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The Magnetic-Field Structure in a Stationary Accretion Disk

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Abstract—The magnetic-field structure in regions of stationary, planar accretion disks around active galactic nuclei where general-relativistic effects can be neglected (from 10 to 200 gravitational radii) is considered. It is assumed that the magnetic field in the outer edges of the disk, which forms in the magnetosphere of the central black hole during the creation of the relativisitic jets, corresponds to the field of a magnetic dipole perpendicular to the plane of the disk. In this case, the azimuthal field component B_φ in the disk arises due to the presence of the radial field B_ρ and the azimuthal velocity component U_φ. The value of the magnetic field at the inner radius of the disk is taken to correspond to the solution of the induction equation in a diffusion approximation. Numerical solutions of the induction equation are given for a number of cases.

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1. INTRODUCTION

The magnetic-field structure in a stationary accretion disk is determined by boundary conditions and the distribution of the U_z, U_ρ, and U_φ velocities of the accreting material. We used a cylindrical coordinate system where the z axis is perpendicular to the plane of the accretion disk, ρ is the distance from the z axis in the plane of the disk, and φ is the azimuthal angle. We considered the magnetic-field structure in a planar accretion disk around a supermassive black hole at the center of an active galactic nucleus. Various models for such accretion disks are concerned with the values for these velocities and their dependence on z and ρ. The aim of our study was to develop a comparatively simple, partially analytical and partially numerical method for computing the magnetic-field components corresponding to given velocity components $U_z(z,\rho)$, $U_\rho(z,\rho)$, and $U_\varphi(z,\rho)$.

The distributions of the density, temperature, and pressure of the disk matter are described in the standard accretion-disk model of Shakura and Syunyaev [1]. Various models for accretion disks around neutron stars were presented in the 1970s (see, e.g., [2]). Most subsequent accretion-disk models [3–8] also describe the evolution of the magnetic field, and, in particular, the transition of the disk to a stationary state. Usually, numerical solutions of a full system of magnetohydrodynamical equations are presented.

The magnetic field $\mathbf{B}(z,\rho,\varphi)$ is usually described in terms of the vector potential $\mathbf{A}(z,\rho,\varphi)$, ($\mathbf{B} = \nabla \times \mathbf{A}$), which is usually taken to have the form $\mathbf{A}(z,\rho) = \Psi(z,\rho)\mathbf{e}_\varphi$. In this case, $B_z = \rho^{-1}\partial(\rho\Psi)/\partial\rho$, $B_\rho = -\partial\Psi/\partial z$, $B_\varphi = 0$.

In models for a thin accretion disk ($B_z(\rho,z) \approx B_z(\rho)$), one boundary condition is provided by the assumption that the disk is located in a uniform external magnetic field $B_z^{(0)}$. In this case, $\Psi_\infty(\rho) = B_z^{(0)}\rho/2$ [9]. According to [6], the condition $\Psi_\infty(\rho) = B_z^{(0)}\rho/2$ is satisfied only in the limit $z \to \infty$, while a component $B_\rho(h)$ of the external magnetic field also exists at the disk surface ($z = h$). The solution of the Biot–Savart equations for the potential $\Psi(z,\rho)$ lead to the dependence $B_{z,\max} \sim \rho^{-2}B_z^{(0)}$. The component $B_\rho(z,\rho)$ is taken to be $B_\rho(\rho,z) \approx B_\rho(h)z/h$.

It was assumed in [10] that the magnetic field near the black hole could be presented as the sum of a jet-like field along the disk rotation axis and a poloidal field in the disk at $\rho_{in} < \rho < \rho_{out}$. It was also assumed that $B_z(h,\rho) \sim \rho^{-n}$, with various values $n = 3, 4, 4.5$. The evolution of the magnetic field for the case when the accretion disk was located in the dipolar magnetic field of a young star was considered in [4]. Reviews of studies of accretion disks are presented in [11, 12]. Numerical solutions to the magnetohydrodynamical equations with a number of specific conditions imposed at the disk boundary are given in [12], where it was assumed that the radial velocity component U_ρ was zero at the boundary of

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the disk. It is interesting that Bisnovatyi—Kogan and Lovelace [12] present the characteristic value $\beta_c \sim \rho$ for stationary accretion (β is the ratio of the plasma pressure to the magnetic pressure).

We considered a stationary induction equation for the magnetic field, assuming that the magnetic field at the boundary of the planar accretion disk corresponds to the field of a magnetic dipole oriented perpendicular to the plane of the disk along the z axis. In this case, both the normal component $B_z \equiv B_{\|}$ and the horizontal component B_ρ of the magnetic field at the boundary are given by the corresponding components of the magnetic-dipole field. The azimuthal component of the dipolar magnetic field B_φ at the disk boundary is zero. The appearance of an azimuthal field B_φ in the disk is related mainly to winding of the component B_ρ due to the azimuthal velocity U_φ.

The central problem is the assumption that the magnetic field at the boundary of a planar accretion disk around a black hole is dipolar. At first glance, this contradicts the assertion that "a black hole has no hair;" i.e., it cannot possess its own magnetic field. However, the black hole possesses a magnetosphere, whose radius can exceed the event-horizon radius. A model of a force-free, stationary, axially symmetric magnetosphere of a Kerr black hole with a poloidal magnetic field was developed in [13]. The extraction of energy from a rotating black hole, leading to the generation of the observed powerful relativistic jets, is possible precisely in this case. The magnetospheric currents that generate the magnetic field are proportional to the angular velocity of the rotation of the black hole in this case. Numerical methods for computing such a magnetosphere were developed in [14]. This yielded various magnetic-field configurations, including dipolar [7].

We considered the simplest case when there are no electric currents near the disk boundary. In this case, both the $B_\rho(h, \rho)$ and $B_z(h, \rho)$ components correspond to those of a dipolar magnetic field. Note that we did not consider the amplification of the magnetic field by the dynamo mechanism.

