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CLASSICAL PROBLEMS OF LINEAR ACOUSTICS AND WAVE THEORY

High-Frequency Wave Diffraction by an Impedance Segment at Oblique Incidence

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Abstract—The plane problem of high-frequency acoustic wave diffraction by a segment with impedance boundary conditions is considered. The angle of incidence of waves is assumed to be small (oblique). The paper generalizes the method previously developed by the authors for an ideal segment (with Dirichlet or Neumann boundary conditions). An expression for the directional pattern of the scattered field is derived. The optical theorem is proved for the case of the parabolic equation. The surface wave amplitude is calculated, and the results are numerically verified by the integral equation method.

Keywords: diffraction by an impedance strip, parabolic equation of diffraction theory, optical theorem
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INTRODUCTION

The plane problem of diffraction by a segment is one of the classical problems of diffraction theory. The geometry under consideration represents a section of the three-dimensional problem of diffraction by a strip, which is of practical value for acoustics and radiophysics. Many papers consider a segment with ideal boundary conditions. A brief review of these papers can be found in [1]. In addition, in [1], the problem of diffraction by an ideal segment was solved by us in the parabolic equation approximation for the case of oblique incidence. Note that one of the first attempts to apply the parabolic equation approach to the problem of diffraction by an ideal strip was made in [2]. In this paper, we extend the results obtained in [1] to the case of impedance boundary conditions.

The problem of diffraction by an impedance segment has been far less studied, and no exact solution to this problem has been obtained. An attempt to consider the problem analytically was made in [3, 4]. In a number of publications, the problem was solved by various approximate methods. In [5, 6], the problem was solved by constructing a diffraction series; in [7, 8], by numerically solving a pair of integral equations; in [9–11], by using a hybrid technique combining analytic and numerical methods; in [12], by the approximate Wiener–Hopf method; and in [13], by modified physical optics theory. The results obtained in the cited publications are cumbersome and mainly derived by time-consuming calculations. In this paper, we propose a simple solution to the impedance segment problem, which is valid in the high-frequency approximation for the case of oblique incidence and relatively small impedance values.

FORMULATION OF THE PROBLEM

Let, on the (x, y) plane, the total field $\tilde{u}(x, y)$ satisfy the Helmholtz equation

$$\Delta\tilde{u} + k^2\tilde{u} = 0 \tag{1}$$

everywhere except for the segment $y = 0$, $-a < x < 0$ (see Fig. 1), on the sides of which the following impedance boundary conditions are satisfied:

$$\frac{\partial\tilde{u}}{\partial n} + \eta\tilde{u} = 0, \tag{2}$$

where n is the normal vector to the strip (it points upward in the upper half-plane and downward in the lower half-plane) and η is the impedance. The impedance obeys the condition of no energy radiation:

$$\mathrm{Im}[\eta] \geq 0. \tag{3}$$

The time dependence is chosen so that a wave propagating in the positive direction has the form $\exp(ikx)$, i.e., the time dependence of all the variable quantities has the form $\exp(-i\omega t)$. We assume that the wavenumber k has a small positive imaginary part according to the principle of ultimate absorption.

The total field is the sum of the incident field $\tilde{u}^{\mathrm{in}}$ and the scattered field $\tilde{u}^{\mathrm{sc}}$:

$$\tilde{u} = \tilde{u}^{\mathrm{in}} + \tilde{u}^{\mathrm{sc}}, \tag{4}$$

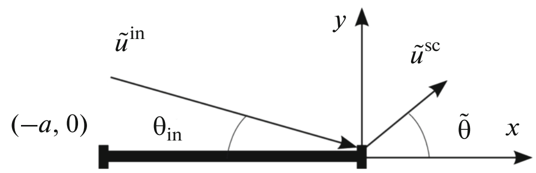

Fig. 1. Geometry of problem.

where

$$\tilde{u}^{\rm in} = \exp\{ikx\cos\theta_{\rm in} - iky\sin\theta_{\rm in}\}. \quad (5)$$

Here, $\theta_{\rm in}$ is the angle of incidence. In addition, it is necessary to satisfy the Meixner conditions at the vertices and the Sommerfeld radiation conditions at infinity.

We introduce the directional pattern of the scattered field $f(\tilde{\theta}, \theta_{\rm in})$:

$$\tilde{u}^{\rm sc}(x, y) = f(\tilde{\theta}, \theta_{\rm in})\sqrt{\frac{k}{2\pi ir}}e^{ikr} + o\left((kr)^{-1/2}\right), \quad \tan\tilde{\theta} = \frac{y}{x}, \quad r = \sqrt{x^2 + y^2}. \quad (6)$$

TRANSITION TO THE PARABOLIC APPROXIMATION

Let us consider high-frequency wave diffraction at oblique incidence; i.e., we assume that the following conditions are satisfied:

$$ka \gg 1, \quad \theta_{\rm in} \ll 1. \quad (7)$$

We study the wave process in which the wave propagation direction is nearly parallel to the x axis and the angular spectrum of waves is sufficiently narrow. In this case, the parabolic approximation is valid [14]. The transition to the parabolic approximation is as follows. From the total field, we separate the oscillating factor

$$\tilde{u} = \exp(ikx)u,$$

and replace the Helmholtz equation by the parabolic equation

$$\left(2ik\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2}\right)u = 0. \quad (8)$$

The parabolic approximation is a standard tool of diffraction theory [15, 16]. Applicability of the parabolic approximation to problems of diffraction by screens is discussed in more detail in [17].

