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CLASSICAL PROBLEMS OF LINEAR ACOUSTICS AND WAVE THEORY

High-Frequency Plane Wave Diffraction by an Ideal Strip at Oblique Incidence: Parabolic Equation Approach

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Abstract—The problem of diffraction of a high-frequency plane wave by a strip with ideal boundary conditions is considered for the case of oblique incidence. The study is based on the parabolic approximation, which is used to construct an expression for the directional pattern in terms of single quadratures. A similar result is obtained using the embedding formula. It is shown that the derived expression approximates the classical Michaeli result. A proof of the optical theorem for the parabolic problem is presented.

Keywords: diffraction by a strip, reflection method, parabolic equation of diffraction theory, embedding formula
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INTRODUCTION

The problem of diffraction by a strip with ideal boundary conditions is one of the simplest canonical problems of the diffraction theory. This problem was solved by separation of variables in [1, 2]. The solution was represented as a series of Mathieu functions. Unfortunately, in the high-frequency case, the series converges rather slowly and laborious computations should be performed to achieve acceptable accuracy.

Many authors tried, but failed, to obtain a solution analogous to the Sommerfeld solution for a half-plane [3, 4].

In the high-frequency approximation, from the practical viewpoint, it is convenient to use the diffraction series approach [5–7]. If the strip is large compared to the wavelength of the incident wave, it is sufficient to take into account only several first terms of the Schwarzschild series. In addition, for high frequencies, the diffraction problem can be considered in terms of geometric diffraction theory [8]. However, in the case of oblique incidence, one of the vertices of the strip is in the half-shadow zone of the other and geometric diffraction theory cannot be applied directly. In [9], the aforementioned difficulties were overcome by cumbersome computations.

In this paper, we consider the problem of diffraction of a high-frequency plane wave by a strip with ideal boundary conditions (Neumann or Dirichlet) in the case of oblique incidence. Our study is motivated by a talk given by I.V. Andronov at the seminar "Diffraction and Propagation of Waves," held under the supervision of V.M. Babich at the St. Petersburg Department of the Steklov Institute of Mathematics, Russian Academy of Sciences. The author constructed the solution for the case of a strip as the limiting case of the solution to the problem of diffraction by a thin elliptical cylinder [10, 11]. Separation of variables and rather subtle properties of special functions were used. We note that the technique developed earlier by us (the properties of the parabolic equation of diffraction theory for the case of diffraction by branched surfaces and the embedding formula method) makes it possible to obtain a solution without difficulty by using the error function alone. The result obtained in this way agrees numerically with the data reported by I.V. Andronov.

The formula closest to our result was given in [12]. There, the field that formed at the strip was determined using physical diffraction theory. As a result, the formula obtained for the directivity lacked symmetry under permutation of the angle of incidence and the scattering angle (i.e., it was nonreciprocal). Next, the formula was made reciprocal by an artificial procedure. In our paper, we actually derive anew the result reported in [12] by using a simpler and more general parabolic equation technique.

The structure of the paper is as follows. First, we formulate a stationary problem for the Helmholtz equation. We change to the parabolic approximation. By directly solving the parabolic equation, we construct an expression for the directivity of the scattered field in terms of unique quadratures. We prove the optical theorem for the parabolic equation and calculate the total scattering cross section. In the Appendix, we show that the expression obtained for the directivity is a particular case of the so-called "embedding formula."

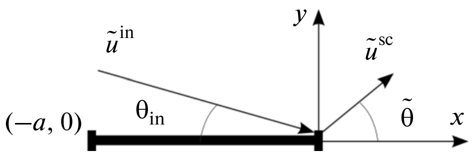

Fig. 1. Geometry of problem.

FORMULATION OF THE PROBLEM

We assume that the total field $\tilde{u}(x,y)$ satisfies the Helmholtz equation

$$\Delta\tilde{u} + k^2\tilde{u} = 0 \quad (1)$$

on the entire (x, y) plane except for the segment $y = 0$, $-a < x < 0$ (Fig. 1), on the sides of which either the Neumann boundary conditions

$$\frac{\partial\tilde{u}}{\partial n} = 0 \quad (2)$$

or the Dirichlet boundary conditions

$$\tilde{u} = 0, \quad (3)$$

are satisfied. The time dependence is chosen so that the wave propagating in the positive direction has the form $\exp(ikx)$. We assume that the wavenumber k has a small positive imaginary part in accordance with the limiting absorption principle.

The total field is represented as the sum of the incident field $\tilde{u}^{\text{in}}$ and the scattered field $\tilde{u}^{\text{sc}}$:

$$\tilde{u} = \tilde{u}^{\text{in}} + \tilde{u}^{\text{sc}}, \quad (4)$$

where

$$\tilde{u}^{\text{in}} = \exp\{ikx\cos\theta_{\text{in}} - iky\sin\theta_{\text{in}}\}. \quad (5)$$

Here, θ_{in} is the angle of incidence. In addition, it is necessary to satisfy the Meixner conditions at the vertices and the Sommerfeld radiation conditions at infinity. We introduce the directivity of the scattered field $f(\tilde{\theta})$:

$$\tilde{u}^{\text{sc}}(x,y) = f(\tilde{\theta},\theta_{\text{in}})\sqrt{\frac{k}{2\pi i\sqrt{x^2+y^2}}}e^{ik\sqrt{x^2+y^2}} + o\left(\left(k\sqrt{x^2+y^2}\right)^{-1/2}\right), \quad \tan\tilde{\theta} = \frac{y}{x}. \quad (6)$$