2. INDUCTION EQUATION

In general, the induction equation for the magnetic field depends on time, and has the form [15, 16]

$$\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) \qquad (1)$$
$$+ \nabla \times \alpha_t \mathbf{B} - \nabla \times (D_\sigma + D_t) \nabla \times \mathbf{B}.$$

Here, $\mathbf{B}(\mathbf{r}, t)$ is the vector magnetic field, $D_\sigma + D_t$ the total diffusion coefficient, equal to the sum of the diffusion coefficients $D_\sigma = c^2/4\pi\sigma$, where σ is the electrical conductivity of the plasma and c is the speed of light, and D_t, due to turbulent motions of the gas. The diffusion coefficient D_t is related to the characteristic scale of the turbulent motions R_0, the characteristic velocity of turbulent vortices U_0, and the characteristic decay time for these vortices τ_0. If $U_0\tau_0/R_0 \ll 1$, then $D_t \approx R_0^2/\tau_0$. By virtue of the symmetry of the problem, D_t can depend on the distance ρ, $D_t = D_t(\rho)$. For the case $U_0\tau_0/R_0 \gg 1$, $D_t \approx U_0 R_0$ [16, 17]. The coefficient $\alpha_t \sim \langle \mathbf{u}\nabla \times \mathbf{u} \rangle$ describes the amplification of the magnetic field (the "α effect"). Here, $\mathbf{u}$ is the fluctuational component of the velocity field and $\mathbf{U}$ is the mean velocity ($\langle \mathbf{u} \rangle = 0$). Further, we will consider the case when there is no α effect; i.e., $\alpha_t = 0$. Moreover, we assumed that $D_\sigma \ll D_t$. A similar treatment of this question is given in [15]. Recall that $\nabla \cdot \mathbf{B} = 0$. We considered the induction equation in a kinematic approximation, where the kinetic energy of the gas is much greater than the energy of the magnetic field ($\rho_{\text{gas}} U^2/2 \gg B^2/(8\pi)$), where ρ_{gas} is the density of the disk matter. The possible realization of this situation in a standard accretion disk was considered in [18], based on an analysis of data on the wavelength dependence of the observed degree of polarization for a number of active galactic nuclei [19]. In this approximation, we can neglect the dependence of the gas velocity on the magnetic field. The stationary nature of the gas flows means that we can neglect the term $\partial \mathbf{B}/\partial t$. Physically, stationary gas flows are realized far from the central parts of the accretion disk when the magnetic-dipole moment is oriented perpendicular to the plane of the disk (the magnetic field is not stationary when the magnetic moment is oriented at some angle to the disk). We have in our case

$$\nabla^2 \mathbf{B}(\mathbf{r}) = \frac{1}{D_t}(-\text{rot}(\mathbf{U} \times \mathbf{B}) \qquad (2)$$
$$+ \nabla D_t \times \text{rot}\mathbf{B}) = \frac{1}{D_t}(\mathbf{B}\text{div}\mathbf{U} + (\mathbf{U}\nabla)\mathbf{B}$$
$$- (\mathbf{B}\nabla)\mathbf{U} + \nabla D_t \times \text{rot}\mathbf{B}) \equiv \mathbf{S}(\mathbf{r}).$$

We used a cylindrical coordinate system with the z axis perpendicular to the plane of the disk, ρ being the distance from the center of the disk in the plane of the disk, and φ being the azimuthal angle. In the axially symmetric case we considered, all quantities depend only on z and ρ. In this case, Eq. (2) can be written in terms of the components B_z, B_ρ, and B_φ in the following form:

$$\left(\frac{\partial^2}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} \right) B_z \qquad (3)$$
$$= \frac{1}{D_t(\rho)}\left[B_z \frac{U_\rho}{\rho} + \frac{\partial}{\partial \rho}(B_z U_\rho) + U_z \frac{\partial B_z}{\partial z} \right.$$

$$- B_\rho \frac{\partial U_z}{\partial \rho} - \frac{\partial D_t(\rho)}{\partial \rho}\left(\frac{\partial B_z}{\partial \rho} - \frac{\partial B_\rho}{\partial z}\right)\Bigg] \equiv S_z(z,\rho),$$

$$\left(\frac{\partial^2}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} - \frac{1}{\rho^2}\right) B_\rho = \frac{1}{D_t(\rho)}\left(B_\rho \frac{U_\rho}{\rho}\right.$$

$$\left. + \frac{\partial}{\partial z}(B_\rho U_z) + U_\rho \frac{\partial B_\rho}{\partial \rho} - B_z \frac{\partial U_\rho}{\partial z}\right) \equiv S_\rho(z,\rho),$$

$$\left(\frac{\partial^2}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} - \frac{1}{\rho^2}\right) B_\varphi$$

$$= \frac{1}{D_t(\rho)}\left(B_\rho \frac{U_\varphi}{\rho} + \frac{\partial}{\partial z}(B_\varphi U_z) + \frac{\partial}{\partial \rho}(U_\rho B_\varphi)\right.$$

$$\left. - B_z \frac{\partial U_\varphi}{\partial z} - B_\rho \frac{\partial U_\varphi}{\partial \rho} - \frac{\partial D_t(\rho)}{\partial \rho}\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\varphi)\right)$$

$$\equiv S_\varphi(z,\rho).$$

Murphy and Pessah [20] show that the turbulence in accretion disks has an anisotropic character. However, we assume in our model that the diffusion coefficient can be taken to be approximately isotropic.

3. GENERAL SOLUTION OF EQ. (2) IN THE AXIALLY SYMMETRIC CASE

Using a Hankel transform, we can write formula (3) in the form

$$\left(\frac{d^2}{dz^2} - x^2\right) \tilde{B}_{z,\rho,\varphi}(z,x) = \tilde{S}_z(z,x), \tilde{S}_{\rho,\varphi}(z,x), \tag{4}$$

where

$$\tilde{B}_z(z,x) = \int_0^\infty d\rho\rho J_0(x\rho) B_z(z,\rho), \tag{5}$$

$$\tilde{B}_{\rho,\varphi}(z,x) = \int_0^\infty d\rho\rho J_1(x\rho) B_{\rho,\varphi}(z,\rho). \tag{6}$$

Here, $J_0(x\rho)$ and $J_1(x\rho)$ are Bessel functions of zero and first order. The inverse Hankel transforms determine $B_z(z,\rho)$, $B_\rho(z,\rho)$, and $B_\varphi(z,\rho)$:

$$B_{\rho,\varphi}(z,\rho) = \int_0^\infty dx x J_1(x\rho) \tilde{B}_{\rho,\varphi}(z,x), \tag{7}$$

$$B_z(z,\rho) = \int_0^\infty dx x J_0(x\rho) \tilde{B}_{z,x}(z,x).$$

Substituting (7) into (5) and (6) leads to the known relation

$$\int_0^\infty d\rho\rho J_n(x\rho) J_n(x'\rho) = \delta(x - x')/x, \tag{8}$$

where $\delta(x - x')$ is the Dirac delta function and $n = 0, 1, \ldots$.

