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The Generic Solution with Isotropic Big Bang*

V. A. Belinski**

ICRANet, 65122 Pescara, Italy and IHES, F-91440 Bures-sur-Yvette, France

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Abstract—It is shown that inclusion of the shear stresses to the cosmological evolution can stabilize the Friedmann Big Bang. This results in the existence of the generic solution with isotropic cosmological singularity.

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1. INTRODUCTION

Observations show that the early Universe was isotropic, homogeneous, and thermally balanced. A number of authors [1–3] expressed the point of view that the initial cosmological singularity should also be in conformity with these properties. In other words, the singularity should be isotropic which ensures that the solution is increasingly well approximated dynamically by a Friedmann model as the singularity is approached. But it is well known that the Big Bang in an exact Friedmann model is unstable. This instability is due to the sharp anisotropy (in general of an oscillatory character [4]) which develops unavoidably near the cosmological singularity. Then the spacetime cannot start expanding isotropically at the beginning unless there is an artificial fine-tuning of unknown origin. However, an intuitive understanding suggests that anisotropy can be damped by shear viscosity, which might result in the existence of a *generic solution with an isotropic singularity*.

To search for the solution of this type, it would be inappropriate to use just the Eckart or Landau–Lifshitz approaches to the relativistic hydrodynamics with dissipative processes. These theories are physically acceptable provided the characteristic times of the macroscopic motions of the matter (like periods of cosmological oscillations) are much bigger than the time of relaxation of the medium to the equilibrium state. However, it might happen that this is not the case near the cosmological singularity since all characteristic macroscopic times (first of all periods of oscillations) in this region go to zero.[1]

The reason why the Eckart and Landau–Lifshitz approaches [5, 6] become unacceptable when internal relaxation times of the material and characteristic external times reach the same order of magnitude is existence in these theories of the supraluminal propagation of exitations of the viscous (and heat) stresses. These effects are of no importance for the "normal" physical scales around us but they turn into the real pathologies in extreme situations (as, for example, near cosmological singularity) when they can not be neglected more. In such extreme cases one needs a macroscopic theory which takes into account Maxwell's relaxation times on the same footing as all other transport coefficients. In a literal sense such a theory does not exist, however, it can be constructed in an approximate form for the cases when a medium does not deviate too much from equilibrium and relaxation times do not noticeably exceed the characteristic macroscopic times.[2] It is reasonable to expect that these conditions will be satisfied automatically for a generic solution (if it exists) near an isotropic singularity describing the beginning of the thermally balanced Friedmann Universe accompanied by the

*The text was submitted by the author in English.

**E-mail: belinski@icra.it

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[1] We use the synchronous system where $-ds^2 = -dt^2 + g_{\alpha\beta}dx^\alpha dx^\beta$. It is known that in synchronous time t periods of cosmological oscillations indeed go to zero and their frequencies go to infinity.

[2] This approximative character is due to the fact that no exact theory of this kind can be deduced from kinetic. However, in the vicinity to the cosmological singularity there is no any kinetic. This follows from the circumstance that in superdense state can not be any notion of particles (L.D. Landau, 1953). In this situation one can take a liberty to consider such approximation as an exact phenomenological theory of a medium without any microscopic structure (Ya.B. Zeldovich, private discussion, 1976).

arbitrary infinitesimally small corrections. It turns out that this indeed is the case for construction we are proposing in this paper.

The main target of the efforts of many authors (starting from the first idea of Cattaneo [7] up to the final formulation of the generalized relativistic theory of a dissipative fluid by Israel and Stewart [8, 9]) was to bring the theory into line with relativistic causality, that is, to eliminate the supraluminal propagation of the thermal and viscous excitations. This was done by including into the theory the Maxwell's relaxation times.

One of the first applications of the Israel–Stewart theory to the problems of cosmological singularity was undertaken in 1979 in paper [10]. In this work the stability of the Friedmann models under the influence of the shear viscosity has already been investigated and it was found that relativistic causality and stability of the Friedmann singularity are in contradiction to each other. Then the final conclusion was "Relativistic causality precludes the stability of isotropic collapse. An isotropic singularity cannot be the typical initial or final state." However, in my recent paper [11] it was shown that this "no go" conclusion was too hasty, since it was the result of a range for the dependence of the shear viscosity coefficient on the energy density that was too restricted.

