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Vibrations of a Rectangular Orthotropic Plate with Free Edges: Analysis and Solution of an Infinite System

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Abstract—A new asymptotically exact solution is obtained for the problem of transverse vibrations of a rectangular orthotropic plate with free edges. The general solution to the vibration equation is constructed as the sum of Fourier series with unknown coefficients, which are related by a homogeneous quasi-regular infinite system of linear algebraic equations. Analysis of the infinite system makes it possible to determine the power-law asymptotics for a nontrivial solution to the system, which makes it possible to calculate the natural vibration frequencies and to construct the corresponding eigenmodes. Examples of numerical calculations for real materials are presented.

Keywords: rectangular orthotropic plate, Chladni figures, infinite system of linear equations, asymptotics

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INTRODUCTION

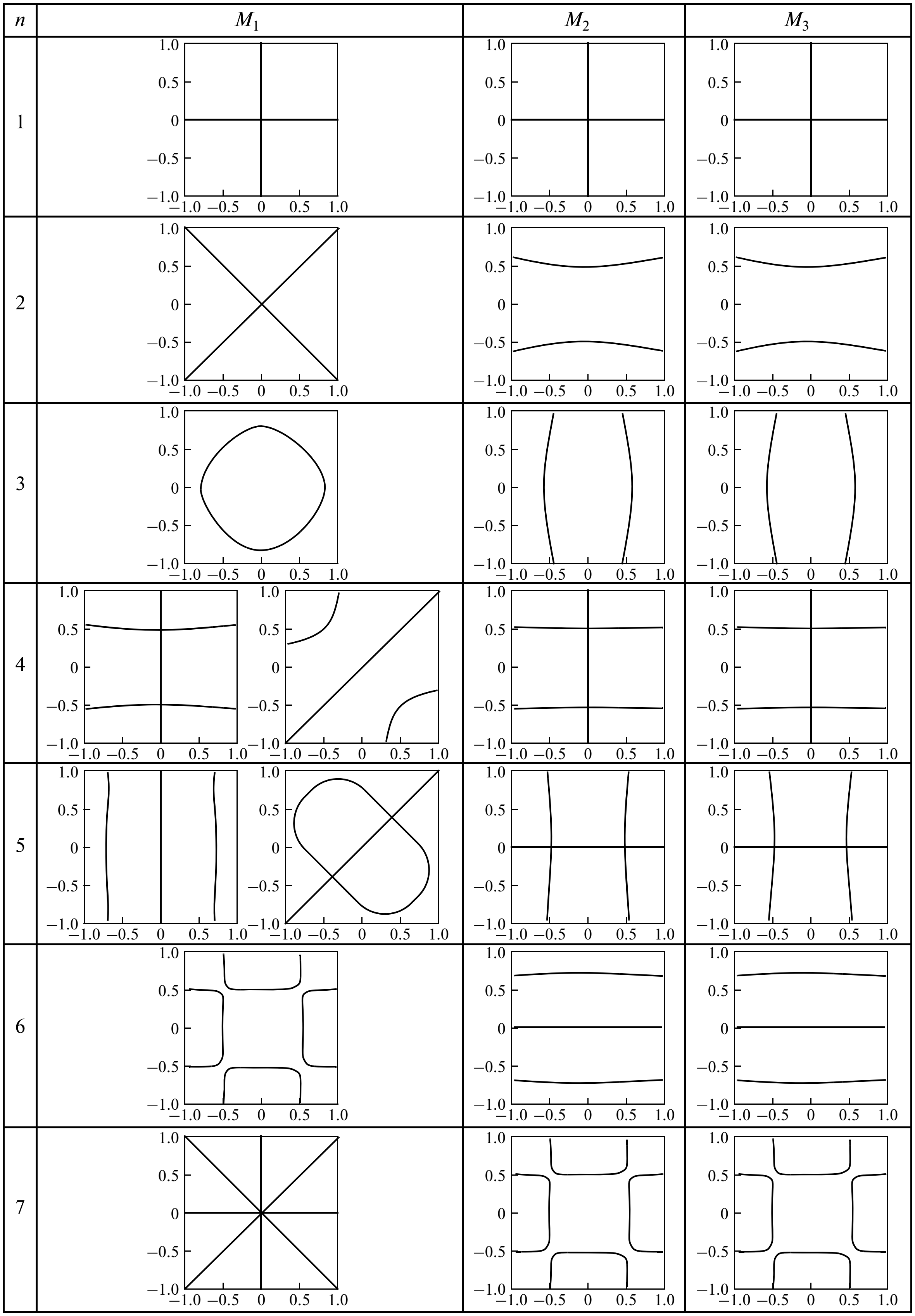
The problem of free transverse vibrations of a rectangular plate is one of the oldest classical problems. Early in the 19th century, owing to aesthetics, experimental demonstrations of Chladni figures [1] attracted public attention to the problem of plate vibrations. In turn, attempts to describe the problem in mathematical terms stimulated the development of mathematical physics.

