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ACOUSTIC SIGNAL PROCESSING AND COMPUTER SIMULATION

Minimal Polynomial Method for Estimating Parameters of Signals Received by an Antenna Array

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Abstract—The effectiveness of the projection minimal polynomial method for solving the problem of determining the number of sources of signals acting on an antenna array (AA) with an arbitrary configuration and their angular directions has been studied. The method proposes estimating the degree of the minimal polynomial of the correlation matrix (CM) of the input process in the AA on the basis of a statistically validated root-mean-square criterion. Special attention is paid to the case of the ultrashort sample of the input process when the number of samples is considerably smaller than the number of AA elements, which is important for multielement AAs. It is shown that the proposed method is more effective in this case than methods based on the AIC (Akaike's Information Criterion) or minimum description length (MDL) criterion.

Keywords: antenna array, projection superresolution methods, estimating the number of signal sources, minimal polynomial of a matrix

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1. INTRODUCTION

At present, methods for estimating the parameters of signal sources with antenna arrays (AAs) are widely applied in different branches of science and technology: hydroacoustics, radio detection and location, radio communications, etc. [1–4]. The angular coordinates of closely spaced sources are estimated by superresolution methods. This term is related to the fact that their use makes it possible to exceed Rayleigh angular resolution limit, which is equal to the width of a ray of the AA. These methods can be divided into two classes—nonparametric and parametric [5–7]. For parametric methods, it is necessary to construct a mathematical signal model based on available a priori data. For example, assumptions are often made about the small angular size of signal sources (discrete sources), the type of wave front (plane, cylindrical, or spherical), and the character of multipath propagation and scattering in the spatial channel. The unknowns are the number of signal sources, their powers, and angular directions, which are numerical parameters in the model. The problem of parametric methods is to estimate these parameters by implementing the input process in the AA. Nonparametric methods do not involve construction of a signal model but are based on direct analysis of the AA input process.

One of popular superresolution methods is the nonparametric Capon method [6, 7]. Among the rather large number of parametric methods, projection methods that construct the matrix projector onto the noise subspace are the most effective. They can include methods of maximum likelihood estimation and MUSIC (MUltiple SIgnal Classification) with its modifications [3, 6, 7], as well as the projection method based on estimation of the degree and roots of the minimal polynomial of the correlation matrix (CM) of the input process in the AA; below, we call it the minimal polynomial method (MPM) [8, 9].

The maximum likelihood estimation and MUSIC methods cannot estimate the number of signal sources, which is one of parameters of the mathematical signal model and must be known or estimated beforehand. Since the likelihood function has no extreme values in the number of sources, it is proposed to modify this function by adding a certain correcting (penalty) function to it. The modified likelihood function has a maximum depending on the number of sources; the position of the maximum is just the estimate of the number of sources. However, there is no statistically rigorous justification for the correcting function. This function is usually formed based on the AIC (Akaike's information criterion) or MDL (minimum description length) [7, 10].

The projection MPM makes it possible to simultaneously estimate the number of sources and construct the matrix projector onto the noise subspace, which can be used to determine the directions of arrival of signals and their powers. The number J of the sources is determined by the degree m of the minimal polynomial of the CM $\mathbf{M}$ of the input process ($J = m - 1$). In practice, however, the estimating CM $\hat{\mathbf{M}}$ of the input process is available. It is obtained from sample of the input process in AA elements. Since the sample has a finite length, the CM $\hat{\mathbf{M}}$ elements are random values and the minimal CM $\hat{\mathbf{M}}$ polynomial consists of N multipliers and coincides with the characteristic $\hat{\mathbf{M}}$ polynomial, where N is the number of AA elements. Therefore, when passing from the exact CM $\mathbf{M}$ to the sample CM $\hat{\mathbf{M}}$, the degree of the minimal polynomial increases from $J+1$ to N and now does not depend on the number of signal sources. In [9], a root-mean-square error (RMSE) functional for the estimate of the degree and roots of the minimal polynomial was proposed and statistically validated. This functional makes it possible to approximate the minimal polynomial of the exact CM $\mathbf{M}$ by a polynomial in which the degree is minimum and the difference from the minimal polynomial does not exceed a given value which can be determined based on a priori information about the intrinsic noise of the AA receivers.

In practice, AAs with a large number of elements are frequently used, when it is difficult to have a long sample of the input process whose number L of sample vectors is larger than the number N of AA elements, e.g., due to the unsteadiness of the signal–noise environment. For this reason, the case of a short sample, when $L < N$, is very important. It is well known [7] that adaptive noise suppression methods are highly effective in this case. It is of interest to analyze the effectiveness of the superresolving MPM under conditions of a short sample of the input process. Moreover, for multielement AAs, the sample can become "ultrashort" when the length L is considerably less than the number N ($L \ll N$).

The signal sources can be both uncorrelated and correlated between each other. For example, in a hydroacoustic channel with multipath signal propagation , one source creates several wave fronts arriving at the AA from different directions. Such situation can be considered as reception of signals from several correlated sources. At the same time, some of signal eigenvalues of the exact CM $\mathbf{M}$ become close to the noise eigenvalue, which impedes estimating the number of sources due to the separation of the eigenvalues of the sample CM $\hat{\mathbf{M}}$ into signal and noise ones.

