a very simple model assuming that matter moves in a radial fashion. The model was improved to take into account angular momentum, dynamical friction, the cosmological constant, shear, and dark energy in several papers [72, 81–119].

Del Popolo [49, 50] studied the effects of shear and rotation in smooth DE models. The effects of shear and rotation were investigated in [49, 50] for smooth dark energy models, in clustering DE cosmologies [69], and in Chaplygin cosmologies [70].

As already reported, in [31] we showed how to solve the equation of motion in the case of the Lemaitre–Tolman model with cosmological constant, angular momentum, and dynamical friction given by

$$\frac{d^2y}{dx^2} = -\frac{A}{2y^2} + \Omega_\Lambda y + \frac{K_j}{y} - \frac{\eta}{H_0} \frac{dy}{dx}, \quad (1)$$

where $K_j = \frac{k}{(H_0 R_0)^2}$, $A = \frac{2GM}{H_0^2 R_0^3}$, and

$$H = H_0 \sqrt{\left(\frac{a_0}{a}\right)^3 \Omega_m + \Omega_\Lambda}, \quad (2)$$

where $J = kR^\alpha$, with $\alpha = 1$, in agreement with [120],¹ k constant, and $\eta/H_0 = 0.5$. The equation is in adimensional coordinates: $y = R/R_0$, $t = x/H_0$.

In the present paper, we use the solution of the quoted equations to obtain a relationship between the velocity $V(R)$, and radius R of a series of group of galaxies, later discussed. The last will be fitted to the data of the quoted groups to get their mass, and turn-around radius.

The V – R relation is obtained as follows. Let's consider Fig. 1 in [31], that we report here. The figure represents the evolution of shell radius for different values of K . The red, cyan, and green lines correspond to $K = -5.737$, $K = -6.2$, and $K = -5.1$, respectively. We report these three cases just for an example. In reality, we obtained a series of curves, from the bottom one to the top one, and considered the intersection with the vertical line.

The vertical line corresponds to $x = 0.964$. Its intersection with the curves, solution of the equations

¹ In that paper $\alpha = 1.1 \pm 0.3$.

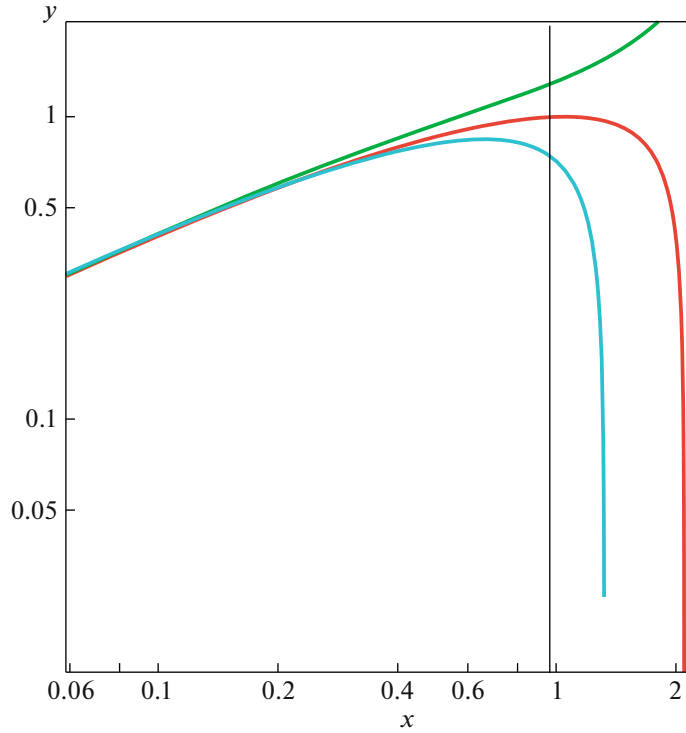


Fig. 1. Evolution of shell radius for different values of K . The red, cyan, and green lines correspond to $K = -5.737$, -6.2 , and -5.1 respectively.

described in the previous section, gives the value $y(x) = y(0.964)$. The solution of the equations of the previous section, also gives the velocity, allowing us to find $u(x) = u(0.964)$. We will get a couple of value (y, u) for each intersection of the vertical line with the curves. This allows us to find a series of points that can be fitted with a relation of the form $u = -b/y^n + by$.

For the model taking into account the cosmological constant, angular momentum, and dynamical friction we may write

$$v = -\frac{1.3436}{y^n} + 1.3436, \quad (3)$$

where $n = 0.9107$. This equation gives the relation between the velocity, v , and the parameter y that as we now show is related to the radius R .

In Fig. 2, we plot the velocity profile (previous equation) using adimensional variables. The points represent the couples (y, u) , obtained as previously described, while the solid line is the fit with the equation $u = -b/y^n + by$.

This can be written in terms of the physical units as

$$v(R) = -bH_0R_0\left(\frac{R_0}{R}\right)^n + bH_0R, \quad (4)$$

where $b = 1.3436$. Substituting in this equation, $R_0 = \left(\frac{2GM}{H_0^2}\right)^{1/3}$, we get

$$v(R) = -b\frac{H_0}{R^n}\left(\frac{2GM}{AH_0^2}\right)^{\frac{n+1}{3}} + bH_0R \quad (5)$$

or

$$v(R) = -\frac{0.66385H_0}{R^n}\left(\frac{GM}{H_0^2}\right)^{\frac{n+1}{3}} + 1.3436H_0R, \quad (6)$$

where $n = 0.9107$. This equation is the relation between the velocity $V(R)$, and radius R that can be used to fit the data shown in Fig. 3.

The previous equation satisfy the condition $v(R_0) = 0$. In the following, we will apply Eq. (6) to some groups of galaxies and clusters. In Table 1, we summarize the parameters of the model.