Exact analytic solutions always attract considerable interest [2, 3]. Still, despite the long history and hundreds of publications devoted to solving the aforementioned problem using different approaches, the problem of transverse vibrations of rectangular plates has an exact solution in the form of Fourier series (a Levy-type solution) in only one case where two opposite sides of the plate are simply supported [4, 5]. In other cases of plate edge conditions, the variables involved in the boundary-value problem cannot be separated. Attempts to overcome this difficulty are being continued to this day. In particular, note [6], where a new method of dual separation of variables was used for a clamped orthothropic plate.

The approach based on the classical separation of variables makes it possible to construct a general solution to the vibration equation in the form of the sum of particular solutions. For arbitrary boundary conditions, this method leads to an infinite system of linear algebraic equations for the series expansion coefficients. This approach was first used in studying the vibrations of rectangular plates with free edges [7]. A more popular modification of this approach to determining the natural frequencies of rectangular plates is the superposition method developed in [8]. In this case, the solution is constructed as the finite sum of particular solutions, which makes it possible to obtain a finite system of equations for the unknown coefficients with the use of an artificial truncation of the infinite series. In [9], the authors use Levy-type solutions in the form of untruncated infinite series for studying flexural vibrations and stability of rectangular plates with arbitrary edge conditions.

The primary importance of a rectangular plate as an element of structure mechanics and engineering applications gave rise to a great number of publications devoted to studying the vibration problem using different approaches. One of them is the Ritz method [10], which was initially proposed for solving the problem of vibrations of a plate with completely free edges. Various modifications of the variational approach provide approximate solutions to a number of vibration and stability problems for rectangular plates [4]. In particular, in [11], the Rayleigh–Ritz method was used to study the effect of a complex load applied in the plane of the plate on plate vibrations and stability. The vibrations and stability of a symmetrically laminated composite rectangular plate under in-plane stresses were studied in [12] by the Rayleigh–Ritz method and the finite strip method. Application of the finite-element method to the study of vibrations of orthotropic plates is described in [13, 14]; application of the Kantorovich method to the same problem is described in [15]; and application of the Green function method is described in [16].

In this paper, the problem of vibrations of an orthotropic rectangular plate with free edges is reduced to a homogeneous infinite system of linear algebraic equa-



First Chladni figures for square plates.

tions. A generalization of the asymptotic expression law proposed by B.M. Koyalovich [17, 18] is used as the basis to determine the power law that describes the decrease of the nontrivial solution to the aforementioned system, which makes it possible to construct an efficient algorithm for calculating the natural frequencies and eigenmodes of the plate.

FORMULATION OF THE PROBLEM AND GENERAL SOLUTION

Let us consider a rectangular orthotropic plate $\{(x,y)\in[-a;a]\times[-b;b]\}$ with a thickness h. According to [19], the elastic properties of the material can be described by four elastic constants, for example, by Young's modulus E_1 along the direction of the x axis, shear modulus G, and two Poisson ratios ν_{12} and ν_{21}. Then, the equation describing free transverse vibrations of the plate in the classical Kirchhoff–Love approximation can be written in terms of the deflection of the plate $w(x,y,t)=W(x,y)e^{i\omega t}$:

$$D_1\frac{\partial^4 W}{\partial x^4}+2D_3\frac{\partial^4 W}{\partial x^2\partial y^2}+D_2\frac{\partial^4 W}{\partial y^4}-D_1\Omega^4 W=0, \quad (1)$$

where $\Omega=\sqrt[4]{\frac{\omega^2\rho h}{D_1}}$ is the dimensionless frequency parameter, ρ is the density of the material, ω is the circular frequency,

$$D_1=\frac{E_1h^3}{12(1-\nu_{12}\nu_{21})};\quad D_2=\frac{\nu_{21}E_1h^3}{12\nu_{12}(1-\nu_{12}\nu_{21})};$$

$$D_3=D_{12}+2D_{66};\quad D_{66}=\frac{Gh^3}{12};\quad D_{12}=\nu_{12}D_2.$$

The boundary conditions for the free edges at the sides $x=\pm a$ have the form

$$M_x=-\left(D_1\frac{\partial^2 W}{\partial x^2}+D_{12}\frac{\partial^2 W}{\partial y^2}\right)=0,$$
$$V_x=-\left(D_1\frac{\partial^3 W}{\partial x^3}+(D_3+2D_{66})\frac{\partial^3 W}{\partial x\partial y^2}\right)=0; \quad (2)$$

at the sides $y=\pm b$:

$$M_y=-\left(D_{12}\frac{\partial^2 W}{\partial x^2}+D_2\frac{\partial^2 W}{\partial y^2}\right)=0,$$
$$V_y=-\left(D_2\frac{\partial^3 W}{\partial y^3}+(D_3+2D_{66})\frac{\partial^3 W}{\partial x^2\partial y}\right)=0. \quad (3)$$

