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NONLINEAR ACOUSTICS

# On Solving Certain Nonlinear Acoustics Problems

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**Abstract**—Previously the authors developed a geometric method for studying and solving nonlinear equations and systems of equations with partial derivatives. This method is used in this paper to obtain a series of exact solutions to certain nonlinear acoustics equations, as well as to reduce the system of Euler equations to systems of common differential equations.

*Keywords*: nonlinear equations in partial derivatives, methods of solving differential equations, nonlinear acoustics equations, exact solutions

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Mathematical works frequently appear in physics journals, since a number of actual physical models are based on nonlinear equations. General methods for analyzing them, as is well known, do not exist. To find the partial derivatives, it is necessary to develop different original methods, each of which is not universal. However, the entire set of such methods sometimes makes it possible to obtain solutions having an important physical sense. Papers on this subject have been published in *Acoustical Physics*.

For example, in [1], a method was developed for a priori use of symmetry based on rational complication of nonlinear acoustics models. In [2], Darboux transformation is used to find partial solutions to the inhomogeneous Burgers equation. Some other approaches to finding solutions to nonlinear equations are described in [3, 4].

Exact solutions occupy a special place. In certain, primarily exceptional, cases, exact solutions make it possible to solve a conceptual problem or obtain characteristics of a phenomenon of interest to a researcher. Quite frequently, exact solutions make it possible to reveal essential features of a complex physical process. Finally, as a rule, exact solutions are suitable as "tests" for evaluating results obtained when using numerical, approximate, or asymptotic methods.

This paper proposes and demonstrates, using specific equations, a number of approaches to obtain such exact solutions. They are as follows:

(1) An equation that describes propagation of finite perturbations in a relaxing medium [5]:

$$\tau\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x}-\frac{\varepsilon}{c_0^2}v\frac{\partial v}{\partial y}\right]+\left[\frac{\partial v}{\partial x}-\frac{\varepsilon}{c_0^2}v\frac{\partial v}{\partial y}\right]=\frac{m\tau}{2c_0}\frac{\partial^2 v}{\partial y^2}. \quad (1)$$

Here, $\tau = \text{const}$ is the characteristic relaxation time, $\varepsilon = \text{const}$ is a nonlinear parameter, $c_0 = \text{const}$ is the sound velocity, $m = c_\infty^2/c_0^2 - 1$, $c_\infty$ is the "frozen" sound velocity, $x$ is the spatial coordinate, $t$ is time, $y = t - x/c_0$ is the traveling coordinate, and $v$ is velocity.

Equation (1) is closely related to the integro-differential equations widely used lately to describe waves in biological tissues and geological structures [5]. Work [6] is devoted to finding solutions to such equations using group analysis methods.

(2) An equation that is used in modified nonlinear-acoustic approach [7]:

$$\left[1+\frac{3\gamma-1}{4}\frac{\rho'}{\rho_0}\right]\frac{\partial\rho'}{\partial x} + \left[-\frac{\gamma+1}{2c_0}\frac{\rho'}{\rho_0}-\frac{(\gamma+1)(\gamma-3)}{4c_0}\left(\frac{\rho'}{\rho_0}\right)^2\right]\frac{\partial\rho'}{\partial\tau} \quad (2)$$
$$=\frac{b}{2c_0^3\rho_0}\left[1+\frac{\gamma-3}{2}\frac{\rho'}{\rho_0}\right]\frac{\partial^2\rho'}{\partial\tau^2}-\frac{5b}{4c_0^2\rho_0}\frac{\partial^2\rho'}{\partial x\partial\tau}+\frac{(\gamma-1)b}{4c_0^3\rho_0^2}\left(\frac{\partial\rho'}{\partial\tau}\right)^2.$$

Here $\gamma$ is the adiabatic index, $\rho_0 = \text{const}$, $c_0 = \text{const}$, $b = \text{const}$, $\rho' = \rho - \rho_0$, $\rho$ is the density of the medium.

(3) An equation describing in the second approximation propagation of restricted beams in lossless media (Khokhlov–Zabolotskaya equation [8]):

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\rho'}{\partial x}-\frac{\varepsilon}{c_0\rho_0}\rho'\frac{\partial\rho'}{\partial\tau}\right)=\frac{c_0}{2}\left(\frac{\partial^2\rho'}{\partial y^2}\right). \quad (3)$$

Here, $c_0 = \text{const}$, $\varepsilon = \text{const}$, $\rho_0 = \text{const}$, $\rho' = \rho - \rho_0$, and $\rho$ is the density of the medium.

The basis of obtaining exact solutions is a geometric method, developed by the authors, that uses nonlinear differential equations and a system in partial derivatives [9–14]. Let us briefly describe the idea of the method.