TRANSITION TO THE PARABOLIC APPROXIMATION

Now, we consider high-frequency wave diffraction in the case of oblique incidence; i.e., we assume that the following conditions are satisfied:

$$ka \gg 1, \quad \theta_{\text{in}} \ll 1. \quad (7)$$

We study the wave process where a wave with a narrow angular spectrum propagates in a direction nearly parallel to the x axis. In this case, the parabolic approximation is valid [13]. The transition to the parabolic approximation is as follows. From the total field, we separate the oscillating factor

$$\tilde{u} = \exp(ikx)u,$$

and replace the Helmholtz equation by the parabolic equation

$$\left(2ik\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2}\right)u = 0. \quad (8)$$

Parabolic approximation is a standard tool of diffraction theory [14, 15]. The applicability of the parabolic equation to the problems of diffraction by screens is considered in more detail in [16].

In the parabolic approximation, the incident wave has the form

$$u^{\text{in}} = \exp\left\{-ikx\frac{\theta_{\text{in}}^2}{2} - iky\theta_{\text{in}}\right\}. \quad (9)$$

We solve the problem as follows. It is necessary to find a solution to Eq. (8) on the plane with the eliminated scatterer (the segment $y = 0$, $-a < x < 0$) so that the solution is continuous everywhere except for the scatterer, continuous on one side and differentiable at the scatterer except for its ends, and bounded near the ends. The solution should coincide with incident wave (9) in the region $x < -a$ and satisfy the Neumann or Dirichlet boundary conditions (given by Eqs. (2) and (3), respectively) at the sides of the scatterer.

The main problem consists in determining the directivity. Let us construct a parabolic analog of Eq. (6). At fixed a and k far from the strip, the scattered field can be represented as

$$u^{\text{sc}}(x,y) = \hat{S}^{N,D}(\theta,\theta_{\text{in}})g(x,y) + o((kx)^{-1/2}), \quad \theta = y/x. \quad (10)$$

Here,

$$g(x,y) = \sqrt{\frac{k}{2\pi ix}}\exp\left\{\frac{iky^2}{2x}\right\} \quad (11)$$

is the Green's function of the parabolic equation of the infinite plane. By analogy with the definition of the directional pattern in the case of the Helmholtz equation, we call $\hat{S}^{N,D}$ the directional pattern. We denote the directional patterns corresponding to the Dirichlet and Neumann boundary conditions by $\hat{S}^D$ and $\hat{S}^N$, respectively. From comparison of Eq. (10) with Eq. (6), we obtain the relation between the directional patterns $f(\tilde{\theta})$ and $\hat{S}^{N,D}(\theta,\theta_{\text{in}})$:

$$\hat{S}^{N,D}(\theta,\theta_{\text{in}}) \approx f(\tilde{\theta},\theta_{\text{in}}). \quad (12)$$

The approximate nature of the formula is due to the fact that the parabolic approximation is valid for only a narrow region of the angular spectrum. Moreover, the formula $\theta \approx \tilde{\theta}$ is satisfied for small angles only.

SOLUTION OF THE PARABOLIC EQUATION

The main advantage of the parabolic equation is that it simplifies description of wave propagation along the x axis. Indeed, within any strip $x' < x < x''$ without any obstacles, the field $u(x, y)$ is described by the integral formula

$$u(x,y) = \int_{-\infty}^{\infty} u(x',y')g(x-x',y-y')dy', \quad (13)$$

where $g(x, y)$ is given by Eq. (11). An important property of Eq. (13) is that it ensures the field continuity in x, namely:

$$\lim_{x\to x'+0} u(x,y) = u(x',y). \quad (14)$$

Formula (13) allows us to represent the solution to the problem under study in terms of quadratures. Let u_1 be the field in the region $y \ge 0$, $-a \le x \le 0$, u_2 be the field in the region $y \le 0$, $-a \le x \le 0$,; and u_3 be the field in the region $x \ge 0$. Then, we have

$$u_1(x,y) = \int_{-\infty}^{\infty} \psi_1(y')g(x+a,y-y')dy',$$
$$\psi_1(y) = \begin{cases} u^{\text{in}}(-a,y), & y>0,\\ \pm u^{\text{in}}(-a,-y), & y<0, \end{cases} \quad (15)$$

$$u_2(x,y) = \int_{-\infty}^{\infty} \psi_2(y')g(x+a,y-y')dy',$$
$$\psi_2(y) = \begin{cases} \pm u^{\text{in}}(-a,-y), & y>0,\\ u^{\text{in}}(-a,y), & y<0, \end{cases} \quad (16)$$

$$u_3(x,y) = \int_{-\infty}^{\infty} \psi_3(y')g(x,y-y')dy',$$
$$\psi_3(y) = \begin{cases} u_1(0,y), & y>0,\\ u_2(0,y), & y<0. \end{cases} \quad (17)$$

Here, the plus signs correspond to the Neumann boundary conditions, and the minus signs, to the Dirichlet boundary conditions. The fact that Eqs. (15)–(17) represent the solution to the problem can be directly verified. The functions ψ_1 and ψ_2 are constructed by the reflection method.