The effectiveness of the MPM for solving the superresolution problem was analyzed in [9] for uncorrelated sources. It was shown that this method ensures a considerably higher resolution than the Capon method. However, it is interesting to compare its effectiveness with that of the MUSIC method. The question of estimating the number of signal sources with the MPM was not considered in [9] either, and cases of an ultrashort sample of the input process and correlated sources were not analyzed. This paper studies these problems.

2. MATHEMATICAL SIGNAL MODEL

The main aim of the work is to compare the effectiveness of the MPM and AIC or MDL criteria in estimating the number of signal sources at the input of a multielement AA in the case of an ultrashort sample of the input process, when the number of samples is considerably less than the number of AA elements. For this purpose, it is sufficient to consider a relatively simple signal model in which it is supposed that the AA receives narrowband signals from J discrete (point) sources. Then, the sample $\mathbf{X}(l)$ of the vector of the input process in an N-element AA at the lth time instant can be represented as

$$\mathbf{X}(l) = \sum_{j=1}^{J} a_j(l)\mathbf{S}_j + \mathbf{Z}(l), \tag{1}$$

where $a_j(l)$ is the complex amplitude of the jth source in the elements of the AA, which is assumed to be a Gaussian noise process; $\mathbf{S}_j$ is the vector of the amplitude–phase distribution of the signal from the jth source; and $\mathbf{Z}(l)$ is the vector of additive Gaussian noise of the receivers with a zero mean and variance σ_0^2. Below, without any loss of generality, we assume $\sigma_0^2 = 1$. The sample vectors $\mathbf{X}(l)$ are taken with a time interval reciprocal to the bandwidth of the receivers $\Delta t \approx (1/\Delta f)$ to ensure their mutual uncorrelatedness.

The statistical relation of the sources is specified by a matrix $\mathbf{B}$ with elements $\mathbf{B}_{jq} = \left\langle a_j(l)a_q^*(l)\right\rangle = \sqrt{v_j v_q}\rho_{jq}$, where v_j is the power of a signal from the jth source in the AA elements, ρ_{jq} is the coefficient of correlation between the complex amplitudes of the jth and qth sources, $(\cdot)^*$ is the complex conjugation, and $\langle\cdot\rangle$ is the statistical average. In the case of uncorrelated sources, the matrix $\mathbf{B}$ is diagonal: $\mathbf{B} = \mathrm{diag}\{v_1, v_2, ..., v_J\}$.

The statistical properties of the set of Gaussian complex quantities $\mathrm{X}(l)$ are determined by the CM of the input process $\mathbf{M} = \langle \mathbf{X}(l)\mathbf{X}(l)^H\rangle$ ($(\cdot)^H$ is Hermitean conjugation), which is as follows [5–7]:

$$\mathbf{M} = \mathbf{I} + \mathbf{S}^{(0)}\mathbf{B}\mathbf{S}^{(0)H}, \tag{2}$$

where $\mathbf{I}$ is the unit CM of uncorrelated intrinsic noise and $\mathbf{S}^{(0)} = [\mathbf{S}_1, \mathbf{S}_2, ..., \mathbf{S}_J]$ is the matrix of wave fronts of signal sources in AA elements. The columns of the matrix $\mathbf{S}^{(0)}$ are vectors $\mathbf{S}_j$ ($j = 1, 2, ..., J$) depending on the angular position of the corresponding signal sources with respect to the AA. In the case of uncorrelated sources, the CM is a sum of CMs of individual sources and, instead of (2), we have

$$\mathbf{M} = \mathbf{I} + \sum_{i=1}^{J} v_i\mathbf{S}_i\mathbf{S}_i^H. \tag{3}$$

In practice, instead of the exact CM (2), its maximum likelihood estimate $\hat{\mathbf{M}}$ by L sample vectors $\mathbf{X}(l)$

of the input process in AA elements is used. It is equal to [11]

$$\hat{\mathbf{M}} = \frac{1}{L}\sum_{l=1}^{L}\mathbf{X}(l)\mathbf{X}(l)^H. \quad (4)$$

Elements of the sample CM are random numbers and the probability of the appearance of multiple eigenvalues is negligibly small. The sample of the input process can be long or short depending on the relation between the number L of samples of the input process and number N of AA elements.

Let us specify the vector $\mathbf{S}_j$ of the amplitude–phase distribution of the jth signal in AA elements. We assume that the sources are point-like and are positioned sufficiently far from the AA aperture, i.e., the wave fronts are plane. For an AA of an arbitrary configuration, the nth component of the vector $\mathbf{S}_j$ is $(S_j)_n = \exp[-j(\mathbf{k}\cdot\boldsymbol{\rho}_n)]$, where the vector $\boldsymbol{\rho}_n$ specifies the position of the nth AA element in a three-dimensional coordinate system (x, y, z), the origin of which is brought into coincidence with the first element, $\mathbf{k}$ is the wavevector specifying the wave propagation direction, and $(\mathbf{k}\cdot\boldsymbol{\rho}_n)$ is the dot product of the vectors $\mathbf{k}$ and $\boldsymbol{\rho}_n$. For a linear and equidistant AA, $(S_j)_n = \exp[j(n-1)u_j]$, $u_j = 2\pi(d/\lambda)\sin\varphi_j$, where φ_j is the angular coordinate of the jth source (the coordinate is reckoned from the normal to the AA), d is the AA period, and λ is the wavelength.