The upper surface of the accretion disk is located at $z = h$, and the lower surface at $z = -h$. The magnetic dipole moment $\mathbf{m} = m\mathbf{e}_z$ is directed along the z axis. The dipolar magnetic field is given by the formula

$$\mathbf{B}(z,\rho) = m(3z\mathbf{r} - \mathbf{e}_z r^2)/r^5, \quad r^2 = \rho^2 + z^2. \tag{9}$$

The values of the magnetic-field components at the $z = h$ and $z = -h$ planes are

$$B_z(\pm h,\rho) = m\frac{2h^2 - \rho^2}{(h^2 + \rho^2)^{5/2}}, \quad B_\rho(\pm h,\rho) = \pm m\frac{3h\rho}{(h^2 + \rho^2)^{5/2}}. \tag{10}$$

Using tables from [21, 22], we obtain

$$\tilde{B}_z(\pm h,x) = mxe^{-xh}, \quad \tilde{B}_\rho(\pm h,x) = \pm mxe^{-hx}. \tag{11}$$

According to the general theory of boundary-value problems [23, 24], the solution of (4) can be represented as a sum of the solution of the homogeneous equation (i.e., $\mathbf{S} = 0$) that satisfies the boundary conditions at $z = \pm h$ and the solution of the inhomogeneous equation ($\mathbf{S} \neq 0$) that is equal to zero at the boundary; i.e.,

$$\tilde{\mathbf{B}}(z,x) = \tilde{\mathbf{B}}_{\text{hom}}(z,x) + \tilde{\mathbf{B}}_{\text{inhom}}(z,x), \tag{12}$$

with

$$\tilde{\mathbf{B}}_{\text{hom}}(\pm h,x) = \tilde{\mathbf{B}}_{\text{dipole}}(\pm h,x), \quad \tilde{\mathbf{B}}_{\text{inhom}}(\pm h,x) = 0. \tag{13}$$

It can easily be verified that

$$\tilde{B}_{z,\text{hom}}(z,x) = m\frac{\cosh xz}{\cosh xh}xe^{-hx}, \quad \tilde{B}_{\rho,\text{hom}}(z,x) = m\frac{\sinh xz}{\sinh xh}xe^{-hx}. \tag{14}$$

According to (7) and (14), the explicit forms of

$B_{z,\text{hom}}(z,\rho)$ and $B_{\rho,\text{hom}}(z,\rho)$ can be expressed as integrals:

$$B_{z,\text{hom}}(z,\rho) = m\int\limits_0^\infty dx x^2 J_0(x\rho)\frac{\cosh xz}{\cosh xh}e^{-hx}, \quad (15)$$
$$B_{\rho,\text{hom}}(z,\rho) = m\int\limits_0^\infty dx x^2 J_1(x\rho)\frac{\sinh xz}{\sinh xh}e^{-hx}.$$

The value of $\tilde{\mathbf{B}}_{\text{inhom}}(z,x)$ can be expressed in terms of a Green's function $\tilde{G}(z,z',x)$:

$$\tilde{\mathbf{B}}_{\text{inhom}}(z,x) = \int\limits_{-h}^{h} dz'\tilde{G}(z,z',x)\tilde{\mathbf{S}}(z',x), \quad (16)$$

where the Green's function has the form

$$\tilde{G}(z,z',x) \equiv \tilde{G}(z',z,x) \quad (17)$$
$$= \frac{\theta(z-z')\sinh x(z-h)\sinh x(z'+h) + \theta(z'-z)\sinh x(z+h)\sinh x(z'-h)}{x\sinh 2xh}.$$

Here, $\theta(z)$ is a step function,

$$\theta(z>0) = 1, \quad \theta(z<0) = 0, \quad (18)$$
$$\frac{d}{dz}\theta(z) = \delta(z).$$

It can easily be verified that $\tilde{G}(z,z',x)$ satisfies the equation

$$\left(\frac{d^2}{dz^2} - x^2\right)\tilde{G}(z,z',x) = \delta(z-z') \quad (19)$$

and the boundary condition $\tilde{G}(\pm h,z',x) = 0$.

Using the transformations (5), (6), and (7) yields a formula for $\mathbf{B}_{\text{inhom}}(z,\rho)$ in the form

$$\mathbf{B}_{\text{inhom}}(z,\rho) \quad (20)$$
$$= \int\limits_{-h}^{h} dz' \int\limits_0^\infty d\rho'\rho' G(z,z',\rho,\rho')\mathbf{S}(z',\rho'),$$

where the Green's function $G(z,z',\rho,\rho')$ is

$$G(z,z',\rho,\rho') \quad (21)$$
$$= \int\limits_0^\infty dx x J_0(x\rho)J_0(x\rho')\tilde{G}(z,z',x)$$

for calculations of $B_z(z,\rho)$ and

$$G(z,z',\rho,\rho') \quad (22)$$
$$= \int\limits_0^\infty dx x J_1(x\rho)J_1(x\rho')\tilde{G}(z,z',x)$$

for calculations of $B_{\rho,\varphi}(z,\rho)$. We then add $\tilde{\mathbf{B}}_{\text{hom}}(z,x)$ and $\mathbf{B}_{\text{hom}}(z,\rho)$ to the right-hand and left-hand sides of (16) and (20). This yields the following integral equations for $\tilde{\mathbf{B}}(z,x) = \tilde{\mathbf{B}}_{\text{hom}}(z,x) + \tilde{\mathbf{B}}_{\text{inhom}}(z,x)$:

$$\tilde{\mathbf{B}}(z,x) = \tilde{\mathbf{B}}_{\text{hom}}(z,x) \quad (23)$$
$$+ \int\limits_{-h}^{h} dz'\tilde{G}(z,z',x)\tilde{\mathbf{S}}(z',x),$$

and for $\mathbf{B}(z,\rho)$:

$$\mathbf{B}(z,\rho) = \mathbf{B}_{\text{hom}}(z,\rho) \quad (24)$$
$$+ \int\limits_{-h}^{h} dz' \int\limits_0^\infty d\rho'\rho' G(z,z',\rho,\rho')\mathbf{S}(z',\rho').$$

It is easy to see that the components $\tilde{\mathbf{B}}(z,x)$ and $\mathbf{B}(z,\rho)$ satisfy Eqs. (4) and (3), respectively. The boundary conditions are satisfied both due to the presence of the free terms $\tilde{\mathbf{B}}_{\text{hom}}(z,x)$ and $\mathbf{B}_{\text{hom}}(z,\rho)$ and because the Green's function is zero at the boundaries $z = \pm h$.

Note that, in the approximation $hx \ll 1$ ($\rho \gg h$), the Green's function $\tilde{G}(z,z',x)$ acquires the form

$$\tilde{G}(z,z',x) \approx \frac{1}{2h}[\theta(z-z')(z-h)(z'+h) \quad (25)$$
$$+ \theta(z'-z)(z+h)(z'-h)] \equiv G(z,z');$$

i.e., it does not depend on x. Using (8), we can obtain the asymptotic formula for $G(z,z',\rho,\rho')$:

$$G(z,z',\rho,\rho') = G(z,z')\frac{\delta(\rho-\rho')}{\rho}. \quad (26)$$

At large distances ($\rho \gg h$), Eq. (24) takes the form

$$\mathbf{B}(z,\rho) = \mathbf{B}_{\text{hom}}(z,\rho) + \int\limits_{-h}^{h} dz' G(z,z')\mathbf{S}(z',\rho). \quad (27)$$

Strictly speaking, this is not an integral equation, since it is satisfied only for large ρ. However, it can be used to estimate the z-averaged magnetic field values for geometrically thin disks (see Section 6).