The directional pattern is calculated as

$$\hat{S}^{N,D}(\theta,\theta_{\text{in}}) = \int_{-\infty}^{\infty} u^{\text{sc}}(0,y)\exp\{-iky\theta\}dy. \quad (18)$$

The integrand involves the scattered field $u^{\text{sc}} \equiv u - u^{\text{in}}$ on the line $x = 0$. The latter expression follows from Eq. (13). Let us prove this. The field in the region $x > 0$ is represented as

$$u^{\text{sc}}(x,y) = \int_{-\infty}^{\infty} u^{\text{sc}}(0,y')g(x,y-y')dy' = \sqrt{\frac{k}{2\pi ix}}$$
$$\times \int_{-\infty}^{\infty} u^{\text{sc}}(0,y')\exp\left\{\frac{ik(y-y')^2}{2x}\right\}dy' = \sqrt{\frac{k}{2\pi ix}} \quad (19)$$
$$\times \exp\left\{\frac{iky^2}{2x}\right\} \int_{-\infty}^{\infty} u^{\text{sc}}(0,y')\exp\left\{-\frac{ikyy'}{x} + \frac{ik(y')^2}{2x}\right\}dy'.$$

In Eq. (19), we pass to the limit of large values of x, y at a fixed value of $\theta = y/x$. We obtain Eq. (10) with directional pattern (18).

We transform Eqs. (15) and (16) in the region $-a < x \le 0$. Note that

$$\int_{-\infty}^{\infty} u^{\text{in}}(-a,y')g(x+a,y-y')dy' = u^{\text{in}}(0,y). \quad (20)$$

Therefore, we have

$$u_1^{\text{sc}}(x,y) = \int_{-\infty}^{0} \left(\pm u^{\text{in}}(-a,-y') - u^{\text{in}}(-a,y')\right) \times g(x+a,y-y')dy', \quad (21)$$

$$u_2^{\text{sc}}(x,y) = \int_{0}^{\infty} \left(\pm u^{\text{in}}(-a,-y') - u^{\text{in}}(-a,y')\right) \times g(x+a,y-y')dy', \quad (22)$$

where the following scattered fields are introduced: $u_{1,2}^{\text{sc}} \equiv u_{1,2} - u^{\text{in}}$. Combining Eqs. (21) and (22) with Eq. (18), we obtain

$$\hat{S}^{N,D}(\theta,\theta_{\text{in}}) = S(\theta,\theta_{\text{in}}) \mp S(-\theta,\theta_{\text{in}}), \quad (23)$$

$$S(\theta,\theta_{\text{in}}) = -\exp\left\{ika\frac{\theta_{\text{in}}^2}{2}\right\} \times \left(Y(-\theta,\theta_{\text{in}}) + Y(\theta,-\theta_{\text{in}})\right), \quad (24)$$

$$Y(\theta_1,\theta_2) = \int_0^{\infty}\int_0^{\infty} \exp\{ik(\theta_1 y_1 + \theta_2 y_2)\}$$
$$\times g(a, y_1+y_2)dy_1 dy_2 = \frac{1}{2}\exp\left\{-ika\frac{\theta_2^2}{2}\right\} \quad (25)$$
$$\times \int_0^{\infty} \exp\{ik(\theta_1-\theta_2)y_1\}\operatorname{erfc}\left(\left(\theta_2 + \frac{y_1}{a}\right)\sqrt{\frac{ka}{2i}}\right)dy_1,$$

where

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-\tau^2}d\tau.$$

In Eq. (23), the upper sign corresponds to the Neumann boundary condition, and the lower sign, to the Dirichlet boundary condition.

We calculate integral (25) by integration by parts. We obtain

$$Y(\theta_1,\theta_2) = \frac{1}{2ik(\theta_1-\theta_2)}\left(\exp\left\{-ika\frac{\theta_1^2}{2}\right\}\right.$$
$$\left.\times\operatorname{erfc}\left(\theta_1\sqrt{\frac{ka}{2i}}\right) - \exp\left\{-ika\frac{\theta_2^2}{2}\right\}\operatorname{erfc}\left(\theta_2\sqrt{\frac{ka}{2i}}\right)\right). \tag{26}$$

Hence, we have

$$S(\theta,\theta_{\text{in}}) = \frac{1}{2ik(\theta+\theta_{\text{in}})}$$
$$\times\left[\operatorname{erfc}\left(-\sqrt{\frac{ak}{2i}}\theta_{\text{in}}\right) - \operatorname{erfc}\left(\sqrt{\frac{ak}{2i}}\theta_{\text{in}}\right)\right. \tag{27}$$
$$\left.+ e^{\frac{iak}{2}(\theta_{\text{in}}^2-\theta^2)}\left\{\operatorname{erfc}\left(-\sqrt{\frac{ak}{2i}}\theta\right) - \operatorname{erfc}\left(\sqrt{\frac{ak}{2i}}\theta\right)\right\}\right].$$

In Eq. (27), the dependences on the angle of incidence and the scattering angle are separated. This formula is a particular case of the so-called "embedding formulas" [17]. These formulas are fairly general. The standard derivation of a embedding formula is based on the application of a differential operator with preset properties to the total field [18]. In the Appendix, we represent the derivation of Eq. (27) as the derivation of the embedding formula. In addition, in the Appendix, we ascribe physical meaning to function S; namely, we introduce it as the directivity on a branched surface.