In the parabolic approximation, the incident wave has the form

$$u^{\rm in} = \exp\left\{-ikx\frac{\theta_{\rm in}^2}{2} - iky\theta_{\rm in}\right\}. \quad (9)$$

We construct a parabolic analog of Eq. (6). With distance from the strip, for fixed a and k, the scattered field has the form

$$u^{\rm sc}(x, y) = S(\theta, \theta_{\rm in})g(x, y) + o((kx)^{-1/2}), \quad \theta = y/x. \quad (10)$$

Here,

$$g(x, y) = \sqrt{\frac{k}{2\pi ix}}\exp\left\{\frac{iky^2}{2x}\right\} \quad (11)$$

is the Green's function of the parabolic equation of an infinite plane. We assume that the quantity S represents the directional pattern of the parabolic problem. From comparison of Eqs. (10) and (6), we obtain the relation between the directional patterns $f(\tilde{\theta}, \theta_{\rm in})$ and $S(\theta, \theta_{\rm in})$:

$$S(\theta, \theta_{\rm in}) \approx f(\tilde{\theta}, \theta_{\rm in}). \quad (12)$$

The approximate nature of the above formula is determined by the fact that the parabolic approximation is valid for only a narrow angular spectrum region. Moreover, the formula $\theta \approx \tilde{\theta}$ is valid for small angles only.

IMPEDANCE BOUNDARY CONDITIONS: THE MALYUZHINETS APPROACH

In monograph [18], a method was developed for reducing impedance boundary problems to problems with Dirichlet conditions. The idea of the method is fairly simple. In our case, it can be described as follows. Instead of the field $u(x, y)$, we consider the field

$$\zeta(x, y) = T_{\pm}[u(x, y)] \equiv \left(\pm\frac{\partial}{\partial y} + \eta\right)u(x, y). \quad (13)$$

The upper sign corresponds to the field in the upper half-plane, and the lower sign, to the field in the lower half-plane. Operators $T_{\pm}$ commute with the parabolic equation operator; i.e., the field $\zeta(x, y)$ satisfies the parabolic equation. By virtue of Eq. (2), $\zeta(x, y)$ satisfies the Dirichlet boundary conditions

$$\zeta(x, 0) = 0, \quad -a < x < 0.$$

Hence, it is necessary to determine the solution to the diffraction problem for a strip with the Dirichlet boundary conditions (such a solution was obtained in [1]) and then to apply the inverse operator $T_{\pm}^{-1}$ to the aforementioned solution.

Instead of directly calculating the inverse operator, we do something simpler; namely, we select a solution that satisfies the impedance boundary conditions.

SOLUTION TO THE PARABOLIC EQUATION

Let us recall the main properties of the parabolic equation. First, since Eq. (8) involves the first derivative with respect to x, the parabolic equation describes only waves propagating from left to right. Second, in any region $x' < x < x''$ without obstacles, the field $u(x, y)$ is described by the integral formula

$$u(x,y) = \int_{-\infty}^{\infty} u(x',y')g(x-x',y-y')dy', \quad (14)$$

where $g(x, y)$ is given by Eq. (11). Hence, if we determine the field $u(x, y)$ on the line $x = 0$, $\infty > y > -\infty$, the problem will be solved. We can naturally separate the plane in four regions. In the region $\infty > y > -\infty$, $-\infty < x \le -a$, we denote the field as u_0, in the region $y \ge 0$, $-a \le x \le 0$, as u_1, in the region $y \le 0$, $-a < x < 0$, as u_2, and in the region $x > a$, as u_3 (see Fig. 2).

Field $u_0(x, y)$ represents only incident plane wave (9). Field $u_1(x, y)$ should satisfy the conditions

$$u_1(-a,y) = \exp\left\{ika\theta_{in}^2/2\right\}\exp\{-iky\theta_{in}\}, \quad y > 0, \quad (15)$$

$$\left(\frac{\partial}{\partial y} + \eta\right)u_1(x,0) = 0, \quad x > -a, \quad (16)$$

while field $u_2(x, y)$ should satisfy the conditions

$$u_2(-a,y) = \exp\left\{ika\theta_{in}^2/2\right\}\exp\{-iky\theta_{in}\}, \quad y < 0, \quad (17)$$

$$\left(-\frac{\partial}{\partial y} + \eta\right)u_2(x,0) = 0, \quad x > -a. \quad (18)$$