Table 1. The constant A , and the fitting parameters b , and n of the velocity-distance ($v-R$) relations

η/H_0	K_J	b	n	A
0.5	1.0	1.34360	0.9105	6.05

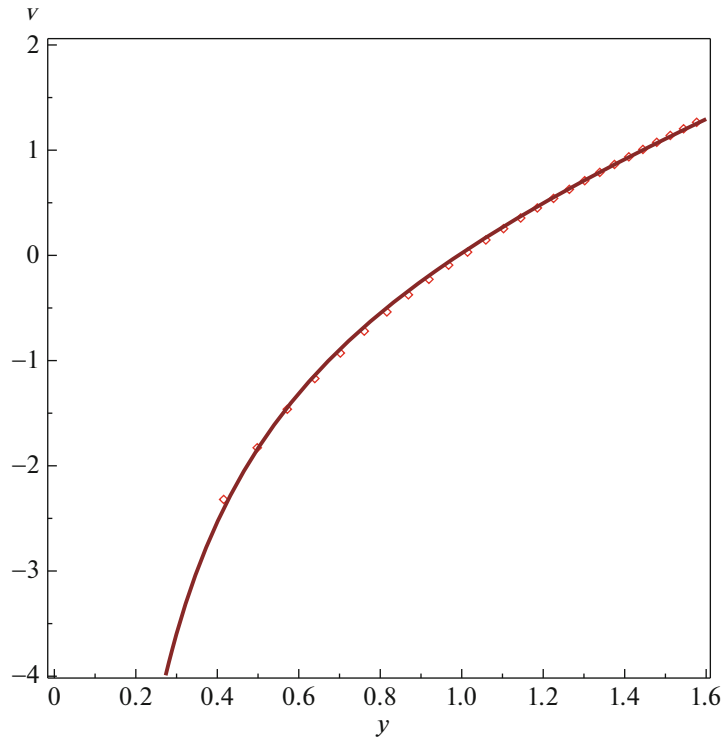


Fig. 2. The velocity profile.

Table 1, as well as Fig. 2 show that including the angular momentum, and dynamical friction steepens the velocity profile, and increases the parameter A . This means that for a given R_0 the mass of the structure increases, while the radius of the zero-gravity surface decreases.

3. APPLICATION TO NEAR GROUPS AND CLUSTERS OF GALAXIES

Now, we will apply Eqs. (6) to near groups and two clusters of galaxies. To this aim, we need for each galaxy its velocity and distance with respect to center of mass. We will use data obtained by [41, 42]. Velocities were transformed from the heliocentric to the Local Group rest frame. The distance can be written as

$$R = \sqrt{D^2 + D_g^2 - 2DD_g \cos \theta}, \quad (7)$$

where the angle θ is the angle between the center of mass and the galaxy, D the distance from the galaxy to the center of mass, and D_g is the distance to the galaxy. Indicating with V , and V_g the center of mass velocities, and that of the galaxy with respect the Local Group rest frame, the velocity difference along the radial direction between both object is

$$V(R) = V_g \cos \alpha - V \cos \beta, \quad (8)$$

being $\alpha = \frac{D \sin \theta}{D_g - D \cos \theta}$ and $\beta = \alpha + \theta$. In the list given by [42] unbound objects, and uncertain distances and

velocities were excluded, and an error of 10% was considered for velocities and distances. In the case of the group M31–MW, the data were obtained from [121], in [41]. We used the data of [42] also for the case of the Virgo cluster.

Figure 3 plots the v – R relationships for the groups studied: the M31–MW group (Fig. 3a), the M81 group (Fig. 3b), the NGC 253 group (Fig. 3c), the IC 342 group (Fig. 3d), the Cen A/M83 group (Fig. 3e), the Virgo cluster (Fig. 3f). The red dots are the data from [41, 42].

Once the Hubble constant, and M are obtained from the fit, the TAR, R_0 is obtained solving $A = \frac{2GM}{H_0^2 R_0^3}$, where A is obtained from the solution of Eq. (1).

3.1. M31–MW

We applied Eq. (6) to the data of Peirani and de Freitas Pacheco [41]. Both the mass and the Hubble parameter were allowed to vary. The results are shown in Table 2. Karachentsev [121] estimated a turn-around radius of 0.94 ± 0.1 Mpc and using the LT model obtained a mass of $1.5 \times 10^{12} M_\odot$, which is much smaller than the estimate of [41] (2.5 ± 0.7) $\times 10^{12} M_\odot$, $R_0 = 1.0 \pm 0.1$ Mpc, and $h = 0.74 \pm 0.04$. The value of the mass is larger than that of [121], that used the LT model. A tendency of the LT models is that of giving higher masses, and smaller h if the effect of the