In general, it is usually simpler to solve the integral equation (24) than the system of differential equations (3). However, in the model we consider further, in which the z dependences of all quantities are neglected, the second equation of (3) is transformed into a separate differential equation for B_ρ, substantially simplifying the solution of the entire system of equations. Therefore, we chose this system of differential equations for our numerical solutions.

4. EXPLICIT EXPRESSIONS FOR $B_{z,\text{hom}}(z,\rho)$ and $B_{\rho,\text{hom}}(z,\rho)$

To obtain explicit formulas for $B_{z,\text{hom}}(z,\rho)$ and $B_{\rho,\text{hom}}(z,\rho)$, we used the Hankel transformations (5) and (6) and the known values of the integrals [21, 22]:

$$\int\limits_0^\infty dx x^2 J_1(x\rho)e^{-ax} = \frac{3a\rho}{(a^2+\rho^2)^{5/2}}, \quad (28)$$

$$\int\limits_0^\infty dx x^2 J_0(x\rho)e^{-ax} = \frac{2a^2-\rho^2}{(a^2+\rho^2)^{5/2}}.$$

Substituting $\tilde{B}_{z,\text{hom}}(z,x)$ [see (14)] into (5) and expanding the $\cosh xh$ function in powers of e^{-xh}, we obtained the following formula for $B_{z,\text{hom}}(z,\rho)$:

$$B_{z,\text{hom}}(z,\rho) = m\Bigg\{\left[\frac{2(2h-z)^2-\rho^2}{((2h-z)^2+\rho^2)^{5/2}} + \frac{2(2h+z)^2-\rho^2}{((2h+z)^2+\rho^2)^{5/2}}\right] - \left[\frac{2(4h-z)^2-\rho^2}{((4h-z)^2+\rho^2)^{5/2}} + \frac{2(4h+z)^2-\rho^2}{((4h+z)^2+\rho^2)^{5/2}}\right] + \left[\frac{2(6h-z)^2-\rho^2}{((6h-z)^2+\rho^2)^{5/2}} + \frac{2(6h+z)^2-\rho^2}{((6h+z)^2+\rho^2)^{5/2}}\right] - \ldots\Bigg\}. \quad (29)$$

The expression in square brackets is symmetric relative to the substitution $z \to -z$. This leads to the property $B_{z,\text{hom}}(-z,\rho) = B_{z,\text{hom}}(z,\rho)$. For $z = \pm h$, we obtain the first expression in (10). When $z = 0$, formula (29) simplifies as follows:

$$B_{z,\text{hom}}(0,\rho) \quad (30)$$

$$= 2m\Bigg\{\frac{2(2h)^2-\rho^2}{((2h)^2+\rho^2)^{5/2}} - \frac{2(4h)^2-\rho^2}{((4h)^2+\rho^2)^{5/2}} + \frac{2(6h)^2-\rho^2}{((6h)^2+\rho^2)^{5/2}} - \ldots\Bigg\}.$$

We can obtain an expression for $B_{\rho,\text{hom}}(z,\rho)$ in the same way:

$$B_{\rho,\text{hom}}(z,\rho) = 3m\rho\Bigg\{\frac{2h-z}{((2h-z)^2+\rho^2)^{5/2}} - \frac{2h+z}{((2h+z)^2+\rho^2)^{5/2}} + \frac{4h-z}{((4h-z)^2+\rho^2)^{5/2}} - \frac{4h+z}{((4h+z)^2+\rho^2)^{5/2}} + \ldots\Bigg\}. \quad (31)$$

The results of the computations using formulas (29) and (31) are presented in Tables 1 and 2.

Equation (3) for B_φ yields the estimate

$$B_\varphi(z,\rho) \approx B_{\rho,\text{hom}}(z,\rho)U_\varphi/\rho. \quad (32)$$

5. MODEL OF A THIN ACCRETION DISK

Many studies have neglected the dependences of quantities on the z coordinate inside the accretion disk—the so-called thin-disk model. Physically, this means that the values of all quantities averaged over the thickness of the disk are used, in particular the averaged magnetic field. The averaged values of the solutions (14) acquire the form

$$\langle\tilde{B}_{z,\text{hom}}(z,x)\rangle_z = \frac{m}{2h}\int\limits_{-h}^{h} dz \frac{\cosh xz}{\cosh xh} x e^{-xh} \quad (33)$$

$$= \frac{m}{h}\tanh(xh)e^{-xh} = \langle B_{z,\text{hom}}(x)\rangle.$$

This means of using $\mathbf{B}_{\text{hom}}(z,\rho)$ at the inner boundary of the disk leaves $\mathbf{B}(z,\rho)$ virtually unchanged at small distances from the inner boundary.

$$\langle B_{\rho,\text{hom}}(z,\rho)\rangle \equiv 0, \quad \langle\tilde{B}_{\rho,\text{hom}}(z,x)\rangle \equiv 0. \quad (34)$$

$\langle B_{\rho,\text{hom}}(\rho)\rangle$ is equal to zero by virtue of the anti-symmetry of the boundary values $B_{\rho,\text{hom}}(h,\rho) = -B_{\rho,\text{hom}}(-h,\rho)$. In particular, in the mid-plane of the accretion disk, $B_{\rho,\text{hom}}(0,\rho) = 0$. Averaging of the anti-symmetric quantity $B_{\rho,\text{hom}}(z,\rho) = -B_{\rho,\text{hom}}(-z,\rho)$ leads to a zero average value. Using formula (7), we obtain for the averaged magnetic field

$$\langle B_{z,\text{hom}}(\rho)\rangle \quad (35)$$

$$= \frac{m}{h^3}\left[\frac{1}{(1+\bar{\rho}^2)^{3/2}} - 2\left(\frac{3}{(3^2+\bar{\rho}^2)^{3/2}} - \frac{5}{(5^2+\bar{\rho}^2)^{3/2}} + \frac{7}{(7^2+\bar{\rho}^2)^{3/2}} - \ldots\right)\right],$$

Table 1. $B_z(\bar{z}, \bar{\rho})$ (in kG), with $\bar{z} = z/h$ and $\bar{\rho} = \rho/h$ calculated using (29) with $m = 4.4 \times 10^{45}$ G cm^3