Let us show that the expressions for the fields

$$u_1(x,y) = \exp\left\{ika\theta_{in}^2/2\right\} \times \int_{-\infty}^{\infty} \psi_1(y')g(x+a,y-y')dy', \quad (19)$$

$$u_2(x,y) = \exp\left\{ika\theta_{in}^2/2\right\} \times \int_{-\infty}^{\infty} \psi_2(y')g(x+a,y-y')dy', \quad (20)$$

$$\psi_1(y) = \begin{cases} \exp\{-ik\theta_{in}y\}, & y > 0, \\ -\dfrac{\eta - ik\theta_{in}}{\eta + ik\theta_{in}}\exp\{ik\theta_{in}y\} \\ + \dfrac{2\eta}{\eta + ik\theta_{in}}\exp\{-\eta y\}, & y < 0, \end{cases} \quad (21)$$

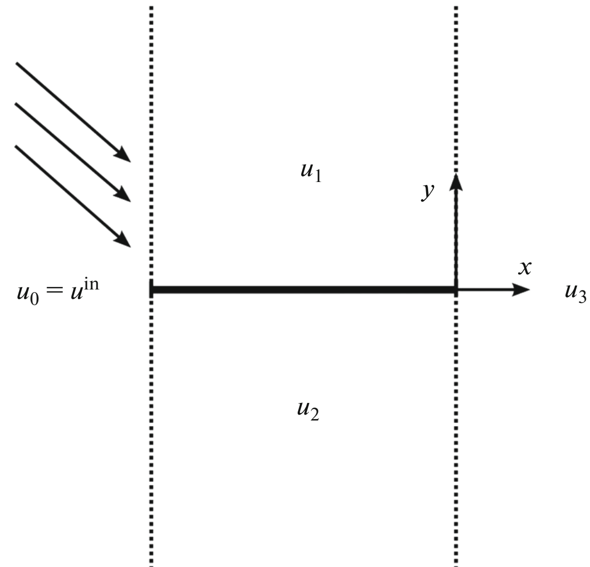

Fig. 2. Geometry of regions u_0, u_1, u_2, and u_3.

$$\psi_2(y) = \begin{cases} -\dfrac{\eta + ik\theta_{in}}{\eta - ik\theta_{in}}\exp\{ik\theta_{in}y\} \\ + \dfrac{2\eta}{\eta - ik\theta_{in}}\exp\{\eta y\}, & y > 0, \\ \exp\{-ik\theta_{in}y\}, & y < 0, \end{cases} \quad (22)$$

satisfy conditions (15)–(18). As an example, we consider $u_1(x, y)$. First of all, despite the presence of exponentially growing factors, convergence of the integrals is ensured by the small positive imaginary part of k and, as a consequence, by the superexponential descent of Green's function (11). Condition (15) is ensured by the first row of Eq. (21) and the fact that superexact formula (18) is continuous in x. Let us proceed to the second condition. We apply operator T_+ to Eq. (19). Note that operator T_+ commutes with the integral operator. Then, we have

$$T_+[\psi_1] = \begin{cases} (\eta - ik\theta_{in})\exp\{-iky\theta_{in}\}, & y > 0, \\ -(\eta - ik\theta_{in})\exp\{iky\theta_{in}\}, & y < 0. \end{cases} \quad (23)$$

Function ψ_1 is continuous (this is ensured by the second term in the second row), and, therefore, $T_+[\psi_1]$ contains no delta function. Function (23) is odd; therefore, we obtain

$$\int_{-\infty}^{\infty} T_+[\psi_1](y')g(x,0-y')dy' = T_+\left[\int_{-\infty}^{\infty} \psi_1(y')g(x,0-y')dy'\right] = 0. \quad (24)$$

Formulas (19) and (20) represent the fields u_1 and u_2. Using the values obtained on the line $x' = 0$, we can calculate the field u_3 according to Eq. (14).

CALCULATION OF THE DIRECTIONAL PATTERN $S(\theta, \theta_{in})$

The directional pattern can be calculated by the formula

$$S(\theta,\theta_{in}) = \int_{-\infty}^{\infty} u^{sc}(0,y)\exp\{-iky\theta\}dy. \quad (25)$$

The latter expression follows from Eq. (14). Let us prove this. The field $u^{sc}(x,y)$ that forms in the region $x > 0$ is expressed as

$$u^{sc}(x,y) = \int_{-\infty}^{\infty} u^{sc}(0,y')g(x,y-y')dy' = \sqrt{\frac{k}{2\pi ix}} \times \int_{-\infty}^{\infty} u^{sc}(0,y')\exp\left\{\frac{ik(y-y')^2}{2x}\right\}dy' = \sqrt{\frac{k}{2\pi ix}} \times \exp\left\{\frac{iky^2}{2x}\right\}\int_{-\infty}^{\infty} u^{sc}(0,y')\exp\left\{-\frac{ikyy'}{x} + \frac{ik(y')^2}{2x}\right\}dy'. \quad (26)$$

Now, in the latter expression, we pass to the limit of large x, y for constant $\theta = y/x$. We obtain Eq. (10) with directional pattern (25).