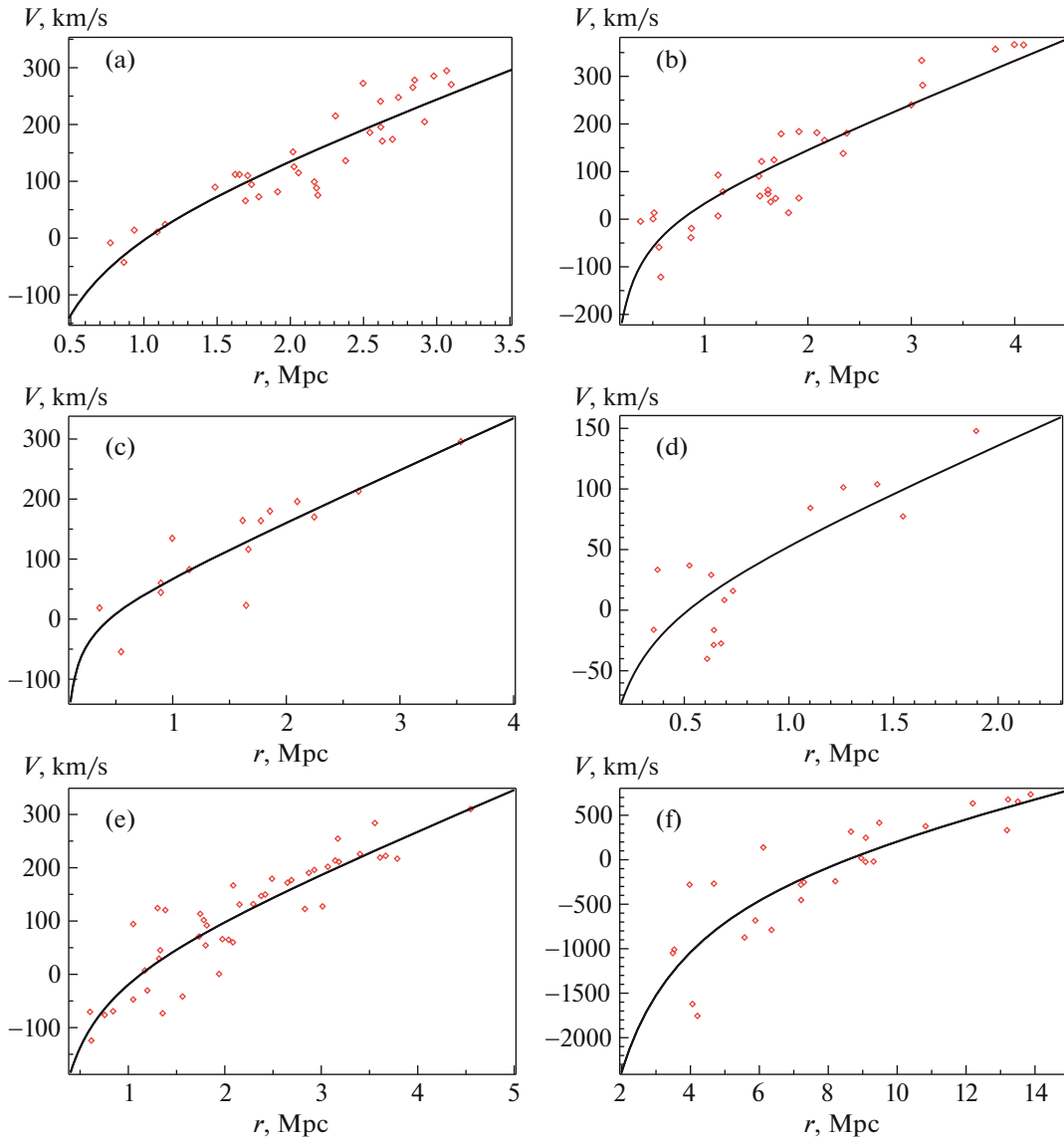


Fig. 3. Velocity–distance plots for the M31–MW group (a), the M81 group (b), the NGC 253 group (c), the IC 342 group (d), the Cen A/M83 group (e), the Virgo cluster (f). The red dots are the data from [41, 42].

cosmological constant, angular momentum, and other effects which contribute with positive terms in the equation of motion are taken into account. In fact, Peirani and de Freitas Pacheco [42] found a values of $h = 0.87 \pm 0.05$ when using the LT model, and $h = 0.73 \pm 0.04$ when taking into account the cosmological constant. Our values of R_0 , mass, and h are in agreement, within the estimated uncertainties with [42]. We recall that the errors, come from the fitting procedure. Similarly to [41] the values of R_0 , the mass and Hubble parameter were obtained using the fit to the data.

3.2. The M81 Group

The M81 group was studied in [121–123]. In [122] it was presented a detailed study of the M81 complex.

From the brightness of the tip of the red giant branch a distance of 3.5 Mpc was estimated, based on Hubble Space Telescope images of different members of the association. In the same paper, distances and radial velocities of about 50 galaxies around and inside M81 estimated an $R_0 = (1.05 \pm 0.07)$ Mpc. Using the LT model, the mass inside R_0 was estimated to be $(1.6 \pm 0.3) \times 10^{12} M_\odot$. In a subsequent work of Kashibade [123] the authors found $R_0 = 0.89 \pm 0.05$ Mpc, in agreement with our estimates, and $M = (1.03 \pm 0.17) \times 10^{12} M_\odot$, a bit smaller than our value. Peirani and de Freitas Pacheco [42] found $M = (0.92 \pm 0.24) \times 10^{12} M_\odot$, in agreement with our value and $h = 0.67 \pm 0.04$, in agreement with our estimate. Our model estimates, gives average values of the mass, M larger than the average of the estimates of [42, 123].

We must recall, that our analysis follow the [41, 42] method, namely we performed a non-linear fit of $V(R)$ – R relationships to the available data, searching for the values of both the mass inside R_0 and the Hubble constant, which minimized the scatter.

This analysis was performed in a similar way to all the other groups.

3.3. The NGC 253 Group

Concerning this group, Karachentsev et al. [124] described the system as small, loose concentrations of galaxies around NGC 300, NGC 253, and NGC 7793. The authors obtained $R_0 = 0.7 \pm 0.1$ Mpc, smaller than our estimates, and $M = (5.5 \pm 2.2) \times 10^{11} M_\odot$, larger than our estimates. By means of the non-linear fit analysis quoted in the previous subsection Peirani [42] found $M = (1.3 \pm 1.8) \times 10^{11} M_\odot$ whose larger uncertainties is probably due to incompleteness in the data. They also found, by means of HST data $h = 0.63 \pm 0.06$. Both their estimates for h , and M , are in agreement with our model.

3.4. The IC342 Group

A study of this group was performed by Karachentsev et al. [125]. Seven dwarf galaxies are associated with the IC 342 group, at an average distance of 3.3 Mpc from the Local Group. The Maffei-I association consists of about eight galaxies with a distance of about 3 Mpc.

According to [125], the group has $R_0 = 0.9 \pm 0.1$, and $M = (1.07 \pm 0.33) \times 10^{12} M_\odot$, both larger than our estimates. Peirani [42] found a smaller value of the mass, $M = (2.0 \pm 1.3) \times 10^{11} M_\odot$, and also R_0 (≈ 0.53 Mpc), while $h = 0.57 \pm 0.10$. Our values of mass, turn-around radius, and h , were obtained using the non-linear fit method (as in [42]) and agree with Peirani estimates [42].

3.5. The CenA/M83 Group

This group was studied by Karachentsev et al. [126, 127]. Direct imaging of dwarf galaxies in the Centaurus A (NGC 5128) showed that these galaxies are concentrated around Cen A and M83 (NGC 5236) and that their distances to the Local Group were 3.8 and 4.8 Mpc, respectively. The value of R_0 around Cen A, using velocities and distances of individual members, was estimated at $R_0 = 1.55 \pm 0.13$ Mpc, and $M = (6.4 \pm 1.8) \times 10^{12} M_\odot$, larger than our estimates. Woodley [128], using different mass indicators found a larger mass ($M = (9.2 \pm 3) \times 10^{12} M_\odot$). Peirani [42] found values 3–4 times smaller ($M = (2.1 \pm 0.5) \times 10^{12} M_\odot$), and $h = 0.57 \pm 0.04$. In our analysis, both M , and h are in agreement with data of [42], and use their non-linear fit method.