Formulas (27) and (23) represent the solution to the problem of diffraction by a strip in the parabolic approximation. It should be noted that the above solution coincides with the solution obtained in [12] accurate to the substitution of $\sin\theta\to\theta$, $\cos\theta\to 1$, which is valid for small angles of incidence and small scattering angles. This is astonishing, because the author of [12] applied an entirely different technique (a combination of the geometric and physical diffraction theories) and used an artificial procedure to obtain a formula satisfying the reciprocity theorem. The method used in [12] is "subtler" than that used by us, because geometric diffraction theory and physical diffraction theory make it possible to correctly determine the directional patterns of edge waves scattered at any angles. At the same time, an evident advantage of the parabolic equation method is the simplicity of describing half-shadow zones. In addition, Eq. (17) makes it possible (if necessary) to determine the uniform field asymptotics. The methods of the geometric and physical diffraction theories make it possible to construct approximate solutions to an exactly formulated diffraction problem, whereas the parabolic equation method provides an exact solution to an approximately formulated problem.

OPTICAL THEOREM FOR THE PARABOLIC EQUATION

An important tool for testing the solution to the diffraction problem is the optical theorem, which relates the total scattering cross section to the forward scattering amplitude. The appearance of negative values in the total scattering cross section points to an error in the solution. In addition, the total scattering cross section is an important characteristic of the diffraction process. Below, we introduce the notion of the total scattering cross section and derive the optical theorem for the parabolic equation. Calculations are performed for a strip with Dirichlet boundary conditions. A strip with Neumann boundary conditions can be considered in a similar way.

We introduce the total scattering cross section as the quantity

$$\Sigma = \int_{-\infty}^{\infty}\left|u^{\text{sc}}(x,y)\right|^2 dy, \tag{28}$$

calculated for a certain fixed x to the right of the scatterer. Using Eq. (13), we can easily show that the result of calculating the above integral is independent of x. From Eq. (18) with allowance for the Parseval equality, we obtain a formula expressing the scattering cross section through the directional pattern:

$$\Sigma = \frac{k}{2\pi}\int_{-\infty}^{\infty}\left|\hat{S}^D(\theta,\theta_{\text{in}})\right|^2 d\theta. \tag{29}$$

Now, we proceed to derivation of the optical theorem. For this purpose, we use the Green's theorem for the parabolic equation.

Theorem. Let, in a certain region Ω, the functions $v(x,y)$ and $w(x,y)$ satisfy the inhomogeneous equations

$$\left(\frac{\partial}{\partial x}+\frac{1}{2ik}\frac{\partial^2}{\partial y^2}\right)v = f(x,y),$$
$$\left(-\frac{\partial}{\partial x}+\frac{1}{2ik}\frac{\partial^2}{\partial y^2}\right)w = h(x,y). \tag{30}$$

Then, the following equality is satisfied:

$$\int_{\partial\Omega}[(\mathbf{v}\cdot\mathbf{n})w-(\mathbf{w}\cdot\mathbf{n})v]dl = 2ik\int_{\Omega}[fw-hv]ds, \tag{31}$$

where $\mathbf{n}$ is the outer unit normal to the boundary $\partial\Omega$ and the vector fluxes $\mathbf{v}$ and $\mathbf{w}$ are determined as

$$\mathbf{v} = (ikv,\partial_y v),\quad \mathbf{w} = (-ikw,\partial_y w). \tag{32}$$

The validity of this theorem immediately follows from the divergence theorem.

Before we begin to prove the optical theorem, we use of the Green's theorem to obtain an auxiliary expression for the directivity. We apply the Green's theorem with the functions $v = u^{\text{sc}}$, $w = \overline{u}^{\text{in}}$ through-

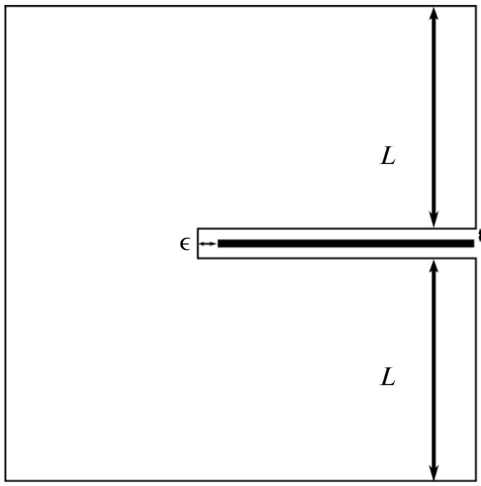

Fig. 2. Region Ω for derivation of Eq. (33)

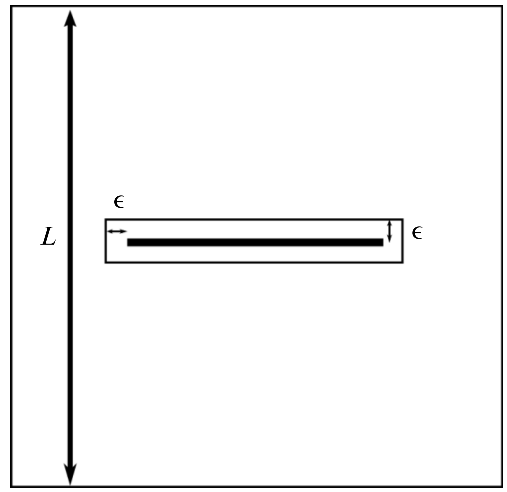

Fig. 3. Region Ω for derivation of optical theorem.