Let a certain physical process be described by a nonlinear equation in partial derivatives $F(\mathbf{x}, u, u_i, u_{ij}, \ldots, u_{i_1 i_2 \ldots i_m}) = 0$, $u = u(\mathbf{x})$, $\mathbf{x} \in R^m$, the subscripts denote differentiation over the corresponding independent variables. The main idea of the geometric method is the assumption that the solution to the equation in partial derivatives depends on one variable (e.g., $u = u(\psi)$ where $\psi = \psi(\mathbf{x})$). Then $\psi(\mathbf{x}) = \text{const}$ is the surface of the level of the function $u(\mathbf{x})$. The change in variable $\psi$ leads to a change in the solution. For such a dependence, the equation in partial derivatives can usually be written as $\sum_k A_k(x, u, u', u'', \ldots, u^{(m)}) B_k(\psi_i, \psi_{ij}, \ldots, \psi_{i_1 i_2 \ldots i_m}) = 0$. Here, a prime denotes differentiation over variable $\psi$. Supposing $B_k(\psi_i, \psi_{ij}, \ldots, \psi_{i_1 i_2 \ldots i_m}) = f_k(\psi)$, where $f_k(\psi)$ are initially arbitrary functions, we determine for which dependences between functions $f_k(\psi)$ the system $B_k(\psi_i, \psi_{ij}, \ldots, \psi_{i_1 i_2 \ldots i_m}) = f_k(\psi)$ is compatible. Then, solving the joint system under certain given initial or boundary-value conditions, we find the form of the function $\psi = \psi(\mathbf{x})$. Substituting functions $f_k(\psi)$, for which system $B_k(\psi_i, \psi_{ij}, \ldots, \psi_{i_1 i_2 \ldots i_m}) = f_k(\psi)$ is compatible, we have the common differential equation (CDE) to solve the equation $\sum_k A_k(x, u, u', u'', \ldots, u^{(m)}) f_k(\psi) = 0$. Solving the CDE and substituting into the obtained solution the earlier determined function $\psi = \psi(\mathbf{x})$, we have the solution to the initial equation in partial derivatives. This geometric method admits a number of modifications. For example, it can be considered that $\psi = u$ [12].

An analogous approach to systems of nonlinear equations in partial derivatives makes it possible to reduce them to CDE systems [13].

Using the above-mentioned equations, we show how, using the described approached, it is possible to obtain exact solutions for conceptual nonlinear acoustics problems described by nonlinear differential equations and systems in partial derivatives also using their reduction to CDE systems.

## ON SOLVING AN EQUATION FOR RELAXING MEDIA

Let us consider Eq. (1). We will use the notation $\frac{\partial v}{\partial x} = v_x$, $\frac{\partial v}{\partial y} = v_y$, $\frac{\partial^2 v}{\partial y^2} = v_{yy}$. We assume that $v_x - \varepsilon v v_y / c_0^2 = f(v)$, where $f(v)$ is a yet unknown function. For this, so that $v$ is the solution to Eq. (1), the following dependence must be fulfilled:

$$\tau v_y f' + f = m\tau v_{yy}/(2c_0). \tag{4}$$

From here on, in this section, a prime denotes differentiation over $v$.

Theorem 1. If function $f(v)$ satisfies the equation

$$f''\left[\frac{2c_0}{m}\left(\frac{\varepsilon}{c_0^2}Cv - 1\right) + C\right] + \frac{4\varepsilon C}{c_0 m}f' - \frac{6\varepsilon}{c_0 m\tau}\left(\frac{\varepsilon C v}{c_0^2} - 1\right) = 0, \tag{5}$$

where $C = \text{const}$, $C > 0$, and in the initial manifold dependence (4) is fulfilled, then the solutions to the equation $v_x - \varepsilon v v_y / c_0^2 = f(v)$ are solutions to Eq. (1)

Proof. Let us show under which conditions system $\tau v_y f' + f = m\tau v_{yy}/(2c_0)$, $v_x - \varepsilon v v_y / c_0^2 = f(v)$ is joint. We write the differential results of the relation $v_x - \varepsilon v v_y / c_0^2 = f(v)$:

$$\begin{aligned} v_{xx} - \varepsilon v v_{yx}/c_0^2 &= f'(v)v_x + \varepsilon v_x v_y / c_0^2, \\ v_{xy} - \varepsilon v v_{yy}/c_0^2 &= f'(v)v_y + \varepsilon v_y^2 / c_0^2. \end{aligned} \tag{6}$$

From relations (4) and (6) we determine the second derivatives of function $v(x, y)$:

$$v_{yy} = 2c_0[\tau v_y f' + f]/(m\tau),$$
$$v_{xy} = v_y f' + \varepsilon v_y^2 / c_0^2 + 2\varepsilon v[\tau v_y f' + f]/(c_0 m\tau),$$
$$v_{xx} = \left(f + \frac{2\varepsilon}{c_0^2} v v_y\right)\left(f' + \frac{\varepsilon}{c_0^2} v_y\right) + \frac{2\varepsilon^2}{c_0^3 m} v^2 \left(v_y f' + \frac{f}{\tau}\right).$$

So that condition (4) and relation $v_x - \varepsilon v v_y / c_0^2 = f(v)$ are satisfied by the same function $v(x, y)$, let us require the equality of the third mixed derivatives and fulfillment of the relation $v_x - \varepsilon v v_y / c_0^2 = f(v)$. We obtain

$$\begin{aligned} v_x &= f + \frac{f[2\varepsilon v f'/(c_0 m) - 6\varepsilon^2 v/(c_0^3 m\tau)]}{f'' + 4\varepsilon f'/(c_0 m)}, \\ v_y &= \frac{(f/m)[2c_0 f'' - 6\varepsilon/(c_0\tau)]}{f'' + 4\varepsilon f'/(c_0 m)}. \end{aligned} \tag{7}$$

Taking into account that $v_{xy} = v_{yx}$, we obtain the equation for determining function $f(v)$:

$$f''\left[\frac{2c_0}{m}\left(\frac{\varepsilon}{c_0^2}Cv - 1\right) + C\right] + \frac{4\varepsilon C}{c_0 m}f' - \frac{6\varepsilon}{c_0 m\tau}\left(\frac{\varepsilon C v}{c_0^2} - 1\right) = 0,$$

which was what needed to be proved.