Table 2. Characteristic parameters of the groups studied

Group	$h(j, \eta)$, km/(Mpc s)	$M(j, \eta)$, $10^{12} M_\odot$	$R(j, \eta)$, Mpc
M31/MW	0.69 ± 0.04	3.57 ± 0.4	1.02 ± 0.1
M81	0.65 ± 0.04	1.398 ± 0.10	0.78 ± 0.05
NGC 253	0.63 ± 0.05	0.244 ± 0.10	0.44 ± 0.1
IC 342	0.55 ± 0.10	0.292 ± 0.10	0.52 ± 0.09
CenA/M83	0.55 ± 0.04	3.015 ± 0.4	1.13 ± 0.08
Virgo	0.59 ± 0.09	1525.55 ± 200	8.56 ± 0.8

3.6. The Virgo Cluster

Concerning Virgo, several estimates for the mass were done by means of the LT model [38, 129], by means of the Virial theorem [39] finding masses smaller than $10^{15} M_\odot$, except [129] who found a value of $1.3 \times 10^{15} M_\odot$. Using the LT model and taking account of the cosmological constant [41] found $M = (1.10 \pm 0.12) \times 10^{15} M_\odot$, $h = 0.65 \pm 0.09$, and $R_0 = 8.6 \pm 0.8$ Mpc. Our estimates are in agreement with the value of h , R_0 of Peirani [41], while the masses are larger than in [41].

In summary, our estimates usually agree with the estimates of [41, 42]. Moving from the LT model to the that taking account of the cosmological constant, and that also taking account of angular momentum, and dynamical friction the values of the cosmological constant decreases, and the opposite happens to the mass, M . Another important issue that is shown by Table 2, is that the values of h are in some cases smaller than the known estimates. In the past decade or so, has been performed dozens of measurements of the Hubble constant, to try to overcome the Hubble constant tension. As clear shown from [130], from the year 2000 the constraints have changed from 72_{-8}^{+8} km/(Mpc s), to the range 67–75 km/(Mpc s). Recent constraints from the gravitational wave signal of GW170817 gives $70.3_{-5.0}^{+5.3}$ km/(Mpc s) [131], $67.4_{-1.2}^{+1.1}$ km/(Mpc s) (DES + BAO + BBN), and 67.5 ± 1.1 km/(Mpc s) [132]. Comparing with our results, CenA/M83 shows values of 59 km/(Mpc s). The last discrepancy with observations may be due to non completeness of the data used in 2008 by [42]. Based on a large-scale survey of the Centaurus group done in 2014–2015, a significant amount of faint dwarf galaxy candidates were discovered [133]. Therefore, the old data used in [42] may contain some selection bias so that the resulting h obtained is systematically smaller.

4. CONSTRAINTS ON THE DM EOS PARAMETER

Recently, the turn-around radius, R_0 has been proposed as a promising way to test cosmological models

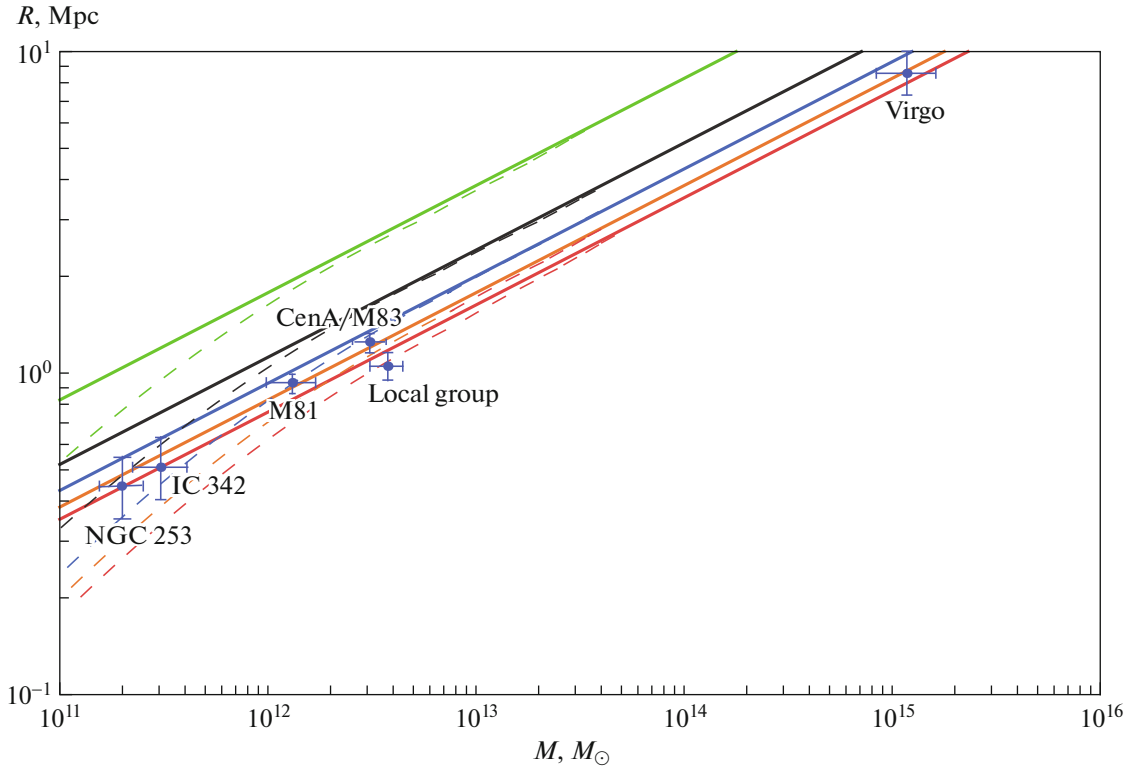


Fig. 4. Mass-radius relation of stable structures for different w . The solid lines from top to bottom represent $w = -0.5$ (solid green line), -1 (black solid line), -1.5 (blue solid line), -2 (pink solid line), -2.5 (red solid line). The dashed lines are the same of the previous lines, but they are obtained in [79]. The dots with error bars, are data as in [45].