$\bar{\rho}\backslash\bar{z}$	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
10.0	−1.999	−1.999	−1.996	−1.992	−1.985	−1.977	−1.968	−1.956	−1.943	−1.928	−1.912
11.0	−1.503	−1.502	−1.500	−1.497	−1.494	−1.489	−1.483	−1.476	−1.467	−1.458	−1.448
12.0	−1.157	−1.157	−1.156	−1.154	−1.152	−1.148	−1.144	−1.140	−1.134	−1.128	−1.122
13.0	−0.910	−0.910	−0.909	−0.908	−0.906	−0.904	−0.902	−0.899	−0.895	−0.891	−0.886
14.0	−0.729	−0.729	−0.728	−0.727	−0.726	−0.725	−0.723	−0.721	−0.718	−0.715	−0.712
15.0	−0.593	−0.592	−0.592	−0.591	−0.591	−0.590	−0.588	−0.587	−0.585	−0.583	−0.581
16.0	−0.488	−0.488	−0.488	−0.488	−0.487	−0.486	−0.485	−0.484	−0.483	−0.481	−0.480
17.0	−0.407	−0.407	−0.407	−0.406	−0.406	−0.405	−0.405	−0.404	−0.403	−0.402	−0.401
18.0	−0.343	−0.343	−0.343	−0.343	−0.342	−0.342	−0.341	−0.341	−0.340	−0.339	−0.338
19.0	−0.292	−0.292	−0.291	−0.291	−0.291	−0.291	−0.290	−0.290	−0.289	−0.289	−0.288
20.0	−0.250	−0.250	−0.250	−0.250	−0.250	−0.249	−0.249	−0.249	−0.248	−0.248	−0.247
21.0	−0.216	−0.216	−0.216	−0.216	−0.216	−0.215	−0.215	−0.215	−0.215	−0.214	−0.214
22.0	−0.188	−0.188	−0.188	−0.188	−0.188	−0.187	−0.187	−0.187	−0.187	−0.186	−0.186
23.0	−0.164	−0.164	−0.164	−0.164	−0.164	−0.164	−0.164	−0.164	−0.163	−0.163	−0.163
24.0	−0.145	−0.145	−0.145	−0.145	−0.144	−0.144	−0.144	−0.144	−0.144	−0.144	−0.144
25.0	−0.128	−0.128	−0.128	−0.128	−0.128	−0.128	−0.128	−0.128	−0.127	−0.127	−0.127
26.0	−0.114	−0.114	−0.114	−0.114	−0.114	−0.114	−0.114	−0.113	−0.113	−0.113	−0.113
27.0	−0.102	−0.102	−0.102	−0.102	−0.102	−0.101	−0.101	−0.101	−0.101	−0.101	−0.101
28.0	−0.091	−0.091	−0.091	−0.091	−0.091	−0.091	−0.091	−0.091	−0.091	−0.091	−0.091
29.0	−0.082	−0.082	−0.082	−0.082	−0.082	−0.082	−0.082	−0.082	−0.082	−0.082	−0.082
30.0	−0.074	−0.074	−0.074	−0.074	−0.074	−0.074	−0.074	−0.074	−0.074	−0.074	−0.074

where $\bar{\rho} = \rho/h$.

Naturally, the averaged values $B_{z,\text{hom}}(\rho)$ do not coincide with the boundary values $B_z(\pm h, \rho)$.

6. AVERAGING OF THE GREEN'S FUNCTIONS

The use of a thin-disk model (neglecting the z dependences) requires the use of averaged Green's functions:

$$\langle G(z, z', \rho, \rho')\rangle_z \tag{36}$$
$$= \frac{1}{2h}\int_{-h}^{h} dz G(z, z', \rho, \rho') \equiv \langle G(z', \rho, \rho')\rangle,$$
$$\langle \tilde{G}(z, z', x)\rangle_z$$
$$= \frac{1}{2h}\int_{-h}^{h} dz \tilde{G}(z, z', x) \equiv \langle \tilde{G}(z', x)\rangle.$$

Using the explicit form of the Green's function $G(z, z', x)$ [see (17)], we obtain

$$\langle \tilde{G}(z, z', x)\rangle_z \equiv \langle \tilde{G}(z', x)\rangle \tag{37}$$
$$= -\frac{1}{2hx^2}\left(1 - \frac{\cosh xz'}{\cosh xh}\right).$$

Note that $\langle \tilde{G}(z', x)\rangle$ is an even function of z'. When averaging the Green's function $G(z, z', \rho, \rho')$ over z, we must use the averaged value of the right-hand side of the induction equation. In this case, the Green's function must be averaged over both z and z'. This yields

$$\langle\langle \tilde{G}(z, z', x)\rangle\rangle_{z,z'} \equiv \langle\langle \tilde{G}(x)\rangle\rangle \tag{38}$$
$$= -\frac{1}{2hx^2}\left(1 - \frac{\tanh xh}{xh}\right).$$

Table 2. $B_\rho(\bar{z}, \bar{\rho})$ (in kG), with $\bar{z} = z/h$ and $\bar{\rho} = \rho/h$ calculated using (31) with $m = 4.4 \times 10^{45}$ G cm^3

$\bar{\rho}\backslash\bar{z}$	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
10.0	0.000	0.060	0.120	0.180	0.239	0.298	0.357	0.415	0.472	0.529	0.585
11.0	0.000	0.041	0.082	0.123	0.163	0.204	0.244	0.284	0.324	0.363	0.401
12.0	0.000	0.029	0.058	0.087	0.115	0.144	0.173	0.201	0.229	0.257	0.284
13.0	0.000	0.021	0.042	0.063	0.084	0.105	0.125	0.146	0.166	0.187	0.207
14.0	0.000	0.016	0.031	0.047	0.062	0.078	0.093	0.109	0.124	0.139	0.154
15.0	0.000	0.012	0.024	0.036	0.047	0.059	0.071	0.083	0.094	0.106	0.117
16.0	0.000	0.009	0.018	0.027	0.037	0.046	0.055	0.064	0.073	0.082	0.091
17.0	0.000	0.007	0.014	0.022	0.029	0.036	0.043	0.050	0.057	0.064	0.071
18.0	0.000	0.006	0.011	0.017	0.023	0.029	0.034	0.040	0.045	0.051	0.057
19.0	0.000	0.005	0.009	0.014	0.018	0.023	0.028	0.032	0.037	0.041	0.046
20.0	0.000	0.004	0.007	0.011	0.015	0.019	0.022	0.026	0.030	0.034	0.037
21.0	0.000	0.003	0.006	0.009	0.012	0.015	0.018	0.022	0.025	0.028	0.031
22.0	0.000	0.003	0.005	0.008	0.010	0.013	0.015	0.018	0.020	0.023	0.025
23.0	0.000	0.002	0.004	0.006	0.009	0.011	0.013	0.015	0.017	0.019	0.021
24.0	0.000	0.002	0.004	0.005	0.007	0.009	0.011	0.013	0.014	0.016	0.018
25.0	0.000	0.002	0.003	0.005	0.006	0.008	0.009	0.011	0.012	0.014	0.015
26.0	0.000	0.001	0.003	0.004	0.005	0.007	0.008	0.009	0.010	0.012	0.013
27.0	0.000	0.001	0.002	0.003	0.005	0.006	0.007	0.008	0.009	0.010	0.011
28.0	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010
29.0	0.000	0.001	0.002	0.003	0.003	0.004	0.005	0.006	0.007	0.008	0.008
30.0	0.000	0.001	0.001	0.002	0.003	0.004	0.004	0.005	0.006	0.007	0.007