Let us now pass to solving the obtained Eq. (5). We write its solution, setting $f' = p(v)$. Function $p(v)$ satisfies the linear equation

$$p'\left[\frac{2c_0}{m}\left(\frac{\varepsilon}{c_0^2}Cv - 1\right) + C\right] + \frac{4\varepsilon C}{c_0 m}p - \frac{6\varepsilon}{c_0 m\tau}\left(\frac{\varepsilon C v}{c_0^2} - 1\right) = 0.$$

Hence,

$$p = \frac{\eta}{\left(\frac{2\varepsilon C}{c_0 m}v + C - \frac{2c_0}{m}\right)^2} + \frac{\frac{6\varepsilon}{c_0 m\tau}\left(\frac{2\varepsilon^2 C^2}{3c_0^3 m}v^3 + \frac{\varepsilon C(Cm - 4c_0)}{2c_0^2 m}v^2 - Cv + \frac{2c_0}{m}v\right)}{\left(\frac{2\varepsilon C}{c_0 m}v + C - \frac{2c_0}{m}\right)^2},$$

$$\eta = \text{const}.$$

Then,

$$f = \varepsilon v^2/(2c_0^2\tau) - (4c_0 + Cm)v/(4Cc_0\tau) + c_0(2c_0 + Cm)/(4C^2\varepsilon\tau). \tag{8}$$

As shown above, solutions satisfying the conditions of theorem 1 satisfy relations (7). Let us substitute in these relations function (8) and solve the obtained system of equations. We obtain

$$\frac{1}{C\tau}x + \frac{1}{\tau}y = -\int \frac{f' + m/(4c_0\tau)}{f}dv. \tag{9}$$

Denoting $\frac{1}{C\tau}x + \frac{1}{\tau}y = z$, we have $v = v(z)$. Substituting $v(z)$ in (1), we obtain the CDE

$$\left(\frac{1}{C} - \frac{\varepsilon}{c_0^2}v\right)v_z - \frac{\varepsilon}{c_0^2}v_z^2 = v_{zz}\left(\frac{m}{2c_0} - \frac{1}{C} + \frac{\varepsilon}{c_0^2}v\right). \tag{10}$$

In Eq. (10), we pass from function $v(z)$ to function $z(v)$. We obtain

$$\left(\frac{1}{C} - \frac{\varepsilon}{c_0^2}v\right)z_v^2 - \frac{\varepsilon}{c_0^2}z_v + z_{vv}\left(\frac{m}{2c_0} - \frac{1}{C} + \frac{\varepsilon}{c_0^2}v\right) = 0. \tag{11}$$

It is easy to check that solution (9) satisfies Eq. (11) and, as a result, Eq. (1) also.

Note. It is possible to write Eq. (1) in the form

$$v_x - \frac{\varepsilon}{c_0^2}vv_y - \frac{\varepsilon\tau}{c_0^2}v_y^2 = \left(\frac{m\tau}{2c_0} + \frac{\varepsilon\tau}{c_0^2}v\right)v_{yy} - \tau v_{xy}$$

and equate both sides of this equation to $f(v)$. Then, similarly to the preceding consideration, we find the function $f(v)$ for which the obtained system of equations

$$v_x - \frac{\varepsilon}{c_0^2}vv_y - \frac{\varepsilon\tau}{c_0^2}v_y^2 = f(v),$$

$$\left(\frac{m\tau}{2c_0} + \frac{\varepsilon\tau}{c_0^2}v\right)v_{yy} - \tau v_{xy} = f(v)$$

is joint. Then, the solution to the first equation of this system will be the solution to Eq. (1) if in the initial manifold the second equation of the system becomes identical.

## ON SOLVING AN EQUATION OF THE MODIFIED NONLINEAR-ACOUSTICS APPROACH

Let us consider Eq. (2). We denote $\rho' = r$ and rewrite Eq. (2) in the form

$$\left(1 + \frac{3\gamma - 1}{4}\frac{r}{\rho_0}\right)r_x - \left[\frac{\gamma + 1}{2}\frac{1}{c_0}\frac{r}{\rho_0} + \frac{(\gamma + 1)(\gamma - 3)}{4}\frac{1}{c_0}\left(\frac{r}{\rho_0}\right)^2\right]r_\tau - \frac{\gamma - 1}{4}\frac{b}{c_0^3\rho_0^2}r_\tau^2 = \frac{b}{2c_0^3\rho_0}\left(1 + \frac{\gamma - 3}{2}\frac{r}{\rho_0}\right)r_{\tau\tau} - \frac{5b}{4c_0^2\rho_0}r_{x\tau}, \tag{12}$$

$$r_x = \frac{\partial r}{\partial x}, \quad r_\tau = \frac{\partial r}{\partial \tau}, \quad r_{\tau\tau} = \frac{\partial^2 r}{\partial \tau^2}, \quad r_{x\tau} = \frac{\partial^2 r}{\partial x \partial \tau}.$$