3. MINIMAL POLYNOMIAL METHOD

The exact CM $\mathbf{M}$ has the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ and the characteristic polynomial $\psi_N(\lambda)$ of degree N. When the number of signal sources is less than the number of AA elements ($J < N$), the polynomial $\psi_N(\lambda)$ has multiple roots. Then, there exists a minimal polynomial $\psi_{J+1}(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_{J+1})$ that has no multiple roots, divides the characteristic polynomial, and has the smallest degree m [12]. The degree of the polynomial $\psi_{J+1}(\lambda)$ is $m = J+1$; i.e., it is determined by the number of signal sources, and its roots are eigenvalues of the CM $\mathbf{M}$ ($\lambda_1 > \lambda_2 > \ldots > \lambda_{J+1}$), which are not equal to each other. The last (smallest) eigenvalue is equal to the power of the intrinsic noise ($\lambda_{J+1} = 1$) and is called the noise eigenvalue; other (signal) eigenvalues depend on the parameters of the signal sources.

The CM $\mathbf{M}$ can be represented as the sum of matrix projectors on the signal subspace (dimension J) and noise subspace (dimension $N-J$): $\mathbf{M} = \mathbf{P}_{\text{signal}} + \mathbf{P}_{\text{noise}}$. The matrix $\mathbf{P}_{\text{noise}}$ can be represented in the form [9]

$$\mathbf{P}_{\text{noise}} = \left[\prod_{p=1}^{J}(\mathbf{M}-\lambda_p\mathbf{I})\right]\left[\prod_{p=1}^{J}(1-\lambda_p)\right]^{-1}. \quad (5)$$

The AA output power fraction corresponding to the subspace of the intrinsic noise during scanning by an AA ray is $|\mathbf{P}_{\text{noise}}\mathbf{S}(\varphi)|^2$, where $\mathbf{S}(\varphi)$ is the vector of an arbitrary direction φ with an nth component of $S(\varphi)_n = \exp[j(n-1)u]$, $u = 2\pi(d/\lambda)\sin\varphi$. Let us form the following inverse function of the angle φ:

$$\eta_{MPM}(\varphi) = [\mathbf{S}^H(\varphi)\mathbf{P}^H_{\text{noise}}\mathbf{P}_{\text{noise}}\mathbf{S}(\varphi)]^{-1}. \quad (6)$$

If the AA is matched with the jth source ($\mathbf{S}(\varphi) = \mathbf{S}_j$), or, in other words, the maximum of the AA directivity diagram is oriented in the direction to this source, the vector $\mathbf{S}(\varphi)$ belongs to the signal subspace and has a zero projection onto the noise subspace. Therefore, the function $\eta_{MPM}(\varphi)$ at the point $\varphi = \varphi_j$ tends to infinity ($\eta_{MPM}(\varphi) \to \infty$ as $\varphi \to \varphi_j$). Using this peak, we can find the angular coordinate of the jth source. The function $\eta_{MPM}(\varphi)$ can be considered the spatial spectrum. Since application of projection methods is usually aimed not at estimating the spectrum but at determining the angular directions of signal sources, the dependence $\eta_{MPM}(\varphi)$ obtained by them for the signal level as a function of the angular coordinate is called the spatial pseudospectrum.

In the case of a long sample of the input process ($L > N$), the sample CM $\hat{\mathbf{M}}$ has N positive eigenvalues $\mu_1 > \mu_2 > \ldots > \mu_N > 0$, not equal to each other. The noise eigenvalue which has multiplicity $N-J$ and is equal to unity for the exact CM $\mathbf{M}$ is split into $N-J$ eigenvalues of the sample CM $\hat{\mathbf{M}}$. The spread of noise eigenvalues increases with a decrease in the sample length. Some of them can be considerably less than unity. The minimal polynomial of the CM $\hat{\mathbf{M}}$ consists of N multipliers and coincides with the characteristic polynomial of the CM $\hat{\mathbf{M}}$ of degree N. Thus, with the passage from the exact CM to the sample CM, the degree of the minimal polynomial increases from $J+1$ to N and, therefore, does not depend anymore on the number of sources J.

IN the case of a short sample ($L < N$), the CM $\hat{\mathbf{M}}$ is degenerate and has L positive eigenvalues and $N-L$ eigenvalues are zero ($\mu_1 > \mu_2 > \ldots > \mu_L > 0, \mu_{L+1} = \mu_{L+2} = \ldots = \mu_N = 0$). The subspace corresponding to the zero eigenvalues is orthogonal to the subspace of sample vectors $\mathbf{X}(l)$. The minimal polynomial of the sample CM $\hat{\mathbf{M}}$ has a degree equal to L and the degree also no longer depends on the number of signal sources.