For the case $hx \ll 1\,(\rho \gg h)$, (37) and (38) are transformed into the formulas

$$\langle \tilde{G}(z', x)\rangle \approx -\frac{1}{4h}(h^2 - z'^2), \tag{39}$$

$$\langle\langle \tilde{G}(x)\rangle\rangle = -\frac{h}{6}. \tag{40}$$

In place of (16), we obtain for the averaged value $\langle \tilde{\mathbf{B}}_{\text{inhom}}(x)\rangle$

$$\langle \tilde{\mathbf{B}}_{\text{inhom}}(x)\rangle = -\frac{1}{2hx^2}\int\limits_{-h}^{h} dz' \left(1 - \frac{\cosh xz'}{\cosh xh}\right)\tilde{\mathbf{S}}(z', x), \tag{41}$$

and for the case $\rho \gg h$

$$\langle \mathbf{B}_{\text{inhom}}(\rho)\rangle = -\frac{1}{4h}\int\limits_{-h}^{h} dz'(h^2 - z'^2)\mathbf{S}(z', \rho). \tag{42}$$

The contribution of the averaged source term $\langle \mathbf{S}(\rho)\rangle$ is given by the expression

$$\langle \mathbf{B}_{\text{inhom}}(\rho)\rangle = -\frac{h}{6}\int\limits_{-h}^{h} dz'\mathbf{S}(z', \rho) \equiv -\frac{h}{3}\langle \mathbf{S}(z', \rho)\rangle. \tag{43}$$

Note that the anti-symmetric part of the source term ($\mathbf{S}_a(z', \rho) = -\mathbf{S}_a(-z', \rho)$) makes no contribution in formulas (41), (42), and (43); i.e., the averaged magnetic field $\langle \mathbf{B}_{\text{inhom}}(\rho)\rangle$ is determined purely by the symmetric part of the source term $\mathbf{S}_s(z', \rho) = \mathbf{S}_s(-z', \rho)$. This is a consequence of the fact that the z-averaged Green's function $\tilde{G}(z, z', x)$ is a symmetric function of z'.

Table 3. $B_z(\bar{z}, \bar{\rho})$ (in kG), taking into account the right-hand side and with $M_8 = 1$, $h = 1.3 \times 10^{13} M_8$ cm, and $m = 4.4 \times 10^{45}$ G cm^3

$\bar{\rho} \backslash \bar{z}$	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
10.0	−1.687	−1.689	−1.696	−1.706	−1.721	−1.740	−1.764	−1.792	−1.826	−1.866	−1.912
11.0	−1.279	−1.281	−1.285	−1.293	−1.304	−1.319	−1.337	−1.359	−1.384	−1.414	−1.448
12.0	−0.998	−0.999	−1.002	−1.008	−1.016	−1.027	−1.040	−1.056	−1.075	−1.097	−1.122
13.0	−0.793	−0.794	−0.797	−0.801	−0.807	−0.815	−0.825	−0.837	−0.851	−0.868	−0.886
14.0	−0.641	−0.642	−0.644	−0.647	−0.652	−0.658	−0.666	−0.675	−0.686	−0.698	−0.712
15.0	−0.526	−0.526	−0.528	−0.530	−0.534	−0.539	−0.545	−0.552	−0.560	−0.570	−0.581
16.0	−0.436	−0.437	−0.438	−0.440	−0.443	−0.447	−0.451	−0.457	−0.463	−0.471	−0.480
17.0	−0.436	−0.437	−0.438	−0.440	−0.443	−0.447	−0.451	−0.457	−0.463	−0.471	−0.480
18.0	−0.310	−0.310	−0.311	−0.312	−0.314	−0.317	−0.320	−0.323	−0.328	−0.333	−0.338
19.0	−0.265	−0.265	−0.266	−0.267	−0.268	−0.270	−0.273	−0.276	−0.279	−0.283	−0.288
20.0	−0.228	−0.228	−0.229	−0.230	−0.231	−0.233	−0.235	−0.237	−0.240	−0.243	−0.247
21.0	−0.198	−0.198	−0.199	−0.199	−0.200	−0.202	−0.203	−0.205	−0.208	−0.211	−0.214
22.0	−0.173	−0.173	−0.173	−0.174	−0.175	−0.176	−0.177	−0.179	−0.181	−0.183	−0.186
23.0	−0.152	−0.152	−0.152	−0.153	−0.153	−0.154	−0.156	−0.157	−0.159	−0.161	−0.163
24.0	−0.134	−0.134	−0.134	−0.135	−0.135	−0.136	−0.137	−0.139	−0.140	−0.142	−0.144
25.0	−0.119	−0.119	−0.119	−0.120	−0.120	−0.121	−0.122	−0.123	−0.124	−0.125	−0.127
26.0	−0.106	−0.106	−0.106	−0.107	−0.107	−0.108	−0.108	−0.109	−0.110	−0.112	−0.113
27.0	−0.095	−0.095	−0.095	−0.095	−0.096	−0.096	−0.097	−0.098	−0.099	−0.100	−0.101
28.0	−0.086	−0.086	−0.086	−0.086	−0.086	−0.087	−0.087	−0.088	−0.089	−0.090	−0.091
29.0	−0.078	−0.078	−0.078	−0.078	−0.078	−0.079	−0.079	−0.079	−0.080	−0.081	−0.082
30.0	−0.071	−0.072	−0.072	−0.072	−0.072	−0.072	−0.072	−0.072	−0.073	−0.073	−0.074

By virtue of the axial symmetry of the problem, the velocity components $U_\rho(z, \rho)$ and $U_\varphi(z, \rho)$ are symmetric relative to the substitution $z \to -z$, while $U_z(z, \rho)$ is anti-symmetric. The magnetic-field components $B_\rho(z, \rho)$ and $B_\varphi(z, \rho)$ are anti-symmetric, while $B_z(z, \rho)$ is symmetric. Since the derivative of a symmetric function with respect to z is an anti-symmetric function and vice versa, we find that the mean value $\langle S_z \rangle$ consists of all the averaged terms of the expression for $S_z(z, \rho)$ in (3). The mean value for the symmetric part is $\langle S_\rho(z, \rho) \rangle = 0$; consequently, in the thin-disk model,

$$B_\rho(\rho) = 0. \tag{44}$$

Similarly, the symmetrical part of $\langle S_\varphi(z, \rho) \rangle = 0$, and

$$B_\varphi(\rho) = 0. \tag{45}$$

7. NUMERICAL SOLUTION OF THE EQUATION WITH THE RIGHT-HAND SIDE

In this section, we present our numerical solution of Eqs. (3). We assumed that $U_z = 0$, and that U_ρ and U_φ do not depend on z. We made a translation to the parameters $\bar{\rho} = \rho/h$, $\bar{z} = z/h$, $\bar{D}_t = D_t/h^2$, $\bar{U}_\varphi = U_\varphi/h$, and $\bar{U}_\rho = U_\rho/h$. The equations of (3) then acquire the form

