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Equilibrium Configurations of Rotating White Dwarfs at Finite Temperatures*♯

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Abstract—In this work, cold and hot, static and rotating white dwarf stars are investigated within the framework of classical physics, employing the Chandrasekhar equation of state. The main parameters of white dwarfs such as the central density, pressure, total mass and radius are calculated fulfilling the stability criteria for hot rotating stars. To construct rotating configurations the Hartle approach is involved. It is shown that the effects of finite temperatures become crucial in low-mass white dwarfs, whereas rotation is relevant in all mass range. The simultaneous accounting for temperature and rotation is critical in the calculation of the radii of white dwarfs. The results obtained in this work can be applied to explain a variety of observational data for white dwarfs from the Sloan Digital Sky Survey Data Releases.

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1. INTRODUCTION

Compact objects are the end products of stellar evolution and they are subdivided into the basic three categories: white dwarfs (WDs), neutron stars (NSs) and black holes (BHs), with the exception of exotic and at the same time hypothetical objects such as quark stars, boson stars, gravastars, etc. [1–4]. These objects are called compact because of their large mass and small size and, correspondingly, high density. It is believed that the initial mass is a key factor determining the final fate of a star. For example, WDs are formed from low-mass star progenitors with masses $M \approx (1-8)\, M_\odot$ (solar mass) [5], though the lower and upper bounds of the progenitor mass are not well constrained both from theory and observations [6]. Nonetheless, the upper limit of the mass of a static cold WD without a magnetic field does not exceed the Chandrasekhar mass limit $M \leq 1.44M$ [7].

The ratio of the gravitational radius to the actual radius of an object, the so-called compactness parameter r_g/R for WDs is ∼0.001, for NSs is ∼0.3, for BHs is equal to 1 [2], where $r_g = 2GM/c^2$ is the gravitational (Schwarzschild) radius, G is the gravitational constant, M is the total mass of the object, c is the speed of light in vacuum. From here it is evident that the role of general relativity (GR) becomes more pronounced when the compactness parameter increases. The importance of GR in the case of massive WDs is well-known in the literature [1, 2]. According to [8] and [9], it is necessary to investigate WDs in GR in order to analyze their stability against the relativistic corrections and small perturbations, though they can be neglected for low-mass WDs.

According to the latest observational data by 2017 there are more than 32 000 registered WDs [10], which are splitted into groups and subgroups depending on their mass, temperature, nuclear composition, magnetic field and other physical parameters. The data are available online and are provided with the description and technical details of observations [11–13].

In general, WDs are crucial to understand the accelerated expansion of the universe in terms of type Ia supernova explosions, they can provide independent information about the age of our galaxy and their distribution contains information about star formation history and subsequent evolution. The progenitors of WDs evolve and age on the stage of the main sequence star losing carbon, nitrogen, oxygen, etc. For this very reason they supply a substantial input to the chemical evolution of our Galaxy and possibly they

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can be considered as a key source of life supporting chemical compounds [14].

Currently, there are three major equations of state (EoSs) for describing the degenerate matter of WDs: the classical Chandrasekhar EoS, the Salpeter EoS, and the relativistic Feynman–Metropolis–Teller (RFMT) EoS. The RFMT EoS generalizes the well-known Chandrasekhar and Salpeter EoSs, including the effects of the Coulomb interactions and the local inhomogeneities of the electron distribution within a full relativistic fashion. As a result, the masses of WDs are smaller and the radii are larger than those obtained from the Chandrasekhar's and Salpeter's EoSs. The principal differences, advantages and drawbacks among these EoSs are described in details in [9]. It should also be noted that the polytropic EoSs, widely used in the literature, are only the limiting cases of the Chandrasekhar or Salpeter EoSs in the non-relativistic and relativistic limits [1, 2].

Throughout the paper WDs are studied using the Chandrasekhar EoS [15, 16] at finite-temperatures in classical physics for the sake clarity and simplicity. A similar approach of the inclusion of finite-temperature effects in the RFMT EoS was analyzed in [17]. The main goal of the paper is to investigate the influence of both rotation and finite-temperatures on the structure of WDs. Accounting for such effects makes the theory of WDs be more realistic and practical [15, 16, 18–21].