Such differences in properties of the exact and sample CMs are essential for solving the posed problems and are caused by the appearance of a set of noise eigenvalues instead of a single one in the sample CM (the splitting of the noise eigenvalue). Therefore, the number and angular coordinates of signal sources cannot be estimated using (5). It is necessary first to estimate the minimal polynomial of the sample CM $\hat{\mathbf{M}}$

and then to construct the matrix projector onto the noise subspace.

Let us consider the statistical root-mean-square criterion proposed in [9]. According to this criterion, a matrix polynomial $\mathbf{I}^{(m)}(\widehat{\mathbf{M}})$ of smallest degree with the Euclidean norm not exceeding a certain threshold Th is found for the sample CM $\widehat{\mathbf{M}}$. Using this polynomial, we can obtain an approximation for the minimal polynomial of the sample CM by a polynomial whose degree is minimal and the difference from the characteristic polynomial yielded by it does not exceed (in the root-mean-square sense) a given value determined from a priori information about the CM of the intrinsic noise.

Let us form a functional $I^{(m)}$ equal to

$$
\begin{gathered}
I^{(m)} = \min_{\gamma_n} \left\| \mathbf{I}^{(m)}(\widehat{\mathbf{M}}) \right\|^2, \\
\mathbf{I}^{(m)}(\widehat{\mathbf{M}}) = \prod\nolimits_{n=1}^{m} (\mathbf{I} - \gamma_n \widehat{\mathbf{M}}),
\end{gathered}
\tag{7}
$$

and find first the coefficients γ_n and then the degree of the minimal polynomial of the CM $\widehat{\mathbf{M}}$. The coefficients γ_n are quantities reciprocal to eigenvalues of the sample CM $\widehat{\mathbf{M}}$ ($\mu_n = 1/\gamma_n$) and are found from solving the system of nonlinear equations

$$
\begin{aligned}
\gamma_n = {} & \mathrm{Sp}\left[\widehat{\mathbf{M}} \prod_{i=1, i\neq n}^{m} \left(\mathbf{I} - \gamma_i \widehat{\mathbf{M}}\right)^2 \right] \\
& \times \left\{ \mathrm{Sp}\left[\widehat{\mathbf{M}}^2 \prod_{i=1, i\neq n}^{m} \left(\mathbf{I} - \gamma_i \widehat{\mathbf{M}}\right)^2 \right] \right\}^{-1},
\end{aligned}
\tag{8}
$$

where $\mathrm{Sp}(\cdot)$ is the spur of this matrix.

To solve this system, the iteration procedure beginning from $m = 1$ can be used. We consider the m numbers γ_n calculated for the functional $I^{(m)}$ as the initial approximations for calculating the $(m + 1)$ numbers γ_n for the functional $I^{(m+1)}$.

For $m = 1$, the functional $I^{(1)} = \min_{\gamma_1} \mathrm{Sp}[(\mathbf{I} - \gamma_1 \widehat{\mathbf{M}})^2]$. Hence we have $\gamma_1 = (\mathrm{Sp}\,\widehat{\mathbf{M}})(\mathrm{Sp}\,\widehat{\mathbf{M}}^2)^{-1}$. If the functional $I^{(1)} < \mathrm{Th}$, the iteration process terminates and the estimate for the degree of the minimal polynomial $\widehat{m} = 1$. It means that the AA contains only the intrinsic noise and the estimate of the number of sources is zero ($\widehat{J} = \widehat{m} - 1 = 0$). If $I^{(1)} > \mathrm{Th}$, we continue the iteration process and assign $m = 2$.

At $m = 2$, the functional $I^{(2)} = \min_{\gamma_1,\gamma_2} \times \mathrm{Sp}[(\mathbf{I} - \gamma_1 \widehat{\mathbf{M}})^2 (\mathbf{I} - \gamma_2 \widehat{\mathbf{M}})^2]$. Differentiating $I^{(2)}$ with respect to γ_1 and γ_2 and equating the derivatives to zero, we obtain a system of nonlinear equations:

$$
\begin{aligned}
\gamma_1 &= \frac{\mathrm{Sp}[\mathbf{I} - \gamma_2 \widehat{\mathbf{M}})^2 \widehat{\mathbf{M}}]}{\mathrm{Sp}[(\mathbf{I} - \gamma_2 \widehat{\mathbf{M}})^2 \widehat{\mathbf{M}}^2]}, \\
\gamma_2 &= \frac{\mathrm{Sp}[\mathbf{I} - \gamma_1 \widehat{\mathbf{M}})^2 \widehat{\mathbf{M}}]}{\mathrm{Sp}[(\mathbf{I} - \gamma_1 \widehat{\mathbf{M}})^2 \widehat{\mathbf{M}}^2]}.
\end{aligned}
\tag{9}
$$

For the initial value, we choose γ_1 obtained at the first step of the iteration procedure for $m = 1$ and substitute it into the second formula in (9) to find the coefficient γ_2. Then, we substitute this value γ_2 into the first formula of (9) and find the next approximation of γ_1. Such mutual substitutions are carried out several times until γ_1 and γ_2 are unchanged. Then, the functional $I^{(2)}$ is found and compared with the threshold Th. If $I^{(2)} < \mathrm{Th}$, the iteration procedure stops and a conclusion is made that the degree of the minimal polynomial $\widehat{m} = 2$, i.e., there is a single source ($\widehat{J} = \widehat{m} - 1 = 1$). If $I^{(2)} > \mathrm{Th}$, the procedure continues and $m = 3$ is assigned.