From u_1 and u_2, we select the scattered components (by subtracting u^{in}). We use the fact that, for $x_2 > x_1$,

$$u^{in}(x_2,y) = \int_{-\infty}^{\infty} u^{in}(x_1,y')g(x_2-x_1,y-y')dy'. \quad (27)$$

Then, we obtain

$$u_1^{sc}(x,y) = \exp\left\{ika\theta_{in}^2/2\right\} \times \int_{-\infty}^{0} \psi_1^{sc}(y')g(x+a,y-y')dy', \quad (28)$$

$$u_2^{sc}(x,y) = \exp\left\{ika\theta_{in}^2/2\right\} \times \int_{0}^{\infty} \psi_2^{sc}(y')g(x+a,y-y')dy', \quad (29)$$

$$\psi_1^{sc}(y) = -\frac{\eta - ik\theta_{in}}{\eta + ik\theta_{in}}\exp\{ik\theta_{in}y\} - \exp\{-ik\theta_{in}y\} + \frac{2\eta}{\eta + ik\theta_{in}}\exp\{-\eta y\}, \quad (30)$$

$$\psi_2^{sc}(y) = -\frac{\eta + ik\theta_{in}}{\eta - ik\theta_{in}}\exp\{ik\theta_{in}y\} - \exp\{-ik\theta_{in}y\} + \frac{2\eta}{\eta - ik\theta_{in}}\exp\{\eta y\}. \quad (31)$$

Now, using Eqs. (28)–(31) and (25), we determine the directional pattern

$$S(\theta,\theta_{in}) = \exp\left\{\frac{ika\theta_{in}^2}{2}\right\}\Bigg(-Y(-\theta,\theta_{in}) - \frac{\eta - ik\theta_{in}}{\eta + ik\theta_{in}}Y(-\theta,-\theta_{in}) - Y(\theta,-\theta_{in}) - \frac{\eta + ik\theta_{in}}{\eta - ik\theta_{in}}Y(\theta,\theta_{in}) + \frac{2\eta}{\eta + ik\theta_{in}} \times Y(-i\eta/k,-\theta) + \frac{2\eta}{\eta - ik\theta_{in}}Y(-i\eta/k,\theta)\Bigg), \quad (32)$$

where

$$Y(\theta_1,\theta_2) = \sqrt{\frac{k}{2\pi ia}} \times \int_0^{\infty}\int_0^{\infty} \exp\left\{ik\left(\theta_1 y_1 + \theta_2 y_2 + \frac{(y_1+y_2)^2}{2a}\right)\right\}dy_1 dy_2. \quad (33)$$

Let us calculate integral (33). For this, we represent it in the form

$$Y(\theta_1,\theta_2) = \frac{1}{2}\exp\left\{-ika\frac{\theta_2^2}{2}\right\} \times \int_0^{\infty} \exp\{ik(\theta_1-\theta_2)y_1\}\operatorname{erfc}\left(\left(\theta_2 + \frac{y_1}{a}\right)\sqrt{\frac{ka}{2i}}\right)dy_1, \quad (34)$$

where

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-\tau^2}d\tau.$$

Integrating Eq. (34) by parts, we obtain

$$Y(\theta_1,\theta_2) = \frac{1}{2ik(\theta_1-\theta_2)}\left(\exp\left\{-ika\frac{\theta_1^2}{2}\right\} \times \operatorname{erfc}\left(\theta_1\sqrt{\frac{ka}{2i}}\right) - \exp\left\{-ika\frac{\theta_2^2}{2}\right\}\operatorname{erfc}\left(\theta_2\sqrt{\frac{ka}{2i}}\right)\right). \quad (35)$$

Formulas (32) and (35) give the expression for the directional pattern in terms of single quadratures.

OPTICAL THEOREM

In [1], we introduced the notion of the total scattering cross section for the parabolic problem. We also proved the optical theorem for a strip with ideal boundary conditions. Below, we formulate the optical theorem for a strip with impedance boundary conditions. The theorem is formulated without proof, because the latter is analogous to that given in [1].

Theorem. The total scattering cross section $\Sigma = \int_{-\infty}^{\infty} |u^{sc}(x,y)| dy$, $x > 0$ is related to the forward scattering amplitude $S(-\theta_{in}, \theta_{in})$ as follows:

$$\Sigma = -2\,\mathrm{Re}[S(-\theta_{in}, \theta_{in})] - 2\,\mathrm{Im}[\eta] \int_{-a}^{0} \left(|u(x,+0)|^2 + |u(x,-0)|^2 \right) dx. \quad (36)$$

The second term appearing on the right-hand side of Eq. (36) and taken with the minus sign is called the absorption cross section, which represents the energy absorbed by the strip. The sum of the absorption cross section and the total scattering cross section is called the extinction cross section (see [19, 20]).