2. THE CHANDRASEKHAR EQUATION OF STATE AT ZERO TEMPERATURE

The EoS of degenerate WD matter, in the simplest case, determines the dependence of the total pressure on the total energy density. The substance of WDs consists of electrons and positively charged ions (naked nuclei). The electrons are considered as a fully degenerate electron gas and they are described by the Fermi–Dirac statistics [22]. In the Chandrasekhar approximation, the distribution of electrons, as well as ions, is assumed to be locally constant [9]. Consequently, the condition of local charge neutrality is given by

$$n_e = \frac{Z}{A} n_N, \tag{1}$$

where n_e is the number density of electrons, Z is the number of protons, A is the average atomic weight (mass number), n_N is the number density of nucleons. In a fully degenerate case, all lower energy levels are filled up to some maximum level, called the Fermi level. The number density of the fully degenerate electron gas up to the Fermi level is defined as

$$n_e = \int_0^{p_e^F} \frac{2}{(2\pi\hbar)^3} d^3p = \frac{8\pi}{(2\pi\hbar)^3} \int_0^{p_e^F} p^2 dp = \frac{(p_e^F)^3}{3\pi^2\hbar^3}, \tag{2}$$

where p_e^F is the Fermi momentum of electron, $\hbar$ is the reduced Planck constant. According to the Chandrasekhar approximation the resulting pressure is due to the electron pressure P_e, while the pressure of positively charged nuclei P_N is insignificant, and the energy density is determined by the energy density of nuclei ε_N, while the energy density of degenerate electrons ε_e is negligibly small. Thus, the Chandrasekhar EoS is defined as [7]

$$\varepsilon_{Ch} = \varepsilon_N + \varepsilon_e \approx \varepsilon_N, \quad P_{Ch} = P_N + P_e \approx P_e. \tag{3}$$

The resulting energy of nucleons by definition is given as

$$\varepsilon_N = \frac{A}{Z} M_u c^2 n_e, \tag{4}$$

where $M_u = 1.66604 \times 10^{-24}$ g is the unified atomic mass unit. The ratio of the atomic number to the number of protons is usually denoted in the literature as $\mu = A/Z$ and all calculations in this paper were carried out by adopting $\mu = 2$ for simplicity. The total pressure of electrons is defined as

$$P_e = \frac{1}{3}\frac{2}{(2\pi\hbar)^3} \int_0^{p_e^F} \frac{c^2p^2}{\sqrt{c^2p^2 + m_e^2c^4}} 4\pi p^2 dp = \frac{m_e^4c^5}{8\pi^2\hbar^3}\Big[x_e\sqrt{1+x_e^2}(2x_e^2/3 - 1) + \ln(x_e + \sqrt{1+x_e^2})\Big], \tag{5}$$

where $x_e = p_e^F/(m_e c)$ is the dimensionless Fermi momentum and m_e is the electron mass [9].

3. THE CHANDRASEKHAR EQUATION OF STATE AT FINITE TEMPERATURES

In general, the expression for the electron number density follows from the Fermi–Dirac statistics and, when temperature is taken into account, it is determined as

$$n_e = \frac{2}{(2\pi\hbar)^3} \int_0^{\infty} \frac{4\pi p^2 dp}{\exp\left[\frac{E(p)-\mu_e(p)}{k_BT}\right] + 1}, \tag{6}$$

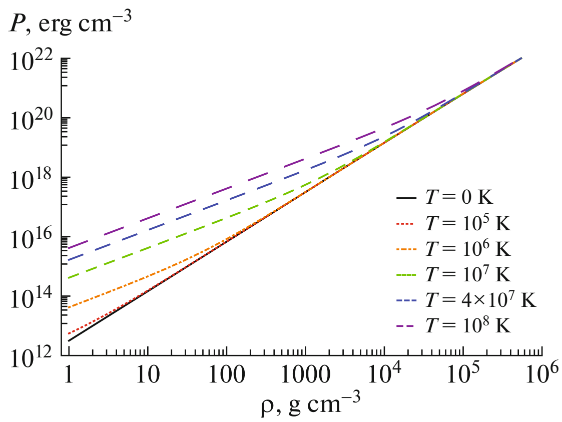

Fig. 1. Total pressure as a function of the mass density for selected temperatures in the range $T = (0\text{–}10^8)$K (colour online).

where $k_B = 1.38 \times 10^{-16}$ erg K^{-1} is the Boltzmann constant, T is the temperature, μ_e is the chemical potential, $E(p) = \sqrt{c^2p^2 + m_e^2c^4} - m_ec^2$ is the kinetic energy, p and m_e are the momentum and the rest mass of an electron, respectively.