As usual, in the vicinity to the singularity where the energy density ε diverges, we approximate the coefficient of viscosity η by the power law asymptotics $\eta \sim \varepsilon^{\nu}$ with some exponent ν. In our old work (due to some more or less plausible thoughts) we choose the values of this exponent from the region $\nu > 1/2$. For these values of ν, the old negative result remains correct, but recently it was made known that the boundary value $\nu = 1/2$ leads to a dramatic change in the state of affairs. It turns out that for this case there exists a window in the space of the free parameters of the theory in which the Friedmann singularity becomes *stable* and at the same time *no supraluminal signals* exist in its vicinity. This possibility was overlooked in our 1979 work.

It is worth adding that the case $\nu < 1/2$ also was analyzed in paper [11] but it is of no interest since it leads to strong instability of a Friedmann singularity independently of the question of relativistic causality.

2. THE OUTLINE OF THE BASIC EQUATIONS

Shear stresses generate an addend S_{ik} to the standard energy-momentum tensor of a fluid:

$$T_{ik} = (\varepsilon + p)u_i u_k + p g_{ik} + S_{ik}, \tag{1}$$

and this additional term has to satisfy the Landau–Lifshitz constraints:

$$S_{ik} = S_{ki}, \quad S^k_k = 0, \quad u^i S_{ik} = 0. \tag{2}$$

Besides we have the usual normalization condition for the 4-velocity:

$$u_i u^i = -1. \tag{3}$$

If the Maxwell's relaxation time τ of the stresses is not zero then do not exists any closed expression for S_{ik} in terms of the viscosity coefficient η and 4-gradients of the 4-velocity (like it was in the Landau–Lifshitz approach [6]). Instead the stresses S_{ik} should be defined [8, 9] from the following differential equation:

$$\begin{aligned} S_{ik} + \tau\left(\delta_i^m + u_i u^m\right)\left(\delta_k^n + u_k u^n\right) S_{mn;l} u^l \\ = -\eta\left(u_{i;k} + u_{k;i} + u^l u_k u_{i;l} + u^l u_i u_{k;l}\right) \\ + \frac{2}{3}\eta\left(g_{ik} + u_i u_k\right) u^l{}_{;l}, \end{aligned} \tag{4}$$

which due to the normalization condition for velocity is compatible with the constraints (2). In case $\tau = 0$ expression for S_{ik}, following from this equation, coincides with that one introduced by Landau and Lifschitz [6]. If the equations of state $p = p(\varepsilon)$, $\eta = \eta(\varepsilon)$, $\tau = \tau(\varepsilon)$ are fixed then the Einstein equations

$$R_{ik} = T_{ik} - \frac{1}{2} g_{ik} T^l_l \tag{5}$$

together with above differential equation for the stresses S_{ik} gives the closed system where from all quantities of interest, that is g_{ik}, u_i, ε, S_{ik} can be found.

Since we are interesting in behaviour of the system in the vicinity to the cosmological singularity where $\varepsilon \to \infty$ the viscosity coefficient η in this asymptotic domain can be approximated by the power law asymptotics

$$\eta = \mathrm{const} \cdot \varepsilon^{\nu}, \tag{6}$$

with some constant exponent ν. Beforehand the value of this exponent is unknown then we need to investigate its entire range $-\infty < \nu < \infty$. As for the relaxation time τ the choice is more definite. It is well known that $\eta/\varepsilon\tau$ represents a measure of velocity of propagation of the shear excitations. Then we model this ratio by a positive constant f (in a more accurate theory f can be a slow varying function on time but in any case this function should be bounded in order to exclude the appearance of the supraluminal signals). Consequently we choose the following model for the relation between relaxation time and viscosity coefficient:

$$\eta = f\varepsilon\tau, \quad f = \mathrm{const}. \tag{7}$$

For the dependence $p = p(\varepsilon)$ we follow the standard approximation with constant parameter γ:

$$p = (\gamma - 1)\,\varepsilon, \quad 1 \leqslant \gamma < 2. \tag{8}$$

Now equations are closed and we can search the asymptotic behaviour of solution in the vicinity to the cosmological singularity. It is convenient to work in the synchronous reference system where the interval is

$$-ds^2 = -dt^2 + g_{\alpha\beta}dx^\alpha dx^\beta. \tag{9}$$

Our task is to take the standard Friedmann solution as background and to find the asymptotic (near singularity) solution of the equations for the linear perturbations around this background in the same synchronous system.