THE SURFACE WAVE

We return to the solution given by Eqs. (28) and (30). The term

$$u_1^{sp}(x,y) = \exp\left\{ika\theta_{in}^2/2\right\} \times \int_{-\infty}^{0} \frac{2\eta}{\eta + ik\theta_{in}} \exp\{-\eta y'\} g(x+a, y-y') dy'$$

corresponds to the surface wave and the surface half-shadow (i.e., the zone of surface wave formation) in the region $-a < x < 0$, $y > 0$. We represent it in the form

$$u_1^{sp}(x,y) = \frac{\eta}{\eta + ik\theta_{in}} \times \exp\left\{\frac{ika\theta_{in}^2}{2}\right\} \exp\left\{i\frac{\eta^2(a+x)}{2k} - \eta y\right\} \times \mathrm{erfc}\left[-\sqrt{\frac{i}{2k(a+x)}}\left(\eta(a+x) + iky\right)\right]. \quad (37)$$

For simplicity, we assume that $\mathrm{Im}[\eta] = 0$. We consider the case of $\mathrm{Re}[\eta] > 0$. We also assume that $ky \ll \eta(a+x)$, i.e., we analyze the field near the segment. For the function $\mathrm{erfc}(z)$, the following asymptotic formula is valid:

$$\mathrm{erfc}(z) = \begin{cases} 2 + \dfrac{e^{-z^2}}{z\sqrt{\pi}}, & \mathrm{Re}[z] \to -\infty, \\ \dfrac{e^{-z^2}}{z\sqrt{\pi}}, & \mathrm{Re}[z] \to \infty. \end{cases} \quad (38)$$

The constant involved in this formula is related to the contribution of the saddle point. In the case of positive impedances, the integration contour appearing in Eq. (38) passes through the saddle point and, hence, we obtain

$$u_1^{sp}(x,y) \approx \frac{2\eta}{\eta + ik\theta_{in}} \exp\left\{\frac{ika\theta_{in}^2}{2}\right\} \times \exp\left\{i\frac{\eta^2(a+x)}{2k} - \eta y\right\} - \frac{2}{\eta + ik\theta_{in}} \exp\left\{\frac{ika\theta_{in}^2}{2}\right\} g(x+a, y). \quad (39)$$

The first term appearing in Eq. (39) represents the surface wave, and the second term represents the edge wave decaying as $(x+a)^{-1/2}$ along the strip. For $\mathrm{Re}[\eta] < 0$, it is necessary to use the second asymptotics from Eq. (38); i.e., in this case, the expression for the field contains only the edge wave whereas the surface wave is not excited. The case of $(x + a) \sim \frac{k}{\eta^2}$ corresponds to the transition region between the asymptotics, i.e., the half-shadow zone. The length of the half-shadow along the x axis is on the order of $\frac{k}{\eta^2}$.

We denote the surface wave amplitude as

$$A = \frac{2\eta}{\eta + ik\theta_{in}}. \quad (40)$$

In Appendix A, the surface wave amplitude is calculated by solving the exact problem by the Wiener–Hopf method. Conditions under which the exact amplitude is close to A are determined.

NUMERICAL RESULTS

The above results were tested for correctness by numerical simulation. A comparison with the exact solution obtained by the integral equation method (see Appendix B) was carried out.

It is well known that the parabolic approximation works well for $\sqrt[4]{ka}\theta_{in} \ll 1$ and $ka \gg 1$. In the case of impedance boundary conditions, another important parameter is the quantity $\eta\sqrt{a/k}$. For $\eta\sqrt{a/k} \ll 1$, the surface wave does not have enough time to develop and the impedance can be considered small. Many numerical experiments were performed within the range of parameters obeying the aforementioned limitations, and in all cases, Eq. (32) appeared to agree well with the exact solution. Moreover, the parabolic approximation also yields results close to exact at the boundary of its domain of applicability. Figure 3 shows the results of numerical simulation for $ka = 200$, $\eta a = 20$, and $\theta_{in} = 0.3$, i.e., for the case where only one out of the three conditions is satisfied: $ka \gg 1$,while $\sqrt[4]{ka}\theta_{in} \sim 1$ and $\eta\sqrt{a/k} \sim 1$. Still, Eq. (32) yields good agreement with the exact solution. Figure 4 shows the results of numerical simulation for $ka = 200$, $\eta a = 60$, and $\theta_{in} = 0.3$. As one would

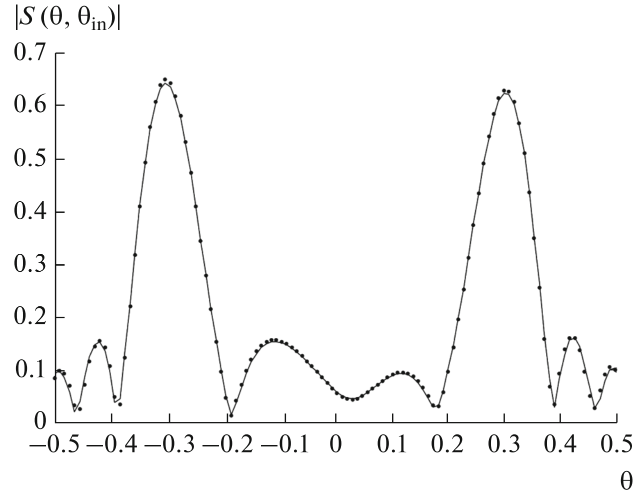

Fig. 3. Magnitude of directional pattern for $ka = 200$, $\eta a = 20$, and $\theta_{in} = 0.3$. The solid line corresponds to results of exact calculations, and the dashed line, to results of calculation by Eq. (32).