This procedure continues until the value of the functional $I^{(m)}$ for a certain $m = \widehat{m}$ becomes less than the threshold. Computational practice shows that the iteration process converges rather quickly. For example, no more than four or five iterations are sufficient to the calculate values of γ_n with an accuracy of 10^{-4} at $m = 4$. The obtained $\widehat{m}$ value is taken as the estimate for the minimal polynomial degree and the obtained numbers γ_n yield estimates of quantities reciprocal to eigenvalues of the sample CM $\widehat{\mathbf{M}}$ ($\mu_n = 1/\gamma_n$). Thus, we have obtained the estimate for the minimal polynomial of the exact CM $\mathbf{M}$. The estimate for the number of sources is determined by the degree of this polynomial ($\widehat{J} = \widehat{m} - 1$).

The threshold Th can be found based on the a priori information about the CM of the intrinsic receiver noise. If the noise has a unit CM, it is a priori known that the degree of the minimal polynomial in the absence of sources ($J = 0$) is $m = 1$. Let the functional $I^{(1)}$ in the presence of noise alone be denoted as $I_0^{(1)}$ and the threshold be chosen equal to $\mathrm{Th} = \left\langle I_0^{(1)} \right\rangle + \chi\sigma_1$, where σ_1 is the root-mean-square deviation of the functional $I_0^{(1)}$, and the parameter χ can be found by specifying the probability of a false alarm caused by the influence of noise. As a rule, the number of AA elements $N^2 \gg 1$. Then, setting $m = 1$ in (7), we have [8]

$$
\left\langle I_0^{(1)} \right\rangle = \frac{N}{(1 + L/N)}, \quad \sigma_1^2 = \frac{2}{(1 + L/N)^2}\left(1 + \frac{2N}{L}\right).
\tag{10}
$$

Note that the threshold depends on the number N of AA elements and number L of samples of the input process. In the case of an ultrashort sample, when $L \ll N$, from (10) we have $\left\langle I_0^{(1)} \right\rangle \approx N$ and $\sigma_1 \approx 2\sqrt{N/L}$. With an increase in the sample length ($L \to \infty$), the threshold Th $\to 0$.

Let us now construct the estimate for the matrix projector onto the noise subspace in the form

$$\hat{\mathbf{P}}_{\text{noise}}^{(MPM)} = \left[\prod_{p=1}^{\hat{J}} (\mathbf{I} - \gamma_p \hat{\mathbf{M}})\right]\left[\prod_{p=1}^{\hat{J}} \left(1 - \frac{\gamma_p}{\gamma_{\hat{J}+1}}\right)\right]^{-1}. \quad (11)$$

Expression (11) passes into (5) as $L \to \infty$ with allowance for the fact that $\hat{J} \to J$, $\hat{\mathbf{M}} \to \mathbf{M}$, $\hat{\gamma}_p \to \gamma_p$, and after easy transformations. Formula (6) for the spatial pseudospectrum takes the form

$$\eta_{MPM}(\varphi) = [\mathbf{S}^H(\varphi)(\hat{\mathbf{P}}_{\text{noise}}^{MPM})^H \hat{\mathbf{P}}_{\text{noise}}^{PMP}\mathbf{S}(\varphi)]^{-1}. \quad (12)$$

4. CORRELATED SIGNAL SOURCES

The expressions for the eigenvalues λ_1 and λ_2 of the exact CM $\mathbf{M}$ in the case of two sources with different powers ($v_1 \neq v_2$) and arbitrary correlation coefficient $\rho_{12} = |\rho_{12}|\exp(j\alpha_{12})$ are rather cumbersome [13]. In superresolution problems, sources of equal power are usually considered. Therefore, we assume that $v_1 = v_2 = v$. Then, the eigenvalues λ_1 and λ_2 are

$$\lambda_{12} = 1 + vN\Big[1 + |g_{12}||\rho_{12}|\cos\alpha_{12} \pm \sqrt{(1 + |g_{12}||\rho_{12}|\cos\alpha_{12})^2 - (1 - |g_{12}|^2)(1 - |\rho_{12}|)^2}\Big], \quad (13)$$

where $g_{12} = \mathbf{S}_2^H \mathbf{S}_1 / N$ is the mismatching factor of wave fronts from two sources.

If the sources are uncorrelated ($|\rho_{12}| = 0$), the signal eigenvalues $\lambda_{1,2} = 1 + vN(1 \pm |g_{12}|)$. If the sources are fully correlated ($|\rho_{12}| = 1$), we have from (13) $\lambda_1 = 1 + 2vN(1 + |g_{12}|\cos\alpha_{12})$ and $\lambda_2 = 1$. Therefore, the second signal eigenvalue becomes equal to the noise eigenvalue. It also follows from (13) that the eigenvalues are invariant to the simultaneous replacement of $|g_{12}|$ by $|\rho_{12}|$ and $|\rho_{12}|$ by $|g_{12}|$. Thus, the infuence of an increase in the degree of source correlation ($|\rho_{12}|$) on the estimate for the number of sources can be interpreted as the influence of the angular approach of the sources ($|g_{12}|$).