Formula (2), taking into account the effects of finite temperatures, can be written in the following alternative form

$$n_e = \frac{8\pi\sqrt{2}}{(2\pi\hbar)^3} m^3 c^3 \beta^{3/2} \times \left[F_{1/2}(\eta,\beta) + \beta F_{3/2}(\eta,\beta)\right], \qquad (7)$$

where

$$F_k(\eta,\beta) = \int\limits_0^\infty \frac{t^k\sqrt{1+(\beta/2)t}}{1+e^{t-\eta}}dt \qquad (8)$$

is the relativistic Fermi–Dirac integral, $\eta = \mu_e/(k_BT)$, $t = E(p)/(k_BT)$ and $\beta = k_BT/(m_ec^2)$ are the degeneracy parameters [17, 23]. Consequently, the total electron pressure for $T \neq 0$ K is given by

$$P_e = \frac{2^{3/2}}{3\pi^2\hbar^3} m_e^4 c^5 \beta^{5/2} \times \left[F_{3/2}(\eta,\beta) + \frac{\beta}{2} F_{5/2}(\eta,\beta)\right]. \qquad (9)$$

The dependence of the total pressure on the total density Eq. (3) at various temperatures $T = (0, 10^5, 10^6, 10^7, 10^8)$ K is plotted in Fig. 1. As one can see, the effects of temperature become noticeable only at lower densities starting from 10^5 g cm^{-3}. For higher densities the thermal effects are negligible.

4. FORMALISM AND STABILITY CRITERIA FOR ROTATING WHITE DWARFS AT FINITE TEMPERATURES

It has been established that for WDs relativistic effects lead only to small perturbations of Newtonian gravity [24]. Consequently, Newton's theory in the low mass region allows one to study sufficiently well the essential physical features of WDs. We use the classical limit of the Hartle–Thorne formalism [25, 26] to analyze perturbatively the structural equations [27]. The basic idea consists in solving Newton's field equation

$$\nabla^2\Phi = 4\pi G\rho, \qquad (10)$$

and the structure equations

$$\frac{dP}{dr} = -\rho\frac{GM}{r^2}, \quad \frac{dM}{dr} = 4\pi r^2\rho, \qquad (11)$$

perturbatively by expanding the radial coordinate as $r = R + \xi$. The structure equations contain the hydrostatic equilibrium condition between gravitational and pressure forces, and the mass balance equation. Hence, here Φ is the gravitational potential, ρ is the matter density related to the energy density as $\varepsilon = c^2\rho$, P is the pressure, $M(r)$ is the mass inside a sphere with radius r, R is the radial coordinate for a spherical configuration and the function $\xi(R,\theta)$ takes into account the deviations from spherical symmetry due to the rotation of the star.

All the important quantities such as the total mass M, equatorial radius R_e, moment of inertia I, angular momentum J, quadrupole moment Q, etc. are then Taylor expanded up to the second order in the angular velocity. Within the Hartle approach, due to an appropriate choice of function ξ, the density ρ and pressure P can be treated as non affected by the rotation of the star. The field and structural equations (10) and (11) can then be integrated numerically to obtain all the important quantities in the preferred approximation [27].

For our analysis it is convenient to introduce the Keplerian angular velocity

$$\Omega_{Kep} = \sqrt{\frac{GM}{R_e^3}}, \qquad (12)$$

because it allows us to calculate all the fundamental parameters at the mass-shedding limit, and to determine the stability region inside which rotating configurations can exist [19].