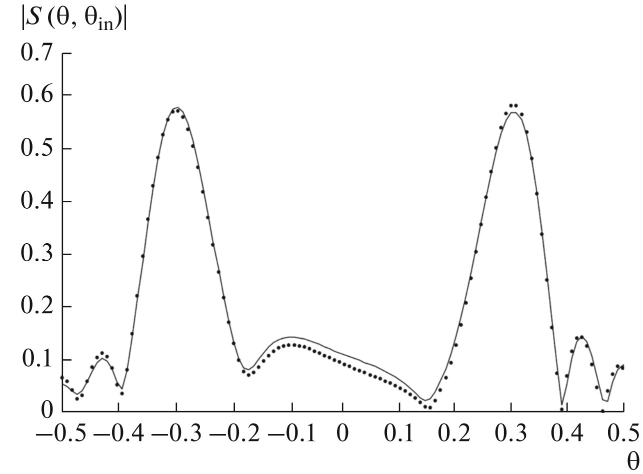

Fig. 4. Magnitude of directional pattern for $ka = 200$, $\eta a = 60$, and $\theta_{in} = 0.3$. The solid line corresponds to results of exact calculations, and the dashed line, to results of calculation by Eq. (32).

expect, with an increase in the impedance value, the accuracy of the parabolic solution decreases.

CONCLUSIONS

In this paper, we extended the ideas described in [1] to the case of diffraction by an impedance strip. We derived an expression in terms of single quadratures for the directional pattern in the parabolic approximation. In addition, we numerically verified the results. We also formulated the optical theorem for the parabolic problem and calculated the surface wave amplitude.

APPENDIX A. SOLUTION OF THE IMPEDANCEHALF-PLANE PROBLEM BY THE WIENER–HOPF METHOD

Below, we obtain an exact solution to the problem of diffraction by an impedance half-plane and calculate the surface wave amplitude. The resulting expression is compared with Eq. (40).

Let the impedance half-line occupy the region $y = 0$, $x > 0$. Let boundary condition (2) be satisfied on the impedance half-plane. We solve the problem for Helmholtz equation (1) with incident wave (5). Using the fact that the impedance is taken as the same on both sides of the half-plane, we represent the scattered field as the half-sum of symmetric and antisymmetric components:

$$\tilde{u}^{sc} = \frac{1}{2}\left(u^s + u^a\right), \tag{42}$$

where

$$u^s(x,-y) = u^s(x,y), \quad u^a(x,-y) = -u^a(x,y).$$

For the symmetric and antisymmetric cases, the incident waves are represented as

$$u^{in,s}(x,y) = \exp\{ikx\cos\theta_{in} - iky\sin\theta_{in}\} + \exp\{ikx\cos\theta_{in} + iky\sin\theta_{in}\},$$

$$u^{in,a}(x,y) = \exp\{ikx\cos\theta_{in} - iky\sin\theta_{in}\} - \exp\{ikx\cos\theta_{in} + iky\sin\theta_{in}\}.$$

For the symmetric problem, the Wiener–Hopf equation has the form

$$U_-(\xi,\xi_*) + K(\xi)V_+(\xi,\xi_*) = -\frac{2i}{\xi+\xi_*}, \tag{43}$$

where

$$K(\xi) = \frac{i\left(\eta + i\sqrt{k^2-\xi^2}\right)}{\eta\sqrt{k^2-\xi^2}}, \tag{44}$$

$$V_+(\xi,\xi_*) = \int_0^\infty \frac{\partial u^s(x,0)}{\partial n} e^{i\xi x}dx, \tag{45}$$

$$U_-(\xi,\xi_*) = \int_{-\infty}^0 u^s(x,0)e^{i\xi x}dx, \tag{46}$$

$$\xi_* = k\cos\theta_{in}. \tag{47}$$

Function $V_+(\xi,\xi_*)$ is analytic with respect to variable ξ in the upper half-plane, and function $U_-(\xi,\xi_*)$ is analytic in the lower half-plane.

An exact solution to Eq. (43) was obtained in [21] by direct factorization of symbol $K(\xi)$. Let us write the formal solution. We assume that

$$K(\xi) = K_+(\xi)K_-(\xi), \tag{48}$$

where function $K_+(\xi)$ is analytic and has no zero values in the upper half-plane and function $K_-(\xi)$ has similar properties in the lower half-plane (in addition, the growth requirements are satisfied, see [22]). Then,

$$\frac{U_-(\xi,\xi_*)}{K_-(\xi)} + K_+(\xi)V_+(\xi,\xi_*) = -\frac{2i}{\xi+\xi_*}\frac{1}{K_-(\xi)} + \frac{2i}{\xi+\xi_*}\frac{1}{K_-(-\xi_*)}, \tag{49}$$

$$V_+(\xi) = -\frac{2i}{\xi+\xi_*}\frac{K_+^{-1}(\xi)}{K_-(-\xi_*)}. \tag{50}$$

The scattered field formed at the impedance plane can be calculated by inversion of Eq. (45):

$$u^{\mathrm{s}}(x,+0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} {}^{-i\xi x}\frac{1}{i\sqrt{k^2-\xi^2}}V_+(\xi)d\xi e. \tag{51}$$