To increase the effectiveness of the projection minimal polynomial method under consideration, we can use the well-known procedure for smoothing the CM: the spatial smoothing technique (SST) [7, 10]. There are several modifications of this procedure. We consider spatial smoothing implemented by AA partitioning into $K = N - Q + 1$ overlapping subarrays consisting of $Q < N$ elements and shifted relative to each other by one element. For each subarray, the CM is estimated and, therefore, K matrices are obtained. Then, the average CM $\mathbf{M}_{SST}$ with dimensions $Q \times Q$ is found.

The form of the CM $\mathbf{M}_{SST}$ is similar to (2), but elements of matrix $\mathbf{B}$ consist of the coefficients $(\rho_{SST})_{jq}$ of correlation between the jth and qth sources [7]:

$$(\rho_{SST})_{jq} = \gamma_{jq}\rho_{jq}, \quad \gamma_{jq} = \frac{\sin[0.5K(u_j - u_q)]}{K\sin[0.5(u_j - u_q)]} \times \exp[j0.5(K-1)(u_j - u_q)]. \quad (14)$$

It follows that $|\gamma_{jq}| < 1$; i.e., the smoothing procedure leads to a decrease in the coefficient of correlation between signal sources at the output of the subarray.

The $Q \times Q$ CM at the output of the ith subarray can be represented as $\mathbf{M}^{\{i\}} = \mathbf{I}_{(Q,N)}\mathbf{M}\mathbf{I}_{(Q,N)}^H$, where $\mathbf{I}_{(Q,N)} = [0_1, \ldots, 0_{i-1}, \mathbf{I}_Q, 0_{i+Q}, \ldots, 0_N]$ is a matrix which has first $i - 1$ and last $N - Q - i + 1$ zero columns and a unit matrix with dimensions $Q \times Q$ in the middle; 0_l is the lth zero column.

Then, it is necessary to replace the CM $\hat{\mathbf{M}}$ in (11) with the smoothed CM $\hat{\mathbf{M}}_{SST}$,

$$(\mathbf{M}_{SST})_{jm} = \frac{1}{K}\sum_{i=1}^{K} \mathbf{M}^{\{i\}} = \frac{1}{K}\sum_{i=1}^{K} \mathbf{I}_{(Q,N)}\mathbf{M}\mathbf{I}_{(Q,N)}^T. \quad (15)$$

Thus, the SST leads to a decrease in the coefficients of correlation between sources at the output of the subarray, which increases the effectiveness of the superresolving methods. However, the dimension of the smoothed CM $\hat{\mathbf{M}}_{SST}$ is smaller than that of the CM $\hat{\mathbf{M}}$, which is equivalent to a decrease in the AA aperture and leads to a decrease in the effectiveness of these methods.

5. ESTIMATING THE NUMBER OF SOURCES BASED ON THE AIC OR MDL CRITERIA

The projection MPM makes it possible to construct the matrix projector onto the noise subspace and to implement the resolution of closely positioned signal sources. At the same time, to use the MUSIC method, one should preliminarily estimate the number of signal sources, which is usually done based on the AIC or MDL criteria. According to these criteria, the position of the minimum of the modified likelihood function with respect to variable J is taken as the estimate for the number of sources. The modified likelihood function has the form [7, 10]

$$F(J) = L(N - J)\log\left(\frac{f_1(J)}{f_2(J)}\right) + f_3(J, L), \quad (16)$$

where the functions $f_1(J)$ and $f_2(J)$ are the arithmetic and geometric means of the noise eigenvalues of the sample CM $\hat{\mathbf{M}}$, respectively, and the penalty function $f_3(J)$ in the case of uncorrelated sources is $f_3(J) = J(2N - J)$ (AIC) and $f_3(J) = 0.5J(2N - J)\log L$ (MDL criterion).

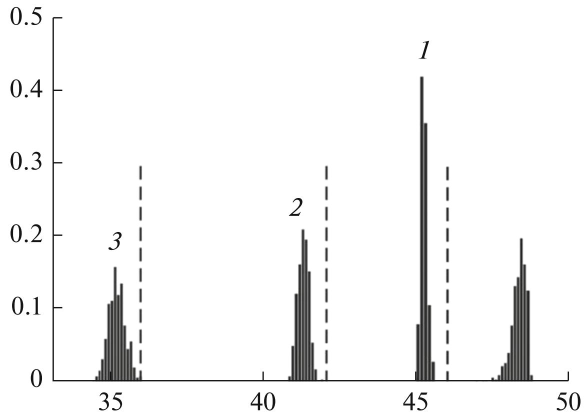

Fig. 1. Histograms of the functionals $I^{(1)}$ and $I^{(2)}$.