$$\bar{D}_t \left(\frac{\partial^2 B_z}{\partial \bar{z}^2} + \frac{\partial^2 B_z}{\partial \bar{\rho}^2} \right) \tag{46}$$
$$+ \frac{\partial B_z}{\partial \bar{\rho}} \left(\frac{\bar{D}_t}{\bar{\rho}} - \bar{U}_\rho + \frac{\partial \bar{D}_t}{\partial \bar{\rho}} \right)$$
$$- B_z \left(\frac{\partial \bar{U}_\rho}{\partial \bar{\rho}} + \frac{\bar{U}_\rho}{\bar{\rho}} \right) - \frac{\partial D_t}{\partial \rho} \frac{\partial B_\rho}{\partial z} = 0,$$
$$\bar{D}_t \left(\frac{\partial^2 B_\rho}{\partial \bar{z}^2} + \frac{\partial^2 B_\rho}{\partial \bar{\rho}^2} \right) + \frac{\partial B_\rho}{\partial \bar{\rho}} \left(\frac{\bar{D}_t}{\bar{\rho}} - \bar{U}_\rho \right)$$

Table 4. $B_\rho(\bar{z},\bar{\rho})$ (kn kG), taking into account the right-hand side and with $M_8 = 1$, $h = 1.3 \times 10^{13} M_8$ cm, and $m = 4.4 \times 10^{45}$ G cm^3

$\bar{\rho}\backslash\bar{z}$	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
10.0	0.000	0.052	0.105	0.159	0.213	0.269	0.327	0.387	0.450	0.516	0.585
11.0	0.000	0.036	0.073	0.110	0.147	0.186	0.225	0.266	0.309	0.354	0.401
12.0	0.000	0.026	0.052	0.078	0.105	0.132	0.160	0.189	0.220	0.251	0.284
13.0	0.000	0.019	0.038	0.057	0.077	0.097	0.117	0.138	0.160	0.183	0.207
14.0	0.000	0.014	0.028	0.043	0.057	0.072	0.088	0.103	0.120	0.137	0.154
15.0	0.000	0.011	0.022	0.033	0.044	0.055	0.067	0.079	0.091	0.104	0.117
16.0	0.000	0.008	0.017	0.025	0.034	0.043	0.052	0.061	0.071	0.080	0.091
17.0	0.000	0.007	0.013	0.020	0.027	0.034	0.041	0.048	0.055	0.063	0.071
18.0	0.000	0.005	0.011	0.016	0.021	0.027	0.033	0.038	0.044	0.050	0.057
19.0	0.000	0.004	0.009	0.013	0.017	0.022	0.026	0.031	0.036	0.041	0.046
20.0	0.000	0.004	0.007	0.011	0.014	0.018	0.021	0.025	0.029	0.033	0.037
21.0	0.000	0.003	0.006	0.009	0.012	0.015	0.018	0.021	0.024	0.027	0.031
22.0	0.000	0.002	0.005	0.007	0.010	0.012	0.015	0.017	0.020	0.023	0.025
23.0	0.000	0.002	0.004	0.006	0.008	0.010	0.012	0.015	0.017	0.019	0.021
24.0	0.000	0.002	0.003	0.005	0.007	0.009	0.010	0.012	0.014	0.016	0.018
25.0	0.000	0.001	0.003	0.004	0.006	0.007	0.009	0.010	0.012	0.014	0.015
26.0	0.000	0.001	0.002	0.004	0.005	0.006	0.008	0.009	0.010	0.012	0.013
27.0	0.000	0.001	0.002	0.003	0.004	0.005	0.007	0.008	0.009	0.010	0.011
28.0	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010
29.0	0.000	0.001	0.002	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.008
30.0	0.000	0.001	0.001	0.002	0.003	0.004	0.004	0.005	0.006	0.007	0.007

$$-B_\rho \frac{\bar{U}_\rho}{\bar{\rho}} - \bar{D}_t \frac{B_\rho}{\bar{\rho}^2} = 0,$$

$$\bar{D}_t \left(\frac{\partial^2 B_\varphi}{\partial \bar{z}^2} + \frac{1}{\bar{\rho}} \frac{\partial B_\varphi}{\partial \bar{\rho}} + \frac{\partial^2 B_\varphi}{\partial \bar{\rho}^2} - \frac{B_\varphi}{\bar{\rho}^2} \right) - \bar{U}_\varphi \frac{B_\rho}{\bar{\rho}} - \frac{\partial}{\partial \bar{\rho}} (\bar{U}_\rho B_\varphi) + B_\rho \frac{\partial \bar{U}_\varphi}{\partial \bar{\rho}} + \frac{\partial \bar{D}_t}{\partial \bar{\rho}} \frac{1}{\bar{\rho}} \frac{\partial}{\partial \bar{\rho}} (\bar{\rho} B_\varphi) = 0.$$

We adopt the velocity components

$$U_\varphi = \sqrt{GM/\rho} \quad \Rightarrow \quad \bar{U}_\varphi = \sqrt{\frac{GM}{\bar{\rho} h^3}}, \qquad (47)$$

where G is the gravitational constant and M the mass of the central object. We adopt for U_ρ the classical value from accretion theory

$$U_\rho = -\frac{\dot{M}}{4h\pi \rho_{\rm gas} \rho} \quad \Rightarrow \quad \bar{U}_\rho = -\frac{\dot{M}}{4h^3 \pi \rho_{\rm gas} \bar{\rho}}, \qquad (48)$$

where $\dot{M} \approx 1.4 \times 10^{25} M_8$ g/s is the accretion rate. Recall that $M_8 = M/10^8\ M_\odot$. Based on the characteristic turbulence scales, we adopted $D_t = |U_\rho| h/3$, i.e., $\bar{D}_t = |\bar{U}_\rho|/3$. This is valid for an accretion disk with Rossby wave instability [25], and also for ADAF models [26]. In these models, the density can be written $\rho_{\rm den} = \Sigma/2h$, where $\Sigma \sim \rho^{-3/2}$ is the surface density [25, 27].

We solved the equations using the grid method of Liebmann [28]. We adopted the solution of the homogeneous equation obtained earlier as the boundary conditions. Tables 3–5 present the results of our computations for B_z, B_ρ, and B_φ for $M_8 = 1$, $h = 1.3 \times 10^{13} M_8$ cm, and $m = 4.4 \times 10^{45}$ G cm^3 [25].