We will assume that $r = r(\psi(x, \tau))$. Then $\psi(x, \tau) =$ const is the surface of the level of function $r$, $r_x = r'\psi_x$, $r_\tau = r'\psi_\tau$, $r_{\tau\tau} = r''\psi_\tau^2 + r'\psi_{\tau\tau}$, $r_{x\tau} = r''\psi_x\psi_\tau + r'\psi_{x\tau}$. From here on, in this section a prime denotes differentiation over independent variable $\psi$. Substituting these expressions in Eq. (12), we obtain the relation

$$\left(1 + \frac{3\gamma - 1}{4}\frac{r}{\rho_0}\right)r'\psi_x - \left[\frac{\gamma + 1}{2}\frac{1}{c_0}\frac{r}{\rho_0} + \frac{(\gamma + 1)(\gamma - 3)}{4}\frac{1}{c_0}\left(\frac{r}{\rho_0}\right)^2\right]r'\psi_\tau - \frac{\gamma - 1}{4}\frac{b}{c_0^3\rho_0^2}(r')^2\psi_\tau^2 = \frac{b}{2c_0^3\rho_0}\left(1 + \frac{\gamma - 3}{2}\frac{r}{\rho_0}\right) \times (r''\psi_\tau^2 + r'\psi_{\tau\tau}) - \frac{5b}{4c_0^2\rho_0}(r''\psi_x\psi_\tau + r'\psi_{x\tau}). \tag{13}$$

Let $\psi_\tau \neq 0$ in expression (13). Dividing each term by $\psi_\tau$, we set

$$\frac{\psi_x}{\psi_\tau} = f_1(\psi), \quad \psi_\tau = f_2(\psi), \quad \frac{\psi_{\tau\tau}}{\psi_\tau} = f_3(\psi), \quad \frac{\psi_{x\tau}}{\psi_\tau} = f_4(\psi). \tag{14}$$

This is sufficient for relation (13) to become a CDE. From relations (14) it follows that $\psi_x = f_1(\psi)f_2(\psi)$, and from the equality of mixed derivatives we find that $\psi = \psi(z)$, where $z = ax + c\tau$, $a =$ const, $c =$ const. However, then it is possible to consider that $r = r(z)$ and Eq. (12) leads to the CDE

$$\left(1 + \frac{3\gamma - 1}{4}\frac{r}{\rho_0}\right)r_z a - \frac{\gamma + 1}{2}\frac{1}{c_0}\frac{r}{\rho_0}r_z c - \frac{(\gamma + 1)(\gamma - 3)}{4} \times \frac{1}{c_0}\left(\frac{r}{\rho_0}\right)^2 r_z c - \frac{\gamma - 1}{4}\frac{bc^2}{c_0^3\rho_0^2}r_z^2 = \frac{b}{2c_0^3\rho_0}\left(1 + \frac{\gamma - 3}{2}\frac{r}{\rho_0}\right)r_{zz}c^2 - \frac{5b}{4c_0^2\rho_0}r_{zz}ac. \tag{15}$$

Let us write some exact solutions to Eq. (15).

(1) If $\gamma = 3$ and $z = c(x + c_0\tau)$, then $r(z) = (2c_0^2\rho_0^2)z/(bc) + k$, $k = \text{const}$.

(2) If $r(z)$ has an inverse function, then, supposing $r_z = p(r)$, we obtain a linear equation, which is satisfied by function $p(r)$:

$$\left[\frac{b}{2c_0^3\rho_0}\left(1+\frac{\gamma-3}{2}\frac{r}{\rho_0}\right)c^2 - \frac{5b}{4c_0^2\rho_0}ac\right]p' + \frac{\gamma-1}{4}\frac{b}{c_0^3\rho_0^2}c^2 p = A(r).$$

Hence we find that

$$z = \int \frac{[(\gamma-3)cr + \rho_0(2c - 5ac_0)]^{(\gamma-1)/(\gamma-3)}\,dr}{\eta + \int[(\gamma-3)cr + \rho_0(2c-5ac_0)]^{2/(\gamma-1)}A(r)dr},$$

$$\eta = \text{const}, \quad A(r) = \left(1 + \frac{3\gamma-1}{4}\frac{r}{\rho_0}\right)a - \left[\frac{\gamma+1}{2}\frac{1}{c_0}\frac{r}{\rho_0} + \frac{(\gamma+1)(\gamma-3)}{4}\frac{1}{c_0}\left(\frac{r}{\rho_0}\right)^2\right]c.$$

In particular, Eq. (2) has a solution in the form $r = M + N/(z + k)$, where $k = \text{const}$,

$$M = 4\rho_0\left[-(3\gamma-1) \pm D^{1/2}\right]\Big/\left(13\gamma^2 - 24\gamma - 29\right),$$

$$D = (5\gamma^3 - 29\gamma^2 + 43\gamma - 35)/[2(\gamma-3)],$$

$$N = [(3\gamma-7)b]\Big/\left[(\gamma+1)(\gamma-3)c_0^2\right],$$

$$z = \frac{(\gamma+1)(\gamma-3)}{(\gamma^2+4\gamma-37)c_0\rho_0}\left[4(2-\gamma)M - \frac{2(\gamma-1)}{\gamma-3}\rho_0\right]x + \tau.$$