The background solution is

$$-ds^2 = -dt^2 \tag{10}$$
$$+ R^2\left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right],$$
$$R = (t/t_c)^{2/3\gamma},$$

$$\varepsilon_{(0)} = 4\left(3\gamma^2 t^2\right)^{-1}, \quad u_0^{(0)} = -1, \tag{11}$$
$$u_\alpha^{(0)} = 0, \quad S_{ik}^{(0)} = 0,$$

where $t > 0$ and t_c is some arbitrary positive constant. We have to deal with the following linear perturbations:

$$\delta g_{\alpha\beta},\ \delta u_\alpha,\ \delta\varepsilon,\ \delta S_{\alpha\beta}. \tag{12}$$

In the linearized version of the equations around the Friedmann solution will appear only these variations. The variations δu_0 and δS_{0k} can not be of the first (linear) order because of the exact relations $u_i u^i = -1$ and $u^i S_{ik} = 0$ and properties $u_0^{(0)} = -1$, $u_\alpha^{(0)} = 0$, $S_{ik}^{(0)} = 0$ of the background. The variations $\delta\tau$ and $\delta\eta$ of the relaxation time and viscosity coefficient, although exist as the first order quantities, will disappear from the linear approximation since they reveal itself only as factors in front of the terms vanishing for the isotropic Friedmann seed.

To find the general solution of equations for small perturbations we apply the technique invented by Lifshitz and used by him to analyze the stability of the Friedmann solution for the perfect liquid [12]. Since all coefficients in the differential equations for perturbations do not depend on spatial coordinates we can represent all quantities of interest in the form of the 3-dimensional Fourier integrals to reduce these equations to the system of the ordinary differential equations in time for the corresponding Fourier coefficients. These coefficients can be expanded in the Lifshitz basis in which the system of equations splits in the three separate and independent subsets (scalar, vectorial and tensorial). However, now equations are more complicated since each type of perturbations contain the terms due to the presence of the shear stresses ($\delta S_{\alpha\beta}$ also consists of the scalar, vector and tensor excitations).

In what follows we will not show these equations and procedure for obtaining their solutions near singularity. An interested reader can find the comprehensive calculations in the article [11]. Below we describe the main results avoiding the mathematical details.

3. SHORT WAVES PULSES

The conformally flat version of the Friedman metric after the transformation $dT = dt/R$ to the time T is $-ds^2 = R^2(T)[-dT^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2]$. Then in the limit of large values of the absolute values k of the wave vectors equations for the Fourier amplitudes of each types of perturbations have solutions of the form of slow varying amplitudes multiplied by the factor $\exp(ivkT)$ but for different types of excitations with different propagation velocity:

$$v_{\text{scalar}}^2 = \gamma - 1 + 4f/3\gamma, \tag{13}$$
$$v_{\text{vector}}^2 = f/\gamma, \quad v_{\text{tensor}}^2 = 1.$$

This result we obtained already in 1979 and it shows that gravitational waves propagate with velocity of light but in order to exclude the supraluminal signals for two other types of perturbations it is necessary to demand $v_{\text{scalar}}^2 < 1$ and $v_{\text{vector}}^2 < 1$. Both of these conditions in the region $1 \leqslant \gamma < 2$ will be satisfied if

$$f < \frac{3}{4}\gamma\,(2 - \gamma)\,. \tag{14}$$

4. DIFFERENT CASES FOR VISCOSITY COEFFICIENT

In case $\nu < 1/2$ the perturbations near singularity ($t \to 0$) contain the strongly divergent mode of the order of $\exp(t^{2\nu-1})$. This mode destroys the background regime. Consequently the values $\nu < 1/2$ are of no interest since in this case does not exists a generic solution of the gravitational equations with the Friedmann singularity.