out the region Ω shown in Fig. 2. Here, $\overline{u}^{\text{in}}$ is a complex conjugate plane wave:

$$\overline{u}^{\text{in}} = \exp\left\{+ikx\frac{\theta_{\text{in}}^2}{2} + iky\theta_{\text{in}}\right\}. \quad (33)$$

We obtain

$$2ik\int_{-\infty}^{\infty} u^{\text{sc}}(0,y)\overline{u}^{\text{in}}(0,y)dy - \int_{-a}^{0}\frac{\partial u^{\text{sc}}}{\partial y}(x,+0)\overline{u}^{\text{in}}(x,0)dx + \int_{-a}^{0}\frac{\partial u^{\text{sc}}}{\partial y}(x,-0)\overline{u}^{\text{in}}(x,0)dx = 0. \quad (34)$$

Here, we took into account the radiation conditions and passed to the limits $\epsilon \to 0$ and $L \to \infty$. We also used Eq. (8). Using Eqs. (16) and (33), we obtain

$$\hat{S}^D(-\theta_{\text{in}},\theta_{\text{in}}) = \frac{1}{2ik} \times \int_{-a}^{0}(\partial_y u^{\text{sc}}(x,+0) - \partial_y u^{\text{sc}}(x,-0))e^{\frac{ikx\theta_{\text{in}}^2}{2}}dx. \quad (35)$$

The result is a formula evident from the viewpoint of the diffraction theory and expressing the directional pattern through the integral of the scattered field over the scatterer surface. In a similar way, it is possible to derive a complex conjugate expression:

$$\overline{\hat{S}}^D(-\theta_{\text{in}},\theta_{\text{in}}) = \frac{-1}{2ik} \times \int_{-a}^{0}(\partial_y \overline{u}^{\text{sc}}(x,+0) - \partial_y \overline{u}^{\text{sc}}(x,-0))e^{-\frac{ikx\theta_{\text{in}}^2}{2}}dx. \quad (36)$$

To prove the optical theorem, we apply the Green's theorem with the functions $v = u^{\text{sc}}$, $w = \overline{u}^{\text{sc}}$ throughout the region Ω shown in Fig. 3. Passing to the limits $\epsilon \to 0$ and $L \to \infty$, taking into account the radiation conditions, and using the fact that, to the left of the strip, the scattered field is zero, from Eqs. (35) and (36) we obtain

$$\Sigma = -2\,\text{Re}[\hat{S}^D(-\theta_{\text{in}},\theta_{\text{in}})]. \quad (37)$$

Thus, we proved the optical theorem for a strip with Dirichlet boundary conditions. For Neumann boundary conditions, Eq. (37) is obtained in a similar way.

Calculating the quantity (27) at the point $(-\theta_{\text{in}},\theta_{\text{in}})$ and eliminating indeterminacies, we obtain

$$\sqrt{\frac{k}{a}}\hat{S}^{D,N}(-\theta_{\text{in}},\theta_{\text{in}}) = -\sqrt{\frac{2i}{\pi}}\exp\left\{\frac{ika\theta_{\text{in}}^2}{2}\right\} - \frac{1}{2}\left(\sqrt{ka}\theta_{\text{in}} \pm \frac{i}{\sqrt{ka}\theta_{\text{in}}}\right) \times \left(\text{erfc}\left(-\sqrt{\frac{ak}{2i}}\theta_{\text{in}}\right) - \text{erfc}\left(\sqrt{\frac{ak}{2i}}\theta_{\text{in}}\right)\right). \quad (38)$$

Here, the upper sign corresponds to the Dirichlet boundary conditions and the lower sign to the Neumann boundary conditions. Substituting Eq. (38) in Eq. (37) and calculating the real part, we easily obtain the expressions for the scattering cross sections. Figure 4 shows the dependences of the scattering cross sections on the angle of incidence for the Dirichlet and Neumann boundary conditions. As one would expect, the scattering cross sections are positive, which indirectly proves the validity of Eq. (27). For small values of $\sqrt{ka}\theta_{\text{in}}$, in the case of Neumann boundary conditions, the scattering cross section tends to zero. In the case of Dirichlet boundary conditions, at the zero angle of incidence, the scattering cross sections is $\Sigma = 4\sqrt{a/\pi k}$. This result coincides with the first term of the series expansion obtained in [19].

CONCLUSIONS

In the parabolic approximation, we constructed directional patterns in terms of single quadratures for