Finally, the inverse β-decay instability determines the critical density which, in turn, defines the onset of instability for a WD to collapse into a NS. Thus the inverse β-decay instability is crucial both for static and rotating configurations. It represents one of the boundaries of the stability region for rotating

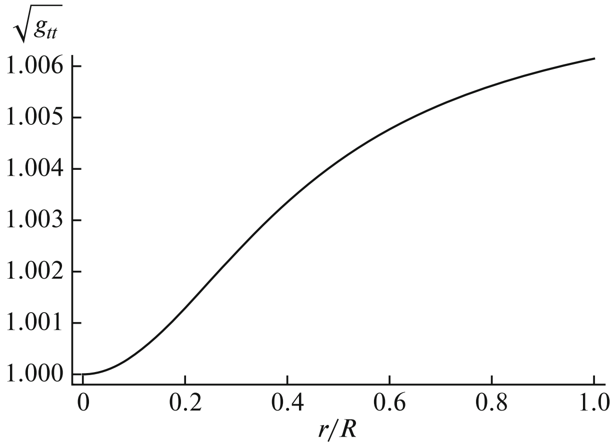

Fig. 2. $\sqrt{g_{tt}}$ as a function of the radial distance for a zero temperature white dwarf with mass $M = 1.44\ M_\odot$ and radius $R = 1000$ km.

WDs [19, 27]. According to [17], the occurrence of the inverse β-decay instability is not affected by the presence of temperature, i.e., it is the same as in the Chandrasekhar EoS $\rho_{crit} = 1.37 \times 10^{11}$ g cm^{-3}. This is related to the fact that the effects of temperature are negligible in the higher density regime.

For the sake of simplicity, throughout the paper we use a uniform temperature profile for isothermal cores of WDs, i.e. WDs without an outer envelop (atmosphere). The atmosphere serves as an isolator and its effect on the structure of WDs can be neglected in this approximation. In order to justify a constant temperature profile within the core, we considered the equilibrium condition for rotating hot relativistic stars, which is given by $T\sqrt{g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2} = constant$ [1], where g_{ik} are the components of the metric tensor in GR and Ω is the angular velocity of the star. For a static star the condition reduces to the well-known Tolman condition $T\sqrt{g_{tt}} = constant$ [28], where T is the local temperature.

In the classical limit $\sqrt{g_{tt}} \approx 1 - \Phi/c^2$, where $\Phi = \Phi(r)$ is the internal Newtonian gravitational potential found from Eq. (10). We constructed $\sqrt{g_{tt}}$ as a function of r/R for a WD with mass 1.44 $M_\odot$ and radius 1000 km in Fig. 2, as an example. One can see that the function $\sqrt{g_{tt}}$ changes slightly from the center to the surface of the isothermal WD core less than 1%. Hence, one can safely use the classical equilibrium condition $T = constant$ for hot WDs. This is the foremost argument to adopt the constant temperature profile. For low mass white dwarfs the function $\sqrt{g_{tt}}$ changes even less than in the previous case, since when the mass decreases, the radius increases and Φ decreases as well. Thus, for the cores of WDs the constant temperature profile is a sound assumption.

5. RESULTS AND DISCUSSION ABOUT ROTATING WHITE DWARFS AT FINITE TEMPERATURES

The Hartle formalism [25–27] was invoked in classical physics to calculate the sought parameters of uniformly rotating WDs employing the Chandrasekhar EoS at finite temperatures. The final results are depicted in Figs. 3 and 4.

Figure 3 shows the equatorial radius as a function of the central density and temperature for both rotating and static WDs. It is obvious that hot WDs possess larger radii than cold ones. For increasing central densities, WDs become more gravitationally bound and spherical. By examining only static WDs one can easily calculate the thickness of a hot non-degenerate layer on top of the cold degenerate one. Consequently, this effect translates also to rotating WDs.

Figure 4 shows the mass–radius relation for hot static and rotating WDs superposed over the estimated mass–radius data points from the Sloan Digital Sky Survey Data Release 4 [29] (brown points).