[43], DE, and disentangle between Λ CDM model, DE, and MG models [43–48].

Pavlidou and Tomaras in [44] calculated R_0 for Λ CDM, while in [45] did the same for smooth DE. According to [48] R_0 is affected by modified gravity (MG) theories. In MG theories [134] found a general relation for R_0 , and [46] found a method to get the same quantities in generic gravitational theories. In [79], we used an extended spherical collapse model (ESCM) introduced, and adopted in [2, 49, 51, 52, 54], to show how R_0 is modified by the presence of vorticity, and shear in the equation of motion. We also showed how the $M - R_0$ plane can be used to put some constraints on the dark energy equation of state (DE EOS) parameter² w , similarly to [44, 45]. The constraints on w depends on the estimated values of the mass and R_0 of galaxies, groups, and clusters. Some data were taken from [45], and others from [41, 42].

With the revised value of mass, M , and R_0 presented in this paper, we recalculate the constraints showed in [79].

² I recall that in cosmology, the equation of state of a perfect fluid is characterized by a dimensionless number w equal to the ratio of its pressure, p , and its energy density, ρ , that is $w = p/\rho$.

Figure 4 plots the mass-radius relation of stable structures for different w . The solid lines from top to bottom represent $w = -0.5$ (solid green line), -1 (black solid line), -1.5 (blue solid line), -2 (pink solid line), -2.5 (red solid line). The dashed lines are the same of the previous lines, but they are obtained using the model in [79]. The dots with error bars, are data obtained in the previous sections.

The constraints to w are reproduced in Table 3. They are different from previous ones obtained by [44, 45] based on the calculation of the mass, M , and R_0 by means of the virial theorem or the LT model.

Table 3. The allowed ranges of w

Stable structure	Range of w
M81	$w \geq -1.5$
IC 342	$w \geq -1$
NGC 253	$w \geq -1$
Cen A/M83	$w \geq -1.5$
Local Group	$w \geq -2$
Virgo	$w \geq -1.5$

5. CONCLUSIONS

In this paper, we used a previous paper result [31] to determine the values of the TAR, R_0 , Hubble parameter value, and mass M of a group of structures. Following [41, 42], we determined a relationship between the velocity and radius, $v-R$ depending on the Hubble parameter, and the structure mass, M , and fitted them to data of the Local Group, M81, NGC 253, IC 342, Cen A/M83, and to the Virgo clusters. Knowing the Hubble parameter value, and the structure mass, the TAR was obtained from the solution of the equation of motion, more precisely determining A

in Eq. (1), and solving the expression $A = \frac{2GM}{H_0^2 R_0^3}$. In

this way, we obtained optimized values of the mass and Hubble constant of the objects studied, and the TAR, R_0 . Our model, and Eq. (1) took into account the cosmological constant, angular momentum and dynamical friction effects. The fit obtained gave values of the masses 30–40% larger than the $v-R$ relationship obtained from the standard LT model, and consequently also R_0 was different from the standard LT model. Knowing the optimized values of the TAR, R_0 , and mass, M , of the studied objects, it was also possible to put new constraints on the DE EoS parameter w . The quoted constraints are different from those obtained in previous papers [44, 45, 79] because our model is improved with respect those used in those papers.