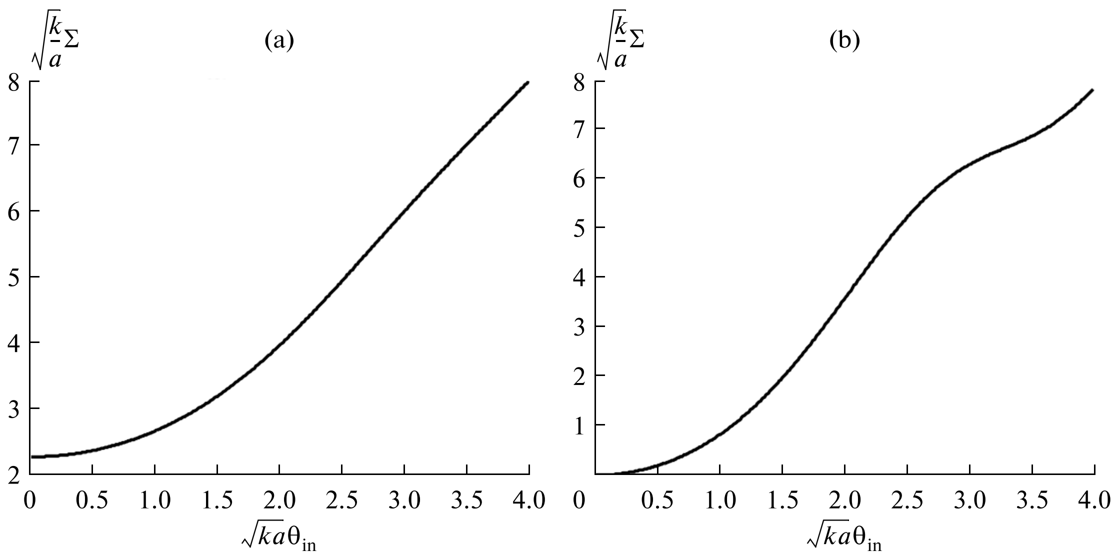

Fig. 4. Scattering cross section for (a) strip with Dirichlet boundary conditions and (b) strip with Neumann boundary conditions.

the problem of diffraction by a strip with Neumann and Dirichlet boundary conditions. The expressions obtained by us coincide in the limit with those given in [12] and are in numerical agreement with the results reported by I.V. Andronov.

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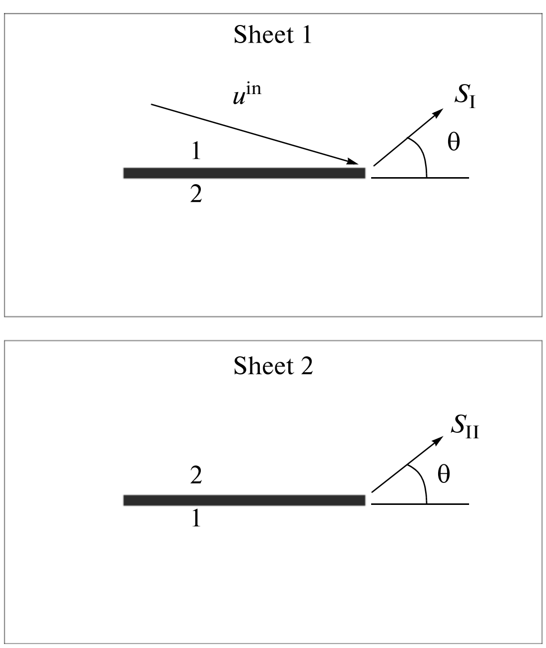

Fig. 5. Two-sheeted surface.

APPENDIX. EMBEDDING FORMULA

Formula (27) was derived by directly solving the parabolic equation and simplifying the result through integration by parts. At the same time, Eq. (27) represents a particular case of the so-called embedding formulas, which are valid for a wide class of problems with piecewise linear boundaries [18]. The significance of the embedding formula is that, instead of the incident plane wave, one considers a point source positioned near one of the corner points of the scatterer. The fields of such sources are called edge Green's functions. The embedding formula expresses the solution to the problem for an incident plane wave through edge Green's functions. Since, in the given case, the boundary Green's functions are calculated in explicit form, the embedding formula yields the solution to the initial problem.

The embedding formula has a simple form and is most easily derived by considering a branched (two-sheeted) surface. Note that the transition to a two-sheeted surface is unrelated to the parabolic approximation and can also be performed in the case of the Helmholtz equation. Following Sommerfeld's ideas, we consider a two-sheeted surface shown in Fig. 5. The surface is cut along the strip, and the numbers 1 and 2 indicate the way of sewing together the edges of the cuts (the edges of the same name should be sewed together). Incident wave (9) is incident on the first sheet only. We introduce the directivities for the first and second sheets $S_I(\theta, \theta_{in}) \equiv S(\theta, \theta_{in})$ and $S_{II}(\theta, \theta_{in})$. From symmetry considerations, it follows that

$$S_{II}(\theta, \theta_{in}) = -S(\theta, \theta_{in}). \quad (39)$$

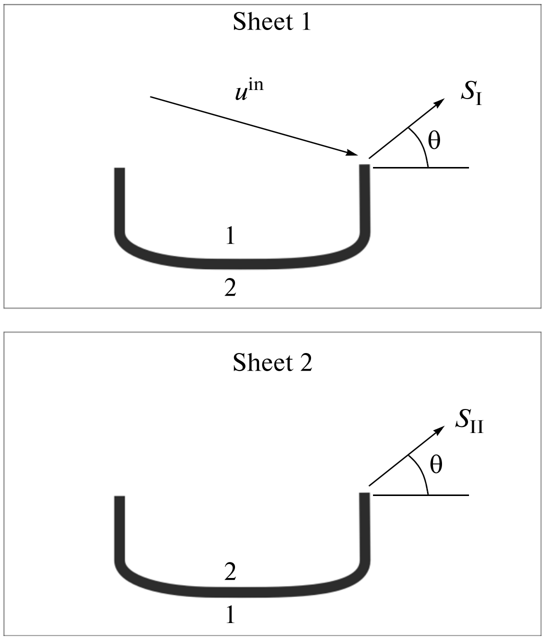

Fig. 6. Deformation of cuts of two-sheeted surface.