For $\nu > 1/2$ near to the singularity $t \to 0$ the solutions for perturbations represent the superposition of two power law modes of orders t^{s_1} and t^{s_2} where exponents s_1 and s_2 are functions on the parameters γ and f. For stability of the Friedmann solution it is necessary for both these exponents to be positive. It turnes out that this condition is equivalent to the restriction $f > \frac{3}{4}\gamma(2-\gamma)$ for the constant f.

However, this restriction is *exactly opposite to the causality condition* (14) which has been obtained previously. Consequently, also for $\nu > 1/2$, assuming the absence of the supraluminal excitations, there is no way to provide stability of the Friedmann solution near singularity. This result has been known already in 1979.

For $\nu = 1/2$ the model contains three arbitrary constans f, γ, β and asymptotic behaviour of the viscosity coefficient η, relaxation time τ and energy density ε are:

$$\eta = \frac{4f}{3\gamma^2\beta}\frac{1}{t}, \quad \tau = \frac{t}{\beta}, \quad \varepsilon = \frac{4}{3\gamma^2}\frac{1}{t^2}. \tag{15}$$

The result is that in the space of parameters f, γ, β there is the window (of finite volume, that is of nonzero measure) in which all time-dependent perturbations tend to zero when $t \to 0$ and also no supraluminal signals exist, that is the causality condition $f < \frac{3}{4}\gamma(2-\gamma)$ take place. This window consists of two different regions: one corresponds to smooth power law behavior and another to damping oscillations (see detailed discription of these regions in paper [11]). This means that in the non-perturbative context a generic solution exists with the following asymptotics for the metric near singularity:

$$g_{\alpha\beta} = R^2\Big(a_{\alpha\beta} + t^{s_1}b^{(1)}_{\alpha\beta} + t^{s_2}b^{(2)}_{\alpha\beta} + t^{s_3}b^{(3)}_{\alpha\beta} + \ldots\Big), \tag{16}$$

where $R = (t/t_c)^{2/3\gamma}$ and exponents s_1, s_2, s_3 are definite functions on the three parameters f, γ, β. The exponent s_3 is always positive while exponents s_1 and s_2 are either positive (smooth behavior) or complex conjugated to each other but with positive real parts (damping oscillations). The additional terms denoted by the triple dots are small corrections which contain the terms of the orders t^{2s_3}, $t^{s_1+s_3}$, $t^{s_2+s_3}$ as well as all their powers and cross products.

In the main approximation the velocity components u_α are the linear superposition of the three powers $t^{s_1+1}, t^{s_2+1}, t^{s_3+1}$ and energy density are going as superposition of t^{-2} and t^{-2+s_3} (energy density never oscillates).

The main addend $a_{\alpha\beta}$ in (16) represents six arbitrary 3-dimensional functions. Each tensor $b^{(1)}_{\alpha\beta}$ and $b^{(2)}_{\alpha\beta}$ consists of the six 3-dimensional functions subjected to the restrictions $a^{\alpha\beta}b^{(1)}_{\alpha\beta} = 0$ and $a^{\alpha\beta}b^{(2)}_{\alpha\beta} = 0$ ($a^{\alpha\beta}$ is inverse to $a_{\alpha\beta}$), consequently $b^{(1)}_{\alpha\beta}$ and $b^{(2)}_{\alpha\beta}$ contain another ten arbitrary 3-dimensional functions. In case of complex conjugated s_1 and s_2 the components $b^{(1)}_{\alpha\beta}$ and $b^{(2)}_{\alpha\beta}$ are complex but in the way to provide reality of the metric tensor. The last term $b^{(3)}_{\alpha\beta}$ and all corrections denoted by the triple dots are expressible in terms of the $a_{\alpha\beta}, b^{(1)}_{\alpha\beta}, b^{(2)}_{\alpha\beta}$ and their derivatives then they do not contain any new arbitrariness.

The shear stresses, velocity and energy density follows from the exact Einstein equations in terms of the metric tensor and its derivatives and all these quantities also do not contain any new arbitrary parameters. In result the solution contains 16 arbitrary 3-dimensional functions the three of which represent the gauge freedom due to the possibility of the arbitrary 3-dimensional coordinate transformations. Then the physical freedom in the solution corresponds to 13 arbitrary functions. This is exactly the number of arbitrary independent physical degrees of freedom of the system under consideration, that is 4 for the gravitational field, 1 for the energy density, 3 for the velocity and 5 for the shear stresses (five because the six components $S_{\alpha\beta}$ follows from the six differential equations of the first order in time with one additional condition $\delta^{\alpha\beta}S_{\alpha\beta} = 0$). Then the solution we constructed is generic.