REFERENCES

1. A. Del Popolo, *Astron. Rep.* **51**, 169 (2007); arXiv: 0801.1091 [astro-ph].
2. A. Del Popolo, *AIP Conf. Proc.* **1548**, 2 (2013).
3. A. Del Popolo, *Int. J. Mod. Phys. D* **23**, 30005 (2014); arXiv: 1305.0456 [astro-ph.CO].
4. A. Del Popolo, M. Le Delliou, and X. Lee, *Galaxies* **6** (3), 67 (2018).
5. P. Bull, Y. Akrami, J. Adamek, T. Baker, et al., *Phys. Dark Universe* **12**, 56 (2016); arXiv: 1512.05356 [astro-ph.CO].
6. A. Del Popolo, M. Le Delliou, and X. Lee, *Phys. Dark Universe* **26**, 100342 (2019); arXiv:1808.02136 [astro-ph.CO].
7. D. N. Spergel, L. Verde, H. V. Peiris, E. Komatsu, et al., *Astrophys. J. Suppl.* **148**, 175 (2003); arXiv: astro-ph/0302209.
8. E. Komatsu, K. M. Smith, J. Dunkley, C. L. Bennett, et al., *Astrophys. J. Suppl.* **192**, 18 (2011).
9. B. Moore, T. Quinn, F. Governato, J. Stadel, and G. Lake, *Mon. Not. R. Astron. Soc.* **310**, 1147 (1999); arXiv: astro-ph/9903164.
10. W. J. G. de Blok, *Adv. Astron.* **2010**, 789293 (2010); arXiv: 0910.3538 [astro-ph.CO].
11. J. P. Ostriker and P. Steinhardt, *Science* (Washington, DC, U. S.) **300**, 1909 (2003); arXiv: astro-ph/0306402.
12. M. Boylan-Kolchin, J. S. Bullock, and M. Kaplinghat, *Mon. Not. R. Astron. Soc.* **415**, L40 (2011); arXiv: 1103.0007 [astro-ph.CO].
13. V. F. Cardone, M. P. Leubner, and A. Del Popolo, *Mon. Not. R. Astron. Soc.* **414**, 2265 (2011); arXiv: 1102.3319 [astro-ph.CO].
14. V. F. Cardone, A. Del Popolo, C. Tortora, and N. R. Napolitano, *Mon. Not. R. Astron. Soc.* **416**, 1822 (2011); arXiv: 1106.0364 [astro-ph.CO].
15. A. Del Popolo, *Astron. J.* **131**, 2367 (2006); arXiv: astro-ph/0609101.
16. A. Del Popolo, *Astron. Astrophys.* **454**, 17 (2006); arXiv: 0801.1086 [astro-ph].
17. D. C. Rodrigues, A. Del Popolo, V. Marra, and P. L. C. de Oliveira, *Mon. Not. R. Astron. Soc.* **470**, 2410 (2017); arXiv: 1701.02698 [astro-ph.GA].
18. J. A. S. Lima, A. Del Popolo, and A. R. Plastino, *Phys. Rev. D* **100**, 104042 (2019); arXiv: 1911.09060 [gr-qc].
19. A. Del Popolo, F. Pace, and D. F. Mota, *Phys. Rev. D* **100**, 024013 (2019); arXiv: 1908.07322 [astro-ph.CO].
20. A. Del Popolo and N. Hioteelis, *J. Cosmol. Astropart. Phys.*, No. 1, 047 (2014); arXiv: 1401.6577 [astro-ph.GA].
21. A. Del Popolo and M. Le Delliou, *J. Cosmol. Astropart. Phys.*, No. 12, 051 (2014); arXiv: 1408.4893 [astro-ph.CO].
22. A. Del Popolo and M. Le Delliou, *Galaxies* **5**, 17 (2017); arXiv: 1606.07790 [astro-ph.CO].
23. M. Raveri, *Phys. Rev. D* **93**, 043522 (2016); arXiv: 1510.00688 [astro-ph.CO].
24. A. V. Astashenok and A. Del Popolo, *Class. Quantum Grav.* **29**, 085014 (2012); arXiv: 1203.2290 [gr-qc].
25. H. E. S. Velten, R. F. vom Marttens, and W. Zimdahl, *Eur. Phys. J. C* **74**, 3160 (2014); arXiv: 1410.2509 [astro-ph.CO].
26. S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989).
27. A. Del Popolo, *Astrophys. J.* **637**, 12 (2006); arXiv: astro-ph/0609100.
28. A. Del Popolo, *Mon. Not. R. Astron. Soc.* **419**, 971 (2012); arXiv: 1105.0090 [astro-ph.CO].
29. A. Del Popolo, *Mon. Not. R. Astron. Soc.* **424**, 38 (2012); arXiv: 1204.4439 [astro-ph.CO].
30. A. Del Popolo, F. Pace, M. Le Delliou, and X. Lee, *Phys. Rev. D* **98**, 063517 (2018); arXiv: 1809.10609 [astro-ph.GA].
31. Del Popolo, *Astron. Rep.* (2020, in press).
32. J. P. Huchra and M. J. Geller, *Astrophys. J.* **257**, 423 (1982).
33. I. Karachentsev, *Astron. J.* **129**, 178 (2005); arXiv: astro-ph/0410065.
34. D. Lynden-Bell, *Observatory* **101**, 111 (1981).
35. A. Sandage, *Astrophys. J.* **307**, 1 (1986).
36. G. Lemaître, *Ann. Soc. Sci. Bruxelles* **53**, 51 (1933).
37. R. C. Tolman, *Proc. Natl. Acad. Sci. U. S. A.* **20**, 169 (1934).
38. G. L. Hoffman, D. W. Olson, and E. E. Salpeter, *Astrophys. J.* **242**, 861 (1980).
39. R. B. Tully and E. J. Shaya, *Astrophys. J.* **281**, 31 (1984).