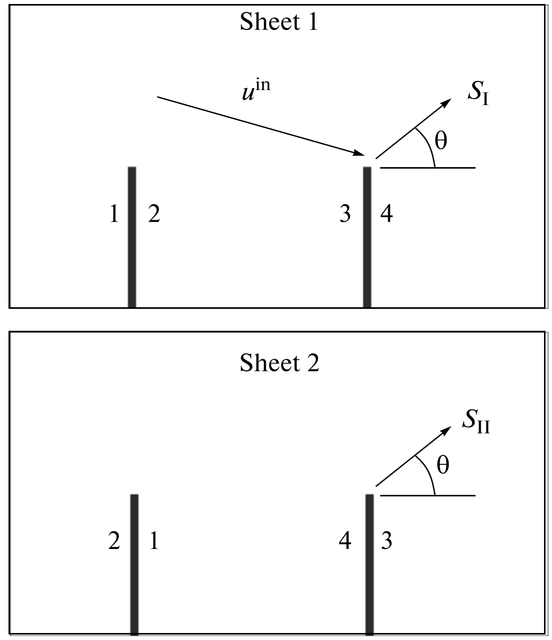

Fig. 7. Alternative representation of two-sheeted surface.

To prove equality (39), it is sufficient, in addition to the initial problem, to consider a symmetric problem with an incident wave on the second sheet and take into account the fact that their sum is trivial.

It is evident that

$$\hat{S}^{N,D}(\theta,\theta_{in}) = S_I(\theta,\theta_{in}) \pm S_{II}(-\theta,\theta_{in}) = S(\theta,\theta_{in}) \mp S(-\theta,\theta_{in}). \tag{40}$$

The upper sign corresponds to a strip with Neumann boundary conditions, and the lower sign, to Dirichlet boundary conditions. We seek $S(\theta,\theta_{in})$. As soon as $S(\theta,\theta_{in})$ is determined, the solutions to the problems with Dirichlet and Neumann boundary conditions are obtained. Note that symmetrization formula (40) coincides with Eq. (23) and, hence, the function S introduced as $S = S_I$, should coincide with the function S introduced by Eq. (24). In the Appendix, we use the definition $S = S_I$ and introduce no new symbol for the same function.

In addition to the representation for the two-sheeted surface shown in Fig. 5, we need one more representation. We deform the cuts as shown in Fig. 6. Increasing the deformation, we arrive at the representation shown in Fig. 7. We introduce a dipole-type edge Green's function. For this purpose, on the two-sheeted surface, we place sources with the strengths +1 and −1, as shown in Fig. 8. The sources are positioned to the right of the point $(-a, 0)$. We denote the edge Green's function (EGF) on the surface by $v(x, y)$. Thus, the EGF satisfies the equation

$$\left(\frac{\partial}{\partial x} + \frac{1}{2ik}\frac{\partial^2}{\partial y^2}\right) v(x, y) = \delta(x - (a + 0))\delta(y) \tag{41}$$

on sheet 1 and the equation

$$\left(\frac{\partial}{\partial x} + \frac{1}{2ik}\frac{\partial^2}{\partial y^2}\right) v(x, y) = -\delta(x - (a + 0))\delta(y) \tag{42}$$

on sheet 2. We introduce no special notations for the field on the first and second sheets, because the field and the directional patterns are always sought on sheet 1.

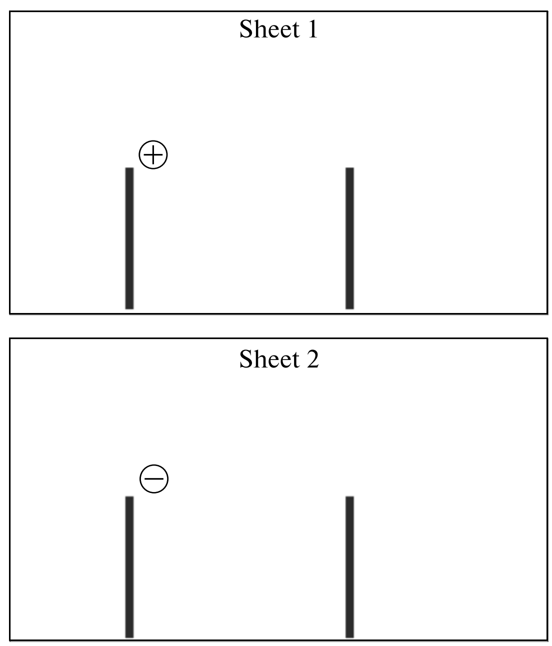

Fig. 8. Determination of boundary Green's functions.

As noted above, the field on sheet 2 can be determined from symmetry considerations.