This result is the generalization of the Lifshitz–Khalatnikov quasi-isotropic solution [13] for the perfect liquid constructed in 1961. However, in case of perfect liquid the isotropic singularity is unstable and Lifshitz–Khalatnikov asymptotics corresponds to the narrow class of particular solutions containing only 3 arbitrary physical 3-dimensional parameters.

5. CONCLUDING REMARKS

1. The results presented show that the viscoelastic material with shear viscosity coefficient $\eta \sim \sqrt{\varepsilon}$ can stabilize the Friedmann cosmological singularity and the corresponding *generic solution of the Einstein equations for the viscous fluid possessing the isotropic Big Bang (or Big Crunch) exists*. Depending on the free parameters f, β, γ of the theory such solution can be either of smooth power law asymptotics near singularity (when both power exponents s_1 and s_2 are real and positive) or it can have the character of damping (in the limit $t \to 0$) oscillations (when s_1 and s_2 have the positive real part and an imaginary part). The last possibility reveal itself as a trace of the chaotic oscillatory regime which is characteristic for the most general asymptotics near the cosmological singularity and which has no any analytical form. The present case show that the shear viscosity can smooth such chaotic behaviour up to the quiet oscillations which have simple asymptotic

expressions in terms of the elementary functions of the type $t^{\mathrm{Re}s}\sin[(\mathrm{Im}s)\ln t]$ and $t^{\mathrm{Re}s}\cos[(\mathrm{Im}s)\ln t]$.

2. In the generic isotropic Big Bang described here some part of perturbations are presented already at the initial singularity $t=0$ which are the three physical components of the arbitrary 3-dimensional tensor $a_{\alpha\beta}(x^1,x^2,x^3)$. Another ten arbitrary physical degrees of freedom are contained in the components of two tensors $b^{(1)}_{\alpha\beta}$ and $b^{(2)}_{\alpha\beta}$ in this formula and they come to the action in the process of expansion. This picture has no that shortage of the classical Lifshitz approach when one is forced to introduce some unexplainable segment between singularity $t=0$ and initial time $t=t_0$ when perturbations arise in such a way that inside this segment it is necessary to postulate without reasons the validity of the exact Friedmann solution free of any perturbations.

3. It might happen that due to the universal growing of all perturbations (in the course of expansion) already before that critical time when equations of state will be changed and will switched off the action of viscosity the perturbation amplitudes will reach the level sufficient for the further development of the observed structure of our Universe. If not we always have that means of escape as inflation phase which can appear in the course of evolution after the Big Bang. Here we are touching another problem. It is known [14, 15] that no inflation can appear without preceding cosmological singularity. Moreover, namely the period of expansion from singularity to an inflationary stage is responsible for the generation of the necessary initial conditions for this inflationary phase. The resolution of all these problems remains to be seen.

4. In our analysis the case of stiff matter ($\gamma=2$) have been excluded. This peculiar possibility should be investigated separately. It is known that for the perfect liquid with stiff matter equation of state a generic solution with isotropic singularity is impossible. The asymptotic of the general solution for this case have essentially anisotropic structure although of the smooth (non-oscillatory) power low character. It might be that viscosity will be able to isotropize such evolution, however, it is not yet clear how the viscous stiff matter should be treated mathematically. The simple way to take $\gamma=2$ in our previous study does not works.

5. Another interesting question is how an evolution directed outwards of a thermally equilibrated state to a non-equilibrium one can be reconciled with the second law of thermodynamics. Indeed, it seems that in accordance with this law no deviation can happen from the background Friedmann expansion since in course of a such deviation entropy must increase but in equilibrium it already has the maximal possible value. The explanation should come from the fact of the presence the superstrong gravitational field. This field is an external agent with respect to the matter itself, consequently, the matter in the Friedmann Universe cannot be considered as an isolated system. It might happen that Penrose [2] is right and the gravitational field possess an intrinsic entropy then this entropy being added to the entropy of matter will bring the situation into line with conventional physics.

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