40. P. Teerikorpi, L. Bottinelli, L. Gouguenheim, and G. Paturel, *Astron. Astrophys.* **260**, 17 (1992).
41. S. Peirani and J. A. de Freitas Pacheco, *New Astron.* **11**, 325 (2006); arXiv: astro-ph/0508614 [astro-ph].
42. S. Peirani and J. A. de Freitas Pacheco, *Astron. Astrophys.* **488**, 845 (2008); arXiv: 0806.4245 [astro-ph].
43. R. C. C. Lopes, R. Voivodic, L. R. Abramo, and L. Sodré, Jr., *J. Cosmol. Astropart. Phys.*, No. 09, 010 (2018); arXiv: 1805.09918 [astro-ph.CO].
44. V. Pavlidou and T. N. Tomaras, *J. Cosmol. Astropart. Phys.*, No. 09, 020 (2014); arXiv: 1310.1920 [astro-ph.CO].
45. V. Pavlidou, N. Tetradis, and T. N. Tomaras, *J. Cosmol. Astropart. Phys.*, No. 05, 017 (2014); arXiv: 1401.3742 [astro-ph.CO].
46. V. Faraoni, M. Lapierre-Léonard, and A. Prain, *J. Cosmol. Astropart. Phys.*, No. 10, 013 (2015); arXiv: 1508.01725 [gr-qc].
47. S. Bhattacharya, K. F. Dialektopoulos, A. Enea Romano, C. Skordis, and T. N. Tomaras, *J. Cosmol. Astropart. Phys.*, No. 07, 018 (2017); arXiv: 1611.05055 [astro-ph.CO].
48. R. C. C. Lopes, R. Voivodic, L. R. Abramo, and L. Sodré, Jr., *J. Cosmol. Astropart. Phys.*, No. 07, 026 (2019); arXiv: 1809.10321 [astro-ph.CO].
49. A. Del Popolo, F. Pace, and J. A. S. Lima, *Int. J. Mod. Phys. D* **22**, 1350038 (2013); arXiv: 1207.5789 [astro-ph.CO].
50. A. Del Popolo, F. Pace, and J. A. S. Lima, *Mon. Not. R. Astron. Soc.* **430**, 628 (2013); arXiv: 1212.5092 [astro-ph.CO].
51. F. Pace, R. C. Batista, and A. Del Popolo, *Mon. Not. R. Astron. Soc.* **445**, 648 (2014); arXiv: 1406.1448 [astro-ph.CO].
52. A. Mehrabi, F. Pace, M. Malekjani, and A. Del Popolo, *Mon. Not. R. Astron. Soc.* **465**, 2687 (2017); arXiv: 1608.07961 [astro-ph.CO].
53. D. C. Rodrigues, V. Marra, A. del Popolo, and Z. Davari, *Nat. Astron.* **2**, 668 (2018); arXiv: 1806.06803 [astro-ph.GA].
54. F. Pace, C. Schimd, D. F. Mota, and A. D. Popolo, *J. Cosmol. Astropart. Phys.*, No. 09, 060 (2019).
55. A. Del Popolo and M. Gambera, *Astron. Astrophys.* **337**, 96 (1998); arXiv: astro-ph/9802214.
56. A. Del Popolo and M. Gambera, *Astron. Astrophys.* **344**, 17 (1999); arXiv: astro-ph/9806044.
57. A. Del Popolo and M. Gambera, *Astron. Astrophys.* **357**, 809 (2000); arXiv: astro-ph/9909156.
58. A. Del Popolo, E. N. Ercan, and Z. Xia, *Astron. J.* **122**, 487 (2001); arXiv: astro-ph/0108080.
59. A. Del Popolo, *Astron. Astrophys.* **387**, 759 (2002); arXiv: astro-ph/0202436.
60. A. Del Popolo, *Mon. Not. R. Astron. Soc.* **336**, 81 (2002); arXiv: astro-ph/0205449.
61. A. Del Popolo, N. Hiotelis, and J. Peñarrubia, *Astrophys. J.* **628**, 76 (2005); arXiv: astro-ph/0508596 [astro-ph].
62. P. J. E. Peebles, *Astrophys. J.* **365**, 27 (1990).
63. E. Audit, R. Teyssier, and J.-M. Alimi, *Astron. Astrophys.* **325**, 439 (1997); arXiv: astro-ph/9704023.
64. A. Del Popolo, M. Deliyergiyev, M. Le Delliou, L. Tolos, and F. Burgio, *Phys. Dark Universe* **28**, 100484 (2020); arXiv: 1904.13060 [gr-qc].
65. Y. Zhou, A. Del Popolo, and Z. Chang, *Phys. Dark Universe* **28**, 100468 (2020).
66. M. H. Chan and A. Del Popolo, *Mon. Not. R. Astron. Soc.* **492**, 5865 (2020); arXiv: 2001.06141 [astro-ph.GA].
67. A. Del Popolo, M. Deliyergiyev, and M. Le Delliou, *Phys. Dark Universe* **30**, 100622 (2020).
68. A. Del Popolo, M. Deliyergiyev, M. Le Delliou, L. Tolos, and F. Burgio, *Phys. Dark Universe* **28**, 100484 (2020); arXiv: 1904.13060 [gr-qc].
69. F. Pace, R. C. Batista, and A. Del Popolo, *Mon. Not. R. Astron. Soc.* **445**, 648 (2014); arXiv: 1406.1448 [astro-ph.CO].
70. A. Del Popolo, F. Pace, S. P. Maydanyuk, J. A. S. Lima, and J. F. Jesus, *Phys. Rev. D* **87**, 043527 (2013); arXiv: 1303.3628 [astro-ph.CO].
71. A. Del Popolo, M. Gambera, and V. Antonuccio-Delogu, *Astron. Astrophys. Trans.* **16**, 127 (1998).
72. A. Del Popolo, *Astrophys. J.* **698**, 2093 (2009); arXiv: 0906.4447 [astro-ph.CO].
73. A. Del Popolo, *Astron. Astrophys.* **448**, 439 (2006).
74. A. Del Popolo and F. Pace, *Astrophys. Space Sci.* **361**, 162 (2016); arXiv: 1502.01947 [astro-ph.GA].
75. A. Del Popolo, *Mon. Not. R. Astron. Soc.* **408**, 1808 (2010); arXiv: 1012.4322 [astro-ph.CO].
76. A. Del Popolo, *J. Cosmol. Astropart. Phys.*, No. 07, 014 (2011), arXiv: 1112.4185 [astro-ph.CO].
77. A. Del Popolo and V. F. Cardone, *Mon. Not. R. Astron. Soc.* **423**, 1060 (2012); arXiv: 1203.3377 [astro-ph.CO].
78. A. Del Popolo, F. Pace, and M. Le Delliou, *J. Cosmol. Astropart. Phys.*, No. 03, 032 (2017); arXiv: 1703.06918.
79. A. Del Popolo, M. H. Chan, and D. F. Mota, *Phys. Rev. D* **101**, 083505 (2020).
80. J. E. Gunn and J. R. Gott III, *Astrophys. J.* **176**, 1 (1972).
81. P. J. E. Peebles, *Astrophys. J.* **155**, 393 (1969).
82. S. D. M. White, *Astrophys. J.* **286**, 38 (1984).
83. B. S. Ryden and J. E. Gunn, *Astrophys. J.* **318**, 15 (1987).
84. B. S. Ryden, *Astrophys. J.* **329**, 589 (1988).
85. L. L. R. Williams, A. Babul, and J. J. Dalcanton, *Astrophys. J.* **604**, 18 (2004); arXiv: astro-ph/0312002.
86. V. Antonuccio-Delogu and S. Colafrancesco, *Astrophys. J.* **427**, 72 (1994).
87. J. A. Fillmore and P. Goldreich, *Astrophys. J.* **281**, 1 (1984).
88. E. Bertschinger, *Astrophys. J. Suppl.* **58**, 39 (1985).
89. Y. Hoffman and J. Shaham, *Astrophys. J.* **297**, 16 (1985).
90. K. Subramanian, R. Cen, and J. P. Ostriker, *Astrophys. J.* **538**, 528 (2000); arXiv: astro-ph/9909279.
91. Y. Ascasibar, G. Yepes, S. Gottlöber, and V. Müller, *Mon. Not. R. Astron. Soc.* **352**, 1109 (2004); arXiv: astro-ph/0312221.