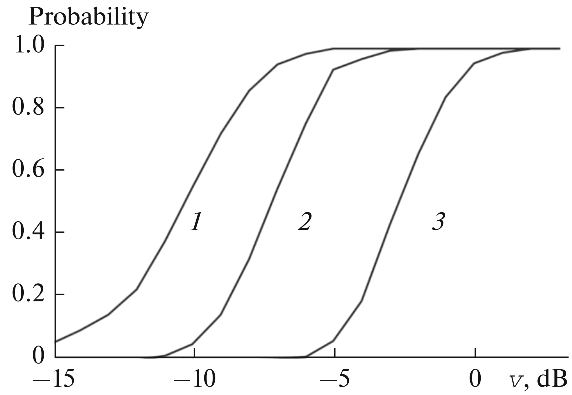

Fig. 2. Probability of correctly estimating the number of sources for different methods.

After estimating $\hat{J}$ of the number of sources, to find their angular coordinates by the MUSIC method, function (6) of the spatial pseudospectrum is used. This function involves the noise projector

$$\hat{\mathbf{P}}_{\text{noise}} = \mathbf{I} - (\hat{\mathbf{U}}_1\hat{\mathbf{U}}_1^H + \hat{\mathbf{U}}_2\hat{\mathbf{U}}_2^H + \ldots + \hat{\mathbf{U}}_{\hat{J}}\hat{\mathbf{U}}_{\hat{J}}^H), \quad (17)$$

where $\hat{\mathbf{U}}_j$ are eigenvectors of the sample CM $\hat{\mathbf{M}}$ corresponding to signal eigenvalues of this matrix [7].

6. SIMULATION RESULTS

Let us consider a linear AA with a number of elements $N = 50$ and period $d = 0.5\ \lambda$. The half-power width of the directional diagram is $\Delta\varphi_{\text{beam}} = 2.0°$. The vector $\mathbf{X}(l)$ of the input process is specified in form (1), where the random value $a_j(l)$ has a zero mean and variance v_j and the power of intrinsic noise is equal to unity. In each experiment we form L samples of the input process by use of (1) and the number of experiments for averaging the simulation results is taken to be 500. Special attention is paid to the problem of estimating the number of signal sources in the case of an ultrashort sample of the input process ($L \ll N$).

1. One signal source. Figure 1 shows the histograms of the functionals $I^{(1)}$ and $I^{(2)}$. The source power $v = 5$ dB and the number of samples $L = 5$, 10, and 20 (histograms *1–3*, respectively, for $I^{(2)}$). The right histogram was constructed for the functional $I^{(1)}$ and is almost similar for all L. Histograms of the functional $I^{(2)}$ are shifted to the left with an increase in L. The dashed lines show the thresholds $\text{Th} = \left\langle I_0^{(1)} \right\rangle + \chi\sigma_1$ obtained using (10) at $\chi = 0.1$. It follows from Fig. 1 that values $I^{(1)} > \text{Th}$ and $I^{(2)} < \text{Th}$ with a probability close to unity. Therefore, the probability of a correct estimate for the number of sources is also close to unity ($\hat{J} = 1$). In what follows, we suppose that $\chi = 0.1$.

Figure 2 shows the comparative probabilities of correctly estimating the number of sources as a function of the source power v for the MPM and MDL and AIC criteria (curves *1–3*, respectively) for length $L = 10$. It is seen that the MPM yields a higher probability of correctly estimating the number of sources in the region of low v ($v < -2$ dB).

Figure 3 presents the RMSD of the estimate for the angular coordinate of the signal source as a function of the power v for the MPM and MUSIC methods (curves *1* and *2*, respectively) for an input process length $L = 10$. The curves were constructed in a region where the probability of a correct estimate exceeds 80%. Only experiments in which a correct estimate for the number of sources was ensured were taken into account ($\hat{J} = 1$). It is seen that both methods give almost equal RMSDs. Curve *3* yields a potential RMSD (Cramér–Rao bounds) of $\sqrt{0.2}\Delta\varphi_{\text{beam}}/\sqrt{v_1 NL}$ [5, 6]. It follows from the figure that the results are close to potentially achievable ones.

2. Two uncorrelated sources. Let Δ denote the normalized angular distance between sources ($\Delta = \Delta\varphi/\Delta\varphi_{\text{beam}}$) and let these sources be arranged symmetrically with respect to the normal to the AA ($\varphi_1 = -\varphi_2$). The power of sources $v = 5$ dB and the number of samples $L = 10$. Figure 4 shows the histograms for the functionals $I^{(1)}$, $I^{(2)}$, and $I^{(3)}$ (histograms *1–3*, respectively). The dashed line shows the threshold $\text{Th} = \left\langle I_0^{(1)} \right\rangle + 0.1\sigma_1$ found using (10). It follows from Fig. 4 that $I^{(1)} > \text{Th}$, $I^{(2)} > \text{Th}$, and $I^{(3)} < \text{Th}$ with a probability close to unity. Therefore, the probability of correctly estimating the number of sources is also close to unity ($\hat{J} = 2$). Figure 5 shows the probabilities of correctly estimating the number of sources depending on their power for an input process length $L = 10$ according to the MPM and MDL and AIC criteria (curves *1–3*, respectively). The parameter Δ is 0.25 (solid curves) and 0.5 (dashed curves). It is seen that the MPM and MDL criterion have an approximately similar effectiveness that exceeds that of the AIC.