Now, we proceed to calculating the surface wave amplitude. We assume that function $K_+^{-1}(\xi)$ has a pole at $\xi = -\tilde{\xi} = -\sqrt{\eta^2+k^2}$. Let the residue at the pole be

$$\operatorname{Re} s[K_+^{-1}(\xi), \xi = -\tilde{\xi}] = M. \tag{52}$$

The contribution to the field because of the aforementioned residue is the surface wave

$$u_\eta^{\mathrm{s}}(x,+0) = e^{i\tilde{\xi}x}\frac{-2}{\eta(\tilde{\xi}-\xi_*)}\frac{M}{K_-(-\xi_*)}. \tag{53}$$

Thus, the surface wave amplitude is

$$\tilde{A}_{\mathrm{s}} = \frac{-2}{\eta(\tilde{\xi}-\xi_*)}\frac{M}{K_-(-\xi_*)}. \tag{54}$$

We expand $K(\xi)$ in a Taylor series at the point $\xi = -\tilde{\xi}$:

$$K(\xi) = (\xi+\tilde{\xi})B_0 + \ldots, \tag{55}$$

where $B_0 = -\frac{\sqrt{\eta^2+k^2}}{\eta^3}$. We assume that θ_{in} and η are small. In addition, function $K_-(\xi)$ is regular near point $-k$. Then,

$$K_-(-\tilde{\xi}) \approx K_-(-\xi_*) \approx K_-(-k)$$

and $M \approx \frac{K_-(-k)}{B} = -K_-(-k)\frac{\eta^3}{\sqrt{\eta^2+k^2}}$.

As a result, we obtain an approximate value of the surface wave amplitude for the antisymmetric problem:

$$\tilde{A}_{\mathrm{s}} = \frac{4\eta^2}{\eta^2+\theta_{\mathrm{in}}^2k^2}. \tag{56}$$

The Wiener–Hopf equation for the antisymmetric problem is

$$\hat{V}_-(\xi,\xi_*) + \hat{K}(\xi)\hat{U}_+(\xi,\xi_*) = -\frac{2i}{\xi+\xi_*}, \tag{57}$$

where

$$\hat{K}(\xi) = i\left(\eta + i\sqrt{k^2-\xi^2}\right), \tag{58}$$

$$\hat{V}_-(\xi,\xi_*) = \int_{-\infty}^{0}\frac{\partial u^{\mathrm{a}}(x,0)}{\partial n}e^{i\xi x}dx, \tag{59}$$

$$\hat{U}_+(\xi,\xi_*) = \int_{0}^{\infty}u^{\mathrm{a}}(x,0)e^{i\xi x}dx. \tag{60}$$

The antisymmetric component of the surface wave amplitude is calculated in a similar way:

$$\tilde{A}_{\mathrm{a}} = -\frac{4i\eta k\theta_{\mathrm{in}}}{\eta^2+\theta_{\mathrm{in}}^2k^2}. \tag{61}$$

For the total surface wave amplitude, we obtain

$$\tilde{A} = \frac{1}{2}\left[\tilde{A}_{\mathrm{s}} + \tilde{A}_{\mathrm{a}}\right] = \frac{2\eta}{\eta + ik\theta_{\mathrm{in}}}. \tag{62}$$

Thus, we have derived an expression for the amplitude, and this expression completely coincides with Eq. (40); i.e., the parabolic approximation adequately describes the surface wave contribution under the condition that θ_{in} and η are small.

APPENDIX B. INTEGRAL EQUATION METHOD

To verify Eq. (32), we solved the diffraction problem numerically by the integral equation method. As in Appendix A, the scattered field was represented as a combination of symmetric and antisymmetric components (see (42)).

In the antisymmetric case, we have the following integral equation (see [23]):

$$\left(\frac{\partial^2}{\partial x^2}+k^2\right)\int_{-a}^{0}G(x-x',0)\nu(x')dx' - \frac{1}{2}\eta\nu(x) = ik\sin\theta_{\mathrm{in}}\exp\{-ikx\cos\theta_{\mathrm{in}}\}, \tag{63}$$

where

$$G(x,y) = -\frac{i}{4}H_0^{(1)}\left(k\sqrt{x^2+y^2}\right). \tag{64}$$

$H_0^{(1)}(z)$ is the Hankel function of the first kind, and $\nu(x)$ is the double-layer potential:

$$u^{\mathrm{a}}(x,y) = -\int_{-a}^{0} \frac{\partial}{\partial y} G(x-x',y)\,\nu(x')dx'. \quad (65)$$

Here, u^{a} is the antisymmetric component of the scattered field $\tilde{u}^{\mathrm{sc}}$. At the strip, the double-layer potential $\nu(x)$ is related to $u^{\mathrm{a}}(x,y)$ by a simple formula:

$$u^{\mathrm{a}}(x,+0) = -\frac{1}{2}\nu(x), \quad -a < x < 0. \quad (66)$$

The antisymmetric part of the scattering pattern is calculated as

$$f^{\mathrm{a}}(\tilde{\theta},\theta_{\mathrm{in}}) = -e^{-i\pi/4} k \sin\tilde{\theta} \int_{-a}^{0} u^{\mathrm{a}}(x,+0) e^{-ikx\cos\tilde{\theta}} dx. \quad (67)$$