92. O. Lahav, P. B. Lilje, J. R. Primack, and M. J. Rees, *Mon. Not. R. Astron. Soc.* **251**, 128 (1991).
93. A. V. Gurevich and K. P. Zybin, *Sov. Phys. JETP* **67**, 1 (1988).
94. A. V. Gurevich and K. P. Zybin, *Sov. Phys. JETP* **67**, 1957 (1988).
95. S. D. M. White and D. Zaritsky, *Astrophys. J.* **394**, 1 (1992).
96. P. Sikivie, I. I. Tkachev, and Y. Wang, *Phys. Rev. D* **56**, 1863 (1997); arXiv: astro-ph/9609022.
97. A. Nusser, *Mon. Not. R. Astron. Soc.* **325**, 1397 (2001); arXiv: astro-ph/0008217.
98. N. Hiotelis, *Astron. Astrophys.* **382**, 84 (2002); arXiv: astro-ph/0111324.
99. M. Le Delliou and R. N. Henriksen, *Astron. Astrophys.* **408**, 27 (2003); arXiv: astro-ph/0307046.
100. P. Zukin and E. Bertschinger, in *Proceedings of the APS April Meeting, February 13–16, 2010*, p. G13.003.
101. Y. Hoffman, *Astrophys. J.* **308**, 493 (1986).
102. Y. Hoffman, *Astrophys. J.* **340**, 69 (1989).
103. S. Zaroubi and Y. Hoffman, *Astrophys. J.* **416**, 410 (1993).
104. F. Bernardeau, *Astrophys. J.* **433**, 1 (1994); arXiv: astro-ph/9312026.
105. J. M. Bardeen, J. R. Bond, N. Kaiser, and A. S. Szalay, *Astrophys. J.* **304**, 15 (1986).
106. Y. Ohta, I. Kayo, and A. Taruya, *Astrophys. J.* **589**, 1 (2003); arXiv: astro-ph/0301567.
107. Y. Ohta, I. Kayo, and A. Taruya, *Astrophys. J.* **608**, 647 (2004); arXiv: astro-ph/0402618.
108. S. Basilakos, *Mon. Not. R. Astron. Soc.* **395**, 2347 (2009); arXiv: 0903.0452 [astro-ph.CO].
109. F. Pace, J.-C. Waizmann, and M. Bartelmann, *Mon. Not. R. Astron. Soc.* **406**, 1865 (2010); arXiv: 1005.0233 [astro-ph.CO].
110. S. Basilakos, M. Plionis, and J. Solá, *Phys. Rev. D* **82**, 083512 (2010); arXiv: 1005.5592 [astro-ph.CO].
111. D. F. Mota and C. van de Bruck, *Astron. Astrophys.* **421**, 71 (2004); arXiv: astro-ph/0401504.
112. N. J. Nunes and D. F. Mota, *Mon. Not. R. Astron. Soc.* **368**, 751 (2006); arXiv: astro-ph/0409481.
113. L. R. Abramo, R. C. Batista, L. Liberato, and R. Rosenfeld, *J. Cosmol. Astropart. Phys.*, No. 11, 012 (2007); arXiv: 0707.2882 [astro-ph].
114. L. R. Abramo, R. C. Batista, L. Liberato, and R. Rosenfeld, *Phys. Rev. D* **77**, 067301 (2008); arXiv: 0710.2368 [astro-ph].
115. L. R. Abramo, R. C. Batista, and R. Rosenfeld, *J. Cosmol. Astropart. Phys.*, No. 07, 040 (2009); arXiv: 0902.3226 [astro-ph.CO].
116. L. R. Abramo, R. C. Batista, L. Liberato, and R. Rosenfeld, *Phys. Rev. D* **79**, 023516 (2009); arXiv: 0806.3461 [astro-ph].
117. P. Creminelli, G. D'Amico, J. Noreña, L. Senatore, and F. Vernizzi, *J. Cosmol. Astropart. Phys.*, No. 03, 027 (2010); arXiv: 0911.2701 [astro-ph.CO].
118. T. Basse, O. Eggers Bj[ae]lde, and Y. Y. Y. Wong, *J. Cosmol. Astropart. Phys.*, No. 10, 038 (2011); arXiv: 1009.0010 [astro-ph.CO].
119. R. C. Batista and F. Pace, *J. Cosmol. Astropart. Phys.*, No. 06, 044 (2013); arXiv: 1303.0414 [astro-ph.CO].
120. J. S. Bullock, T. S. Kolatt, Y. Sigad, R. S. Somerville, et al., *Mon. Not. R. Astron. Soc.* **321**, 559 (2001); arXiv: astro-ph/9908159.
121. I. D. Karachentsev, M. E. Sharina, D. I. Makarov, A. E. Dolphin, et al., *Astron. Astrophys.* **389**, 812 (2002); arXiv: astro-ph/0204507 [astro-ph].
122. I. D. Karachentsev, A. E. Dolphin, D. Geisler, E. K. Grebel, et al., *Astron. Astrophys.* **383**, 125 (2002).
123. I. D. Karachentsev, A. Dolphin, R. B. Tully, M. Sharina, et al., *Astron. J.* **131**, 1361 (2006); arXiv: astro-ph/0511648 [astro-ph].
124. I. D. Karachentsev, E. K. Grebel, M. E. Sharina, A. E. Dolphin, et al., *Astron. Astrophys.* **404**, 93 (2003); arXiv: astro-ph/0302045.
125. I. D. Karachentsev, M. E. Sharina, A. E. Dolphin, and E. K. Grebel, *Astron. Astrophys.* **408**, 111 (2003).
126. I. D. Karachentsev, M. E. Sharina, A. E. Dolphin, E. K. Grebel, et al., *Astron. Astrophys.* **385**, 21 (2002).
127. I. D. Karachentsev, R. B. Tully, A. Dolphin, M. Sharina, et al., *Astron. J.* **133**, 504 (2007); arXiv: astro-ph/0603091.
128. K. A. Woodley, *Astron. J.* **132**, 2424 (2006); arXiv: astro-ph/0608497.
129. P. Fouque, J. M. Solanes, T. Sanchis, and C. Balkowski, *Astron. Astrophys.* **375**, 770 (2001).
130. W. L. Freedman, B. F. Madore, D. Hatt, T. J. Hoyt, et al., *Astrophys. J.* **882**, 34 (2019); arXiv: 1907.05922 [astro-ph.CO].
131. K. Hotokezaka, E. Nakar, O. Gottlieb, S. Nissanke, K. Masuda, G. Hallinan, K. P. Mooley, and A. T. Deller, *Nat. Astron.* **3**, 940 (2019); arXiv:1806.10596 [astro-ph.CO].
132. N. Schöneberg, J. Lesgourgues, and D. C. Hooper, *J. Cosmol. Astropart. Phys.*, No. 10, 029 (2019); arXiv: 1907.11594 [astro-ph.CO].
133. O. Müller, H. Jerjen, and B. Binggeli, *Astron. Astrophys.* **597**, A7 (2017); arXiv: 1605.04130 [astro-ph.GA].
134. S. Capozziello, K. F. Dialektopoulos, and O. Luongo, *Int. J. Mod. Phys. D* **28**, 1950058 (2019); arXiv: 1805.01233 [gr-qc].