Table 5. $B_\varphi(\bar{z}, \bar{\rho})$ (in kG), taking into account the right-hand side and with $M_8 = 1$, $h = 1.3 \times 10^{13} M_8$ cm, and $m = 4.4 \times 10^{45}$ G cm^3

$\bar{\rho}\backslash\bar{z}$	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
10.0	0.000	−0.009	−0.018	−0.028	−0.037	−0.047	−0.057	−0.067	−0.078	−0.077	0.000
11.0	0.000	−0.005	−0.010	−0.016	−0.021	−0.027	−0.032	−0.038	−0.044	−0.044	0.000
12.0	0.000	−0.003	−0.006	−0.009	−0.013	−0.016	−0.019	−0.023	−0.026	−0.026	0.000
13.0	0.000	−0.002	−0.004	−0.006	−0.008	−0.010	−0.012	−0.014	−0.016	−0.016	0.000
14.0	0.000	−0.001	−0.003	−0.004	−0.005	−0.006	−0.008	−0.009	−0.011	−0.010	0.000
15.0	0.000	−0.001	−0.002	−0.003	−0.003	−0.004	−0.005	−0.006	−0.007	−0.007	0.000
16.0	0.000	−0.001	−0.001	−0.002	−0.002	−0.003	−0.004	−0.004	−0.005	−0.005	0.000
17.0	0.000	0.000	−0.001	−0.001	−0.002	−0.002	−0.002	−0.003	−0.003	−0.003	0.000
18.0	0.000	0.000	−0.001	−0.001	−0.001	−0.001	−0.002	−0.002	−0.002	−0.002	0.000
19.0	0.000	0.000	0.000	−0.001	−0.001	−0.001	−0.001	−0.001	−0.002	−0.002	0.000
20.0	0.000	0.000	0.000	0.000	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	0.000
21.0	0.000	0.000	0.000	0.000	0.000	−0.001	−0.001	−0.001	−0.001	−0.001	0.000
22.0	0.000	0.000	0.000	0.000	0.000	0.000	−0.001	−0.001	−0.001	−0.001	0.000
23.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	−0.001	−0.001	0.000
24.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
25.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
26.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
27.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
28.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
29.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Recall that B_z is symmetrical with respect to z, while B_ρ and B_φ are anti-symmetrical.

8. CONCLUSION

We have considered the magnetic-field structure in planar accretion disks assuming a stationary state, when the time derivative in the induction equation can be neglected. The induction equation (1) transforms into a diffusion equatiaon for the magnetic field (2) with sources (right-hand sides of the equaton) related to stationary plasma motions. We have assumed that the magnetic field at the boundaries of the planar disk ($z = \pm h$) coincides with the field of a magnetic dipole located at the center of the disk, with the magnetic dipole oriented perpendicular to the plane of the disk.

The general solution of the induction equation was represented as a sum of the solutions of the homogeneous equation with specified boundary values of the magnetic field and the solution of the equation with its right-hand side and zero boundary conditions. The use of a Hankel transform with Bessel functions of zeroth and first order makes it possible to reduce the problem to solving an ordinary differential equation using a Green's function.

In many models, the half-width of the accretion disk does not depend on the distance ρ when $\rho \gg 3r_g$ [29].

Table 1 shows that the solution of the homogeneous equation (4) leads to a comparatively small increase in the field $B_{z,\text{hom}}(\bar{z}, \bar{\rho})$, compared to the boundary value $B_{z,\text{hom}}(1, \bar{\rho})$. This increase disappears when $\bar{\rho} \gg 1$, and the field $B_{z,\text{hom}}(\bar{z}, \bar{\rho})$ coincides with the boundary value throughout the disk thickness. A negative sign for a magnetic-field component indicates that it is directed opposite to the direction in which the corresponding coordinate increases.

Table 3 shows that the magnetic field $B_z(\bar{z}, \bar{\rho})$ decreases with distance from a boundary. For example, $B_z(0, 10) = 0.882B_z(1, 10)$ when $\bar{\rho} = 10$, and $B_z(0, 30) = 0.959B_z(1, 30)$ when $\bar{\rho} = 30$. Note that the decrease in $B_z(\bar{z}, \bar{\rho})$ for a specified distance $\bar{\rho}$ is not linear in the variable $\bar{z}$.

Table 4 shows that the magnetic field $B_\rho(\bar{z}, \bar{\rho})$ decreases with distance from a boundary, and vanishes when $\bar{z} = 0$. This decrease is not linear, in disagreement with the formula $B_\rho(z, \rho) = B_\rho(h, \rho)z/h$ adopted in [6].

It is interesting that, like the function $\bar{\rho}$, $B_z(0, \bar{\rho})$ decreases more slowly than $B_z(1, \bar{\rho}) \sim \bar{\rho}^{-3}$ when $\bar{\rho} \gg 1$. The same is true for $B_\rho(\bar{z}, \bar{\rho}) \sim \bar{\rho}^{-4}$ (for $\bar{z} = 0.5$).

It is also interesting that the azimuthal field $B_\varphi(\bar{z}, \bar{\rho})$ decreases roughly as $\bar{\rho}^{-6}$ with increasing distance $\bar{\rho}$. Note that including the term with $\partial D_t/\partial\rho$ has virtually no effect on the values of the magnetic-field components.

The magnetic-field values presented in Tables 3–5 could be verified using polarimetric observations.

According to [30–32], the magnetic field strongly influences the polarization arising as a result of scattering of the radiation on electrons. This gives rise to a strong wavelength dependence of the Stokes parameters, as a result of the Faraday rotation of the plane of polarization occuring over a mean-free path of a photon in the scattering medium. The degree of polarization $P_l(B, \mu)$ and polarization position angle χ are defined as follows, if $B_z > B_\rho$ [30–32]:

$$P_l(B, \mu) = \frac{P_l(\mu)}{\sqrt{1 + a^2}}; \quad \tan 2\chi = a, \tag{49}$$

where $\mu = \cos i$, i is the inclination of the region of scattering to the line of sight, and $P_l(\mu)$ is the degree of polarization of the radiation emerging from a plane-parallel atmosphere with no magnetic field [33, 34].

The Faraday-depolarization parameter a has the form

$$a = 0.8\lambda_{\text{own}}^2 \mu B_z(\bar{\rho}(\lambda)), \tag{50}$$

where λ_{own} is the wavelength of the radiation in its own coordinate system, and $\bar{\rho}(\lambda) = R_\lambda/h$. R_λ is the characteristic radius of the accretion disk, which depends on the wavelength.

The decrease in $B_z(\bar{z}, \bar{\rho})$ with distance from a boundary (and the corresponding decrease in $B_\rho(\bar{z}, \bar{\rho})$) could lead to the appearance of a strong wavelength dependence of the polarizaton angle χ, in accordance with (49). Of course, this situation is realized if there is a sufficient magnetic field $B(z)$ over a length of two mean-free paths for a photon, measured from the disk boundary.

We have obtained numerical solutions for the range of accretion-disk radii $R = (10–30)R_g$ (Tables 3–5), which could correspond to the region of generation of hard electromagnetic radiation. Recall that $R_g = GM/c^2 \simeq 1.5 \times 10^5(M/M_\odot)$ cm. Therefore, the detection of a strong wavelength dependence for the degree of polarizaton and polarization position angle is expected for future UV and X-ray polarization measurements.

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