## ON SOLVING THE EQUATION FOR RESTRICTED ACOUSTIC BEAMS

Let us consider Eq. (3). We denote $\rho' = r$. We consider that $r = r(\psi(x, y, \tau))$. Then $\psi(x, y, \tau) = \text{const}$ is the surface of the level of function $r$ and, using the notation $\frac{\partial r}{\partial x} = r_x$, $\frac{\partial r}{\partial y} = r_y$, $\frac{\partial r}{\partial \tau} = r_\tau$, $\frac{\partial^2 r}{\partial x\partial \tau} = r_{x\tau}$, $\frac{\partial^2 r}{\partial \tau^2} = r_{\tau\tau}$, $\frac{\partial^2 r}{\partial y^2} = r_{yy}$, we have (here a prime denotes differentiation over variable $\psi$) $r_x = r'\psi_x$, $r_y = r'\psi_y$, $r_\tau = r'\psi_\tau$, $r_{\tau\tau} = r''\psi_\tau^2 + r'\psi_{yy}$, $r_{\tau\tau} = r''\psi_x\psi_\tau + r'\psi_{x\tau}$. Substituting these expressions into Eq. (3), we obtain

$$r''\left(\psi_x\psi_\tau - \frac{c_0}{2}\psi_y^2\right) - \frac{\varepsilon}{c_0\rho_0}(r'^2 + rr'')\psi_\tau^2 + r'\left(\psi_{x\tau} - \frac{c_0}{2}\psi_{yy}\right) - \frac{\varepsilon rr'}{c_0\rho_0}\psi_{\tau\tau} = 0. \quad (16)$$

Let $\psi_\tau \neq 0$. We divide each term in Eq. (16) by $\psi_\tau^2$ and set

$$(\psi_x\psi_\tau - 0.5c_0\psi_y^2)/\psi_\tau^2 = f(\psi),$$
$$(\psi_{x\tau} - 0.5c_0\psi_{yy})/\psi_\tau^2 = f_1(\psi), \quad \psi_{\tau\tau}/\psi_\tau^2 = f_2(\psi). \quad (17)$$

Then Eq. (3) can be represented in the form

$$r''f - \frac{\varepsilon}{c_0\rho_0}(r'^2 + rr'') + r'f_1 - \frac{\varepsilon}{c_0\rho_0}rr'f_2 = 0.$$

Let us show for which functions $f(\psi)$, $f_1(\psi)$, $f_2(\psi)$ system (17) is joint.

Theorem 2. Let $f \neq \text{const}$. If system of equations (17) is joint, then $f_2 = -f''/f'$, $f_1 = [(3f'^2 - 4ff'') \pm f'^2]/(4f')$, and the second derivatives of function $\psi(\tau, x, y)$ satisfy the dependences

$$\psi_{\tau x} = 0.5c_0f_2\psi_y^2 + (ff_2 + f')\psi_\tau^2, \quad \psi_{y\tau} = f_2\psi_\tau\psi_y,$$
$$\psi_{\tau\tau}/\psi_\tau^2 = f_2(\psi), \quad \psi_{xx} = [0.25c_0^2f_2\psi_y^4 + c_0\psi_\tau^2\psi_y^2(3ff_2 + 3f' - 3f_1) + \psi_\tau^4 f(ff_2 + 2f')]/\psi_\tau^2,$$
$$\psi_{xy} = [0.5c_0f_2\psi_y^3 + (3ff_2 + 3f' - 2f_1)\psi_\tau^2\psi_y]/\psi_\tau,$$
$$\psi_{yy} = f_2\psi_y^2 + 2(ff_2 + f' - f_1)\psi_\tau^2/c_0.$$

Proof. Let us consider the first equation of system (17)

$$(\psi_x\psi_\tau - 0.5c_0\,\psi_y^2)/\psi_\tau^2 = f(\psi). \quad (18)$$

Let us write for Eq. (18) the system of equations of the characteristics

$$\frac{d\tau}{ds} = \psi_x - 2\psi_\tau, \quad \frac{dx}{ds} = \psi_\tau, \quad \frac{dy}{ds} = -c_0\psi_y, \quad \frac{d\psi}{ds} = 0,$$
$$\frac{d\psi_\tau}{ds} = f'\psi_\tau^3, \quad \frac{d\psi_x}{ds} = f'\psi_\tau^2\psi_x, \quad \frac{d\psi_y}{ds} = f'\psi_\tau^2\psi_y. \quad (19)$$

We expand system (19) to second-order products. For this, we write it in more detail $\frac{d\psi_\tau}{ds} = \psi_{\tau\tau}\frac{d\tau}{ds} + \psi_{\tau x}\frac{dx}{ds} + \psi_{\tau y}\frac{dy}{ds} = f'\psi_\tau^3$. Substituting instead of the derivatives from the independent variables their values from (19) and differentiating the obtained expression first over $\tau$ and them sequentially over $x$ and $y$, we obtain the first three equations. Performing similar actions with $\frac{d\psi_y}{ds}$, we obtain the final equations of the system.