According to Eq. (10), the EGF can be represented in the form

$$v(x,y) = V(\theta)g(x,y) + o((kx)^{-1/2}). \quad (43)$$

Here, we introduced the notation $V(\theta)$ for the directional pattern of the EGF. Note that the directional pattern of the EGF depends on a single variable, whereas the directional pattern of the initial problem $S(\theta,\theta_{in})$ depends on two variables. Let us calculate $V(\theta)$. From Eq. (13), it follows that

$$v(0,y) = g(a,y), \quad y > 0, \quad (44)$$

$$v(0,y) = -g(a,y), \quad y < 0. \quad (45)$$

Substituting Eqs. (43) and (44) in Eq. (16), we obtain

$$V(\theta) = \sqrt{\frac{k}{2\pi ia}}\int_0^\infty \exp\left\{\frac{iky^2}{2a}\right\}\left(e^{-iky\theta} - e^{iky\theta}\right)dy$$
$$= \frac{1}{2}\exp\left\{-\frac{ika\theta^2}{2}\right\}\left(\operatorname{erfc}\left(-\theta\sqrt{\frac{ak}{2i}}\right) - \operatorname{erfc}\left(\theta\sqrt{\frac{ak}{2i}}\right)\right). \quad (46)$$

Now, we proceed to derivation of the splitting formula. We consider the field on the two-sheeted surface with the cuts shown in Fig. 7. We apply the following operator to the total field $u(x,y)$:

$$H = \frac{\partial}{\partial y} + ik\theta_{in}. \quad (47)$$

Let us analyze the properties of the field

$$w(x,y) \equiv H[u](x,y). \quad (48)$$

First, the field $w(x,y)$ satisfied the parabolic equation everywhere except for the vicinities of the cuts. This follows from the fact that the operator H commutes with the operator of the equation. Second, the field $w(x,y)$ does not contain the incident wave. This follows from the relation

$$H[u^{in}](x,y) = 0. \quad (49)$$

Finally, in Eq. (47), the derivative with respect to y leads to the appearance of monopole sources at the end points of the cuts. Let us demonstrate it for the vertex $(-a,0)$.

We consider a narrow strip near the cut: $-a < x < -a + \epsilon$, $-\infty < y < \infty$. According to Eq. (13), within the strip, the field can be represented in the form

$$u(x,y) = \lim_{\epsilon\to 0}\int_\epsilon^\infty u(-a,y')g(x+a,y-y')dy'. \quad (50)$$

Integration is performed over the positive axis, because, on the negative axis, the field is zero. We apply the operator H and perform integration by parts:

$$w(x,y) = \lim_{\epsilon\to 0}\int_\epsilon^\infty w(-a,y')g(x+a,y-y')dy' + u(-a,\epsilon)g(x+a,y-\epsilon). \quad (51)$$

Passing to the limit in the latter expression, we obtain

$$w(x,y) = \int_0^\infty w(-a-0,y')g(x+a,y-y')dy' + u(-a-0,0)g(x+a,y). \quad (52)$$

The first term appearing on the right-hand side corresponds to the field without sources, and the second term to the field of a monopole point source characterized by the amplitude $u(-a-0,0)$, and positioned at the point $(-a+0,0)$. A similar procedure can be performed for the second vertex.

To continue the derivation, it is necessary to prove the uniqueness theorem for the parabolic equation. In [16], this was done for a more complicated problem. An equivalent integral statement of the problem was considered, and application of the uniqueness theorem was shown to be correct. In the case under study, the uniqueness can be proved in a similar way.

Thus, as a consequence of the uniqueness of the solution to the diffraction problem, the field $w(x,y)$ should be a linear combination of the fields of point sources:

$$w(x,y) = u(-a-0,0)v(x,y) + u(-0,0)\,g(x,y). \quad (53)$$

Changing to the directivities in the latter expression, we obtain

$$S(\theta,\theta_{in}) = \frac{\exp\{ika\theta_{in}^2/2\}V(\theta) + u(-0,0)}{ik(\theta+\theta_{in})}. \quad (54)$$

Here, we took into account the relation

$$u(-a-0,0) = u^{in}(a,0). \quad (55)$$

Formula (54) represents the splitting formula in its weak formulation. It involves the unknown quantity $u(-0,0)$. To express the field through the directional pattern of the EGF, we use the reciprocity theorem. We consider the problem with a point source of unit strength at the point $(-x',\theta_{in}x')$ on sheet 1 (x' is a large positive number). The field generated by the source is asymptotically close to the field of the incident wave multiplied by $g(x',\theta_{in}x')$. Setting $x'\to\infty$ and applying the reciprocity theorem, we obtain

$$u(-0,0) = V(\theta_{in})\exp\{ika\theta_{in}^2/2\}. \quad (56)$$

The exponential factor appears in the latter expression because, in Eq. (43), the directional pattern of the point source positioned at the vertex $(-a,0)$, is so introduced as if it were at the vertex (0, 0). This was done to simplify subsequent calculations. A mathematically stricter derivation of Eq. (56) can be found in [20], where it is based on the Green's theorem for a parabolic equation.

Substituting Eq. (56) in Eq. (54), we obtain the splitting formula in strong formulation:

$$S(\theta,\theta_{in}) = \exp\{ika\theta_{in}^2/2\}\frac{V(\theta)+V(\theta_{in})}{ik(\theta+\theta_{in})}. \quad (57)$$

Note that, because of the evident symmetry, we have

$$V(-\theta) = -V(\theta), \quad (58)$$

Therefore, the directivity given by Eq. (57) has no singularity at $\theta = -\theta_{in}$, but, in this case, calculation of the limiting value requires application of the L'Hospital rule. Eqs. (57) and (46) represent the result in terms of single quadratures. One can easily see that Eq. (57) coincides with Eq. (27).

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