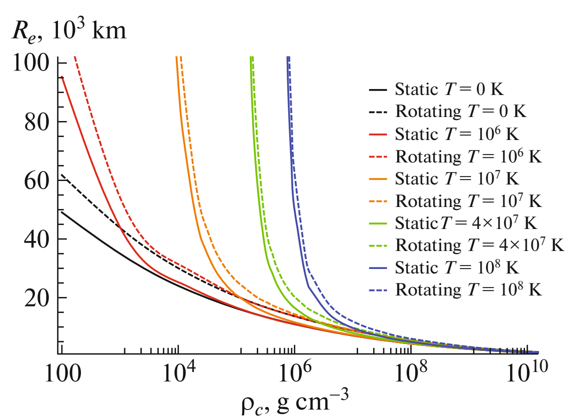

Fig. 3. Radius versus central density (colour online).

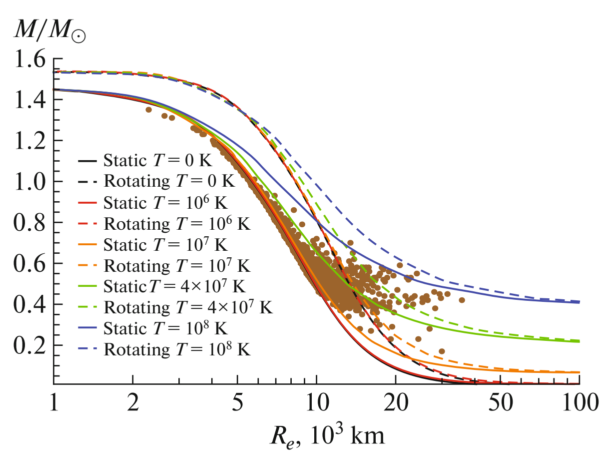

Fig. 4. Mass versus radius (colour online).

It is evident that this relation is very different from the degenerate case, in particular, for small masses and large radii, depending on the temperature of the isothermal core. The data points are consistent with the theoretical mass–radius relation.

From the astrophysical context the mass–radius relations for hot WDs play a pivotal role. As one can see from Fig. 4 for the fixed mass the radius of a WD can be diverse depending on the values of the rotation period and temperature. From observations, unlike the radius of stars, it is relatively easy to measure the mass. Therefore, the calculation of radius is a very delicate problem as the small corrections due to the rotation, GR and the effects of finite temperatures become more dominant in radius but not in mass [30].

In Figs. 3 and 4 rotating WDs are at the Keplerian sequence. All realistic uniformly rotating WDs will be in between the static and mass shedding limit for a fixed temperature. It should be noted, that here we consider only the temperature of the WD isothermal core T_c. The interrelation of the core temperature and the observed effective surface temperature T_{eff} is given via the Koester relation as $T_{eff}^4/g = 2.05 \times 10^{-10} T_c^{2.56}$, where g is the surface gravity [31]. By employing the Koester formula one can show easily that our calculations are compatible and consisted with the observational data for WDs [17, 29].

6. CONCLUSION

Mass–radius and radius–central density relations of static and rotating, cold and hot WDs were calculated using the Chandrasekhar EoS in classical physics. The effects of finite temperatures were accounted for in the EoS. The effects of rotation, such as the deformation of a star, extra mass due to the balance of the centrifugal force and gravity, were investigated within the Hartle formalism.

It was shown that in the construction of a realistic model of WDs the effects of finite temperatures and rotation must be accounted for self-consistently. It was demonstrated that for low-mass WDs the effects of temperature are more prominent. Instead, the rotation affects the structure of WDs in all mass ranges. Consequently, rotation gives an additional degree of freedom for cold and hot white dwarfs, as expected. The mass–radius relations obtained in this work are consistent with observations [29].

Moreover, unlike in previous studies, here the effects of rotation and finite temperatures were considered together in all our calculations. Namely, we considered the temperatures of the isothermal cores of white dwarfs. For comparison with the observed surface temperatures of white dwarfs, the Koester formula must be used, which establishes the interrelation between the effective surface temperature and the temperature of the isothermal cores.

The astrophysical implications of rotating cold and hot WDs are widespread [32–36]. It is clear that the inclusion of the magnetic field and nuclear composition will broaden the applications of WDs to further extent [37–44]. Therefore, it would be interesting to continue our research taking into account the nuclear composition of the WDs matter along with rotation, temperature and magnetic field. This problem will be considered in our future investigations.

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