$$\frac{d\psi_{\tau\tau}}{ds} = -2\psi_{\tau\tau}\psi_{x\tau} + 2f\psi_{\tau\tau}^2 + 2f'\psi_{\tau\tau}\psi_\tau^2 + c_0\psi_{\tau y}^2 + f''\psi_\tau^4 + 3f'\psi_\tau^2\psi_{\tau\tau},$$
$$\frac{d\psi_{\tau x}}{ds} = -\psi_{\tau\tau}\psi_{xx} + 2f\psi_{\tau\tau}\psi_{\tau x} + 2f''\psi_{\tau\tau}\psi_\tau\psi_x - \psi_{\tau x}^2 + c_0\psi_{\tau y}\psi_{xy} + f''\psi_\tau^3\psi_x + 3f'\psi_\tau^2\psi_{\tau x},$$
$$\frac{d\psi_{\tau y}}{ds} = -\psi_{\tau\tau}\psi_{yx} + 2f\psi_{\tau\tau}\psi_{\tau y} + 2f'\psi_{\tau\tau}\psi_\tau\psi_y - \psi_{\tau x}\psi_{\tau y} + c_0\psi_{\tau y}\psi_{yy} + f''\psi_\tau^3\psi_y + 3f'\psi_\tau^2\psi_{\tau y}, \quad (20)$$
$$\frac{d\psi_{yy}}{ds} = -2\psi_{\tau y}\psi_{xy} + 2f\psi_{\tau y}^2$$

$$+ 4f'\psi_{\tau y}\psi_\tau\psi_y + c_0\psi_{yy}^2 + f''\psi_\tau^2\psi_y^2 + f'\psi_\tau^2\psi_{yy},$$

$$\frac{d\psi_{xy}}{ds} = -\psi_{\tau y}\psi_{xx} + 2f\psi_{y\tau}\psi_{\tau x}$$
$$+ 2f'\psi_{\tau y}\psi_\tau\psi_x + \psi_{xy}\psi_{\tau x} + c_0\psi_{yy}\psi_{yx}$$
$$+ f''\psi_\tau^2\psi_x\psi_y + 2f'\psi_\tau\psi_y\psi_{\tau x} + f'\psi_\tau^2\psi_{xy}.$$

Let us require that the second and third equations of system (17) be the first integrals of system (20) We obtain the dependences for which this takes place:

$$-2\psi_{\tau\tau}\psi_{x\tau} + 2f\psi_{\tau\tau}^2 + 5f'\psi_{\tau\tau}\psi_\tau^2 + c_0\psi_{y\tau}^2 + f''\psi_\tau^4$$
$$- 2f_2f'\psi_\tau^4 = 0,$$
$$-0.5c_0[2f\psi_{\tau y}^2 + 4f'\psi_{\tau y}\psi_\tau\psi_y$$
$$+ c_0\psi_{yy}^2 + f''\psi_\tau^2\psi_y^2 + f'\psi_\tau^2\psi_{yy}] - 2f_1f'\psi_\tau^4 \quad (21)$$
$$-\psi_{\tau\tau}\psi_{xx} + 2f\psi_{\tau\tau}\psi_{\tau x} + 2f'\psi_{\tau\tau}\psi_\tau\psi_x - \psi_{\tau x}^2$$
$$+ 2c_0\psi_{y\tau}\psi_{xy} + f''\psi_\tau^3\psi_x + 3f\psi_\tau^2\psi_{\tau x} = 0.$$

To relations (21) we add the differential results of Eq. (18) and the second and third relations of (17):

$$\psi_{\tau\tau}\psi_x + \psi_{x\tau}\psi_\tau - c_0\psi_y\psi_{y\tau} - f'\psi_\tau^3 - 2f\psi_\tau\psi_{\tau\tau} = 0,$$
$$\psi_{\tau x}\psi_x + \psi_\tau\psi_{xx} - c_0\psi_y\psi_{xy} - f'\psi_\tau^2\psi_x$$
$$- 2f\psi_\tau\psi_{\tau x} = 0, \quad (22)$$
$$\psi_x\psi_{y\tau} + \psi_\tau\psi_{xy}$$
$$- c_0\psi_y\psi_{yy} - f'\psi_\tau^2\psi_y - 2f\psi_\tau\psi_{y\tau} = 0,$$
$$(\psi_{x\tau} - 0.5c_0\psi_{yy})/\psi_\tau^2 = f_1(\psi), \quad \psi_{\tau\tau}/\psi_\tau^2 = f_2(\psi).$$

Determining from system (21)–(22) the second derivatives of function $\psi(\tau, x, y)$ and requiring identical fulfillment of all mentioned equations, we find that

$$\psi_{\tau x} = 0.5c_0f_2\psi_y^2 + (ff_2 + f')\psi_\tau^2, \quad \psi_{y\tau} = f_2\psi_\tau\psi_y,$$
$$\psi_{yy} = f_2\psi_y^2 + 2(ff_2 + f' - f_1)\psi_\tau^2/c_0,$$
$$\psi_{xy} = [0.5c_0f_2\psi_y^3 + (3ff_2 + 3f' - 2f_1)\psi_\tau^2\psi_y]/\psi_\tau,$$
$$\psi_{xx} = [0.25c_0^2f_2\psi_y^4 + c_0\psi_\tau^2\psi_y^2(3ff_2 + 3f' - 3f_1)$$
$$+ \psi_\tau^4f(ff_2 + 2f')]/\psi_\tau^2,$$
$$\text{where } f_2 = -f''/f',$$
$$f_1 = [(3f'^2 - 4ff'') \pm f'^2]/(4f'),$$

which was what needed to be proved.