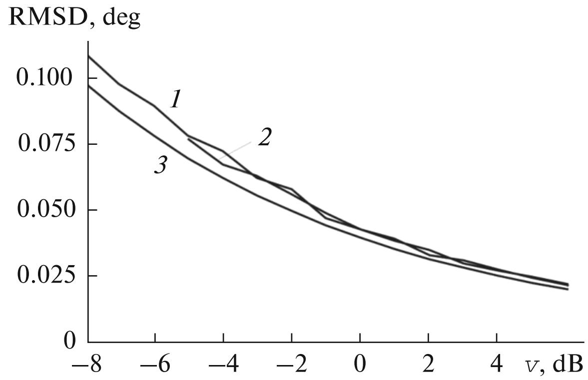

Fig. 3. Root-mean-square deviation of the estimate for the angular coordinate of the source.

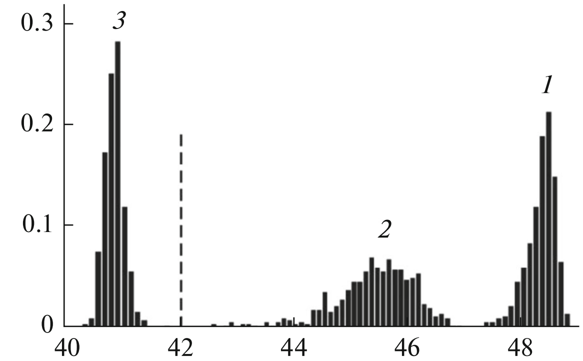

Fig. 4. Histograms of the functionals $I^{(1)}$, $I^{(2)}$, and $I^{(3)}$.

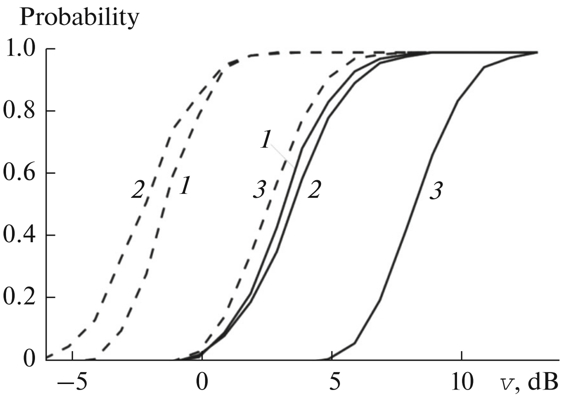

Fig. 5. Probability of correctly estimating the number of sources using different methods.

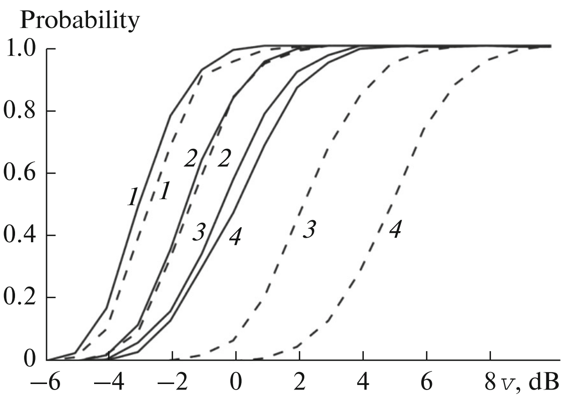

Fig. 6. Probability of correctly estimating the number of sources using the MPM.

3. Two correlated sources. Let us estimate the number of sources using the MPM. Figure 6 shows the probabilities of correctly estimating the number of sources depending on their power v for different absolute values of coefficients of correlation between sources $|\rho_{1,2}|$ = 0, 0.7, 0.9, and 0.95 (curves *1–4*, respectively) for an input process length $L = 10$. The solid curves were obtained using the spatial smoothing procedure; the dashed curves, without it. The phase of the correlation coefficient was specified as constant in each experiment and uniformly distributed in the interval $[0, 2\pi]$ for different experiments. The normalized angular distance between the sources was $\Delta = 1.0$, and the size of the subarray during the spatial smoothing procedure was $K = 30$. The threshold was chosen with allowance for additional averaging of intrinsic noise according to (15). It is seen that the spatial smoothing procedure is effective for high correlation coefficients ($|\rho_{1,2}| > 0.7$), when the effect of the decrease in the correlation coefficient prevails. For lower correlation coefficients, a decrease in the dimension of the smoothed CM is a decisive factor and leads to a decrease in the effectiveness of the smoothing procedure.

6. CONCLUSIONS

The effectiveness of the projection MPM for solving the problem of determining the number of sources of signals acting on an AA with arbitrary configuration has been investigated. The method estimates the degree of the minimal polynomial of the sample CM of the input process in the AA by using a statistically validated root-mean-square criterion. Cases of uncorrelated and correlated sources have been considered. The simulation results are presented for the case of an ultrashort sample of the input process when the number of samples is considerably less than the number of AA elements. It has been shown that the proposed method is more effective as compared to methods based on the AIC and MDL criteria. In the case of correlated sources, using the MPM with the spatial smoothing procedure increases the probability of a correct estimate for the number of sources at high correlation coefficients (>0.7).

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