In the symmetric case, the following integral equation is valid:

$$\frac{1}{2}\mu(x) + \eta\int_{-a}^{0} G(x-x',0)\mu(x')dx' = -\eta\exp\{-ikx\cos\theta_{\mathrm{in}}\}, \quad (68)$$

where $\mu(x)$ is the double-layer potential:

$$u^{\mathrm{s}}(x,y) = \int_{-a}^{0} G(x-x',y)\mu(x')dx', \quad \frac{\partial u^{\mathrm{s}}}{\partial y}(x,+0) = \frac{1}{2}\mu(x), \quad -a < x < a. \quad (69)$$

Here, u^{s} is the symmetric component of the scattered field $\tilde{u}^{\mathrm{sc}}$. The symmetric part of the directional pattern is calculated as

$$f^{\mathrm{s}}(\tilde{\theta},\theta_{\mathrm{in}}) = e^{-i\pi/4} \int_{-a}^{0} \frac{\partial u^{\mathrm{s}}}{\partial y}(x,+0) e^{-kx\cos\theta} dx. \quad (70)$$

Equations (63) and (68) are solved numerically by standard methods, and Eqs. (67) and (70) are used to calculate the symmetric and antisymmetric parts of the directional pattern. The directional pattern $f(\tilde{\theta},\theta_{\mathrm{in}})$ is calculated according to the formula

$$f(\tilde{\theta},\theta_{\mathrm{in}}) = \frac{1}{2}\left[f^{\mathrm{a}}(\tilde{\theta},\theta_{\mathrm{in}}) + f^{\mathrm{s}}(\tilde{\theta},\theta_{\mathrm{in}})\right]. \quad (71)$$

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REFERENCES

1. A. I. Korol'kov and A. V. Shanin, Acoust. Phys. 2016. (in press).
2. A. V. Popov, Akust. Zh. **19**, 594 (1973).
3. A. V. Shanin and A. I. Korolkov, Quart. J. Mech. Appl. Math. **68**, 321 (2015).
4. A. V. Shanin and A. I. Korolkov, Quart. J. Mech. Appl. Math. **68**, 341 (2015).
5. M. I. Herman and J. L. Volakis, Radio Sci. **22**, 335 (1987).
6. A. H. Serbest and G. Uzgören, Radio Sci. **24**, 455 (1989).
7. B. A. Senior, IEEE Trans. Ant. Propag. **27**, 808 (1979).
8. J. L. Volakis and S. S. Bindiganavale, Radio Sci. **31**, 401 (1996).
9. W. D. Burnside, C. L. Yu, and R. J. Marhefka, IEEE Trans. Ant. Propag. **23**, 551 (1975).
10. J. N. Sahalos and G. A. Thiele, IEEE Trans. Ant. Propag. **29**, 780 (1981).
11. T. Ikiz, S. Koshikawa, K. Kobayashi, E. I. Veliev, and A. H. Serbest, J. Electromagn. Waves Appl. **15**, 315 (2001).
12. A. H. Serbest and A. Büyükaksoy, *Some Approximate Methods Related to the Diffraction by Strips and Slits,* Ed. by M. Hashimoto, M. Idemen, and O. A. Tretyakov, (Sci. House, Tokyo, 1993).
13. Y. Z. Umul, Optics Commun. **284**, 1760 (2011).
14. Vinogradova, M.B., Rudenko, O.V., and Sukhorukov, A.P., *Theory of Waves* (Nauka, Moscow, 1979), [in Russian].
15. V. A. Fok, Electromagnetic Wave Diffraction and Propagation Problems (Sov. Radio, Moscow, 1970) [in Russian].
16. G. D. Malyuzhinets, Phys.-Usp. **2**, 749 (1959).
17. A. I. Korol'kov and A. V. Shanin, J. Mathem. Sci. **214**, 302 (2014).
18. G. D. Malyuzhinets, *Generalization of Reflection Method in Theory of Diffraction* (TsNII "RUMB", Leningrad, 1981) [in Russian].
19. S. Ström, in *The Scattered Field. Field Representation and Introduction to Scattering*, Ed. by V. V. Varadan, A. Lakhtakia, and V. K. Varadan, (Elsevier Sci., 1991), pp. 143–149.
20. M. I. Mishchenko, J. Quantitat. Spectr. Radiat. Trans. **101**, 404 (2006).
21. T. B. A. Senior, Proc. R. Soc. London, Ser. A **213**, 436 (1952).
22. B. Noble, *The Wiener-Hopf Technique for the Solution of Partial Differential Equations* (Chelsea, 1958).
23. H. Hönl, A. Maue, and K. Westphal, *Theorie der Beugung,* Encycl. Phys., Crystal Optics Diffraction, Ed. by S. Flügge, (Springer Verlag, Berlin, 1966).

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