If the conditions obtained in theorem 2 are fulfilled, then Eq. (16) takes the form

$$r''f - \frac{\varepsilon}{c_0\rho_0}(r'^2 + rr'')$$
$$+ \frac{r'}{4f'}[(3f'^2 - 4ff'') \pm f'^2] + \frac{\varepsilon}{c_0\rho_0}rr'\frac{f''}{f'} = 0. \quad (23)$$

Here $f(\psi)$ is an arbitrary function. Equation (23) becomes identical for all $c_0$, $\rho_0$, and $\varepsilon$ is $r = A\sqrt{f}$, $A =$ const and in the expression for function $f_1$ a minus sign is chosen. If in the expression for $f_1$ a plus sign is chosen, then relation (23) will become identical for any $c_0$, $\rho_0$, $\varepsilon$ only when $f' = 0$. This case is not considered.

Thus, we have found that the solution to Eq. (3) reduces to the solution to Eq. (23) if the surface of the level $\psi(\tau, x, y) =$ const is determined from Eq. (18). To determine $r = r(\tau, x, y)$ let us turn to solving this equation. Solving system of equations of characteristics (19), we obtain

$$\psi_\tau = \frac{1}{\sqrt{2f'(a-s)}}, \quad \psi_y = \frac{C(\psi,\alpha)}{\sqrt{(a-s)}},$$
$$\psi_x = \frac{C_1(\psi,\alpha)}{\sqrt{a-s}}, \quad C_1 = \frac{C^2c_0f' + f}{\sqrt{2f'}}, \quad a = \text{const}, \quad (24)$$

$$\tau = \frac{\sqrt{2}(f - C^2c_0f')}{\sqrt{f'}}\sqrt{a-s} + g,$$
$$x = -\sqrt{\frac{2}{f'}(a-s)} + g_1, \quad y = 2Cc_0\sqrt{a-s} + g_2. \quad (25)$$

In the general case, it is possible to set $g = g(\psi, \alpha)$, $g_1 = g_1(\psi, \alpha)$, $g_2 = g_2(\psi, \alpha)$; then formulas (25) give the transition to independent variables $s$, $\psi$, $\alpha$ if the relation $\psi \equiv \psi(\tau(s, \psi, \alpha), x(s, \psi, \alpha), y(s, \psi, \alpha))$ is identically fulfilled. This relation will become identical when

$$1 = \psi_\tau\tau_\psi + \psi_x x_\psi + \psi_y y_\psi,$$
$$0 = \psi_\tau\tau_s + \psi_x x_s + \psi_y y_s, \quad 0 = \psi_\tau\tau_\alpha + \psi_x x_\alpha + \psi_y y_\alpha. \quad (26)$$

The second relation (26) is fulfilled identically. Substituting in the first and third relations (26) the values of the corresponding quantities from (24), (25), we arrive at the following necessary conditions:

$$\frac{\partial g}{\partial\psi} + \frac{\partial g_1}{\partial\psi}(C^2c_0f' + f) + \frac{\partial g_2}{\partial\psi}C\sqrt{2f'} = 0,$$
$$\frac{\partial g}{\partial\alpha} + \frac{\partial g_1}{\partial\alpha}(C^2c_0f' + f) + \frac{\partial g_2}{\partial\alpha}C\sqrt{2f'} = 0. \quad (27)$$

Relations (27), in particular, are fulfilled if $g =$ const, $g_1 =$ const, $g_2 =$ const. Then, we assign the functions $f(\psi)$, $C(\psi, \alpha)$. Let us determine $\psi(\tau, x, y)$, excluding $\alpha$ and $s$ from formulas (25). We substitute the found formula $\psi(\tau, x, y)$ into expression $r = A\sqrt{f(\psi(\tau, x, y))}$. We obtain $r = r(\tau, x, y)$, which will satisfy Eq. (3).

Let us consider, for example, one particular case. Let $g = g_1 = g_2 = 0$, $f = M\psi$, $C = K\psi\alpha$. Then, substituting these values into (25) and excluding from the obtained expressions $s$ and $\alpha$, we find that $f = M\psi = \left[y^2/(2c_0x^2) - \tau/x\right]$ and, consequently, $r = A\sqrt{f} = A\sqrt{y^2/(2c_0x^2) - \tau/x}$ is the exact solution to Eq. (3).

Assigning the set of other functions $f$, $C$, $g$, $g_1$, $g_2$ satisfying conditions (27) and excluding from relations (25) variables $s$ and $\alpha$, we obtain other functions $\psi(\tau, x, y)$ and other exact solutions to Eq. (3).

## ON REDUCING A SYSTEM OF EULER EQUATIONS TO A CDE SYSTEM

This method can also be applied to study systems of nonlinear differential equations.

Let us write the system of Euler equations in the form [8]

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V}\right] = -\nabla p,$$
$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V}) = 0, \quad \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}. \qquad (28)$$

Here $\gamma$ = const is the adiabatic index and $\mathbf{V}$ is the velocity vector with components $u$, $v$, $w$.

Let $u = u(\psi)$, $v = v(\psi)$, $w = w(\psi)$, $\rho = \rho(\psi)$. Then system (28) can be represented as

$$\begin{aligned}
&(1 + uf_1 + vf_2 + wf_3)u' + (\gamma p_0/\rho_0^{\gamma}) f_1 \rho^{(\gamma-2)}\rho' = 0,\\
&(1 + uf_1 + vf_2 + wf_3)v' + (\gamma p_0/\rho_0^{\gamma}) f_2 \rho^{(\gamma-2)}\rho' = 0,\\
&(1 + uf_1 + vf_2 + wf_3)w' + (\gamma p_0/\rho_0^{\gamma}) f_3 \rho^{(\gamma-2)}\rho' = 0,\\
&(1 + uf_1 + vf_2 + wf_3)\rho' + \rho f_1 u' + \rho f_2 v' + \rho f_3 w' = 0.
\end{aligned} \qquad (29)$$

Here (assuming that $\psi_t \neq 0$) $f_1(\psi) = \psi_x/\psi_t$, $f_2(\psi) = \psi_y/\psi_t$, $f_3(\psi) = \psi_z/\psi_t$, $f_1(\psi)$, $f_2(\psi)$, $f_3(\psi)$ are arbitrary functions. The prime (') denotes differentiation over $\psi$.

For system (29) to have a nontrivial solution, the determinant for the derivatives should be equal to zero. Equating the determinant to zero, we obtain

$$\rho = \left[\frac{\rho_0^{\gamma}(1 + uf_1 + vf_2 + wf_3)^2}{\gamma p_0 (f_1^2 + f_2^2 + f_3^2)}\right]^{1/(\gamma-1)}. \qquad (30)$$

It is easy to check that the dependences $f_1(\psi) = \psi_x/\psi_t$, $f_2(\psi) = \psi_y/\psi_t$, $f_3(\psi) = \psi_z/\psi_t$ take place if $\psi = \psi(s)$, $s = t + f_1 x + f_2 y + f_3 z$. Then, it can be considered that $\mathbf{V} = \mathbf{V}(s)$, $\rho = \rho(s)$, $f_1(s)$, $f_2(s)$, $f_3(s)$.

Further, we demonstrate how in particular cases, assigning a specific type of arbitrary functions, it is possible to reduce the system of Euler equations to a CDE system. We set $f_1(s) = f_2(s) = f_3(s) = s$; then system (28) reduces to the CDE system

$$[1 + s(u + v + w)]\frac{d\mathbf{V}}{ds} + \left(\frac{\gamma p_0}{\rho_0^{\gamma}}\right) s \rho^{(\gamma-2)} \frac{d\rho}{ds} = 0,$$
$$s = \frac{t}{1 - x - y - z}, \qquad (31)$$

where, according to (30), we have $\rho = \left[\frac{\rho_0^{\gamma}[1 + s(u + v + w)]^2}{3\gamma p_0 s^2}\right]^{1/(\gamma-1)}$.

We set $f_1 = f_2 = f_3 = 1/s$; then system (28) reduces to the CDE system

$$[s \pm (u + v + w)]\frac{d\mathbf{V}}{ds} \pm \left(\frac{\gamma p_0}{\rho_0^{\gamma}}\right) \rho^{(\gamma-2)} \frac{d\rho}{ds} = 0,$$
$$\rho = \left[\frac{\rho_0^{\gamma} s^2 [s \pm (u + v + w)]^2}{\gamma p_0}\right]^{1/(\gamma-1)}, \qquad (32)$$
$$s = \frac{t \pm \sqrt{t^2 + 4(x + y + z)}}{2}.$$

We set $f_1 = s$, $f_2 = 1/s$, $f_3 = 1$; then system (28) reduces to the CDE system

$$\mp(s \mp us^2 - v + sw)\frac{du}{ds} + \left(\frac{\gamma p_0}{\rho_0^{\gamma}}\right)\rho^{(\gamma-2)} s^2 \frac{d\rho}{ds} = 0,$$
$$\mp(s \mp us^2 - v + sw)\frac{dv}{ds} \pm \left(\frac{\gamma p_0}{\rho_0^{\gamma}}\right)\rho^{(\gamma-2)} \frac{d\rho}{ds} = 0,$$
$$\mp(s \mp us^2 - v + sw)\frac{dw}{ds} \mp \left(\frac{\gamma p_0}{\rho_0^{\gamma}}\right)\rho^{(\gamma-2)} s \frac{d\rho}{ds} = 0, \qquad (33)$$
$$\rho = \left[\frac{\rho_0^{\gamma}(s \mp us^2 - v + sw)^2}{\gamma p_0 (s^4 + s^2 + 1)}\right]^{1/(\gamma-1)},$$
$$s = \frac{-(t + z) \pm \sqrt{(t + z)^2 - 4y(x - 1)}}{2(x - 1)}.$$

## CONCLUSIONS

The paper illustrates a geometrical method for obtaining exact solutions to nonlinear acoustics equations using as an example three equations, but the given approach can also be used to solve other nonlinear equations in partial derivatives encountered in nonlinear acoustics. For the considered equations, other exact solutions can also be obtained if we assign other initial surfaces.

Application of the geometrical method to a system of nonlinear Euler equations in partial derivatives made it possible to reduce the system to CDE systems (31), (32), (33). This process can be continued assigning different functions $f_1(\psi)$, $f_2(\psi)$, $f_3(\psi)$.

It should be noted that there exist many methods for obtaining exact solution. The bibliography on this subject would require more than one page. A feature of our approach is that it makes it possible not only to obtain series of exact solutions, but also to note the features of development of the processes [9, 14]. Thus, it is easy to see that to observe the process described by Eq. (2), it suffices to assign initial conditions on one surface of the level and then obtain the solution at other surfaces of the level. Different behavior is observed in the case of Eq. (3). Here, in order to obtain a general picture, it is necessary on each surface of the level to assign something. Such processes exist in nonlinear thermal conductivity, when a perturbation from one point to another propagates according to the type of conical refraction, and which makes it possible to see our approach to obtaining exact solutions [11].

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