The general solution to the problem can be represented as the sum of even and odd components with respect to each of the coordinates:

$$W=\sum_{k,j=0}^{1}W_{kj}, \quad (4)$$

where W_{00} is even in both coordinates, W_{01} is even in x and odd in y, etc.

By the standard separation of variables, it is possible to represent the general solution to vibration equation (1) for each of the symmetry cases as the sum of two Fourier series with unknown coefficients:

$$W_{kj}=\sum_{n=1}^{\infty}(A_nH_j(p_{nk}y)+B_nH_j(\overline{p}_{nk}y))T_k(\alpha_{nk}x)$$
$$+\sum_{n=1}^{\infty}(C_nH_k(q_{nj}x)+D_nH_k(\overline{q}_{nj}x))T_j(\beta_{nj}y), \quad (5)$$

where trigonometric and hyperbolic functions are denoted as

$$T_j(z)=\begin{cases}\cos z, & j=0,\\ \sin z, & j=1,\end{cases}\qquad H_j(z)=\begin{cases}\cosh z, & j=0,\\ \sinh z, & j=1.\end{cases}$$

The separation constants are chosen in a form that ensures completeness of solution (5) at the plate boundary:

$$\alpha_{nj}=\frac{\pi}{a}\left(n-1+\frac{j}{2}\right),\quad \beta_{nj}=\frac{\pi}{b}\left(n-1+\frac{j}{2}\right). \quad (6)$$

Quantities $p_{nk},\overline{p}_{nk}$ and $q_{hj},\overline{q}_{hj}$ are the roots of the following characteristic equations:

$$D_2p^4-2D_3\alpha^2p^2+D_1\left(\alpha^4-\Omega^4\right)=0, \quad (7)$$

$$D_1q^4-2D_3\beta^2q^2+D_2\beta^4-D_1\Omega^4=0. \quad (8)$$

These roots are easily expressed in analytical form:

$$p=\sqrt{\frac{D_3\alpha^2+\sqrt{(D_3^2-D_1D_2)\alpha^4+D_1D_2\Omega^4}}{D_2}},$$
$$\overline{p}=\sqrt{\frac{D_3\alpha^2-\sqrt{(D_3^2-D_1D_2)\alpha^4+D_1D_2\Omega^4}}{D_2}}, \quad (9)$$

$$q=\sqrt{\frac{D_3\beta^2+\sqrt{(D_3^2-D_1D_2)\beta^4+D_1^2\Omega^4}}{D_2}},$$
$$\overline{q}=\sqrt{\frac{D_3\beta^2-\sqrt{(D_3^2-D_1D_2)\beta^4+D_1^2\Omega^4}}{D_2}}. \quad (10)$$

Depending on the signs of the radicands, quantities (9)–(10) can be complex. However, by virtue of the Vieta theorem for Eqs. (7)–(8), the expressions

$$D_1(\alpha_{nk}^2+q_{mj}^2)(\alpha_{nk}^2+\overline{q}_{mj}^2)=D_2(\beta_{mj}^2+p_{nk}^2)(\beta_{mj}^2+\overline{p}_{nk}^2)$$
$$=D_1\alpha_{nk}^4+2D_3\alpha_{nk}^2\beta_{mj}^2+D_2\beta_{mj}^4-D_1\Omega^4 \quad (11)$$

must be real.

The general solution given by Eqs. (4) and (5) exactly satisfies vibration equation (1) and is sufficiently flexible to satisfy any preset boundary conditions. In the case of free edges of the plate, conditions (2) and (3) imposed on the normal reactions V_x and V_y can be exactly satisfied. Indeed, from Eq. (6) it follows that, for any symmetry type, $T_k'(\alpha_{nk}a)=T_j'(\beta_{nj}b)=0$. Then, if the unknown coefficients are chosen in the form

$$A_n = \frac{(-1)^n b\sqrt{D_1}\left(D_2\bar{p}_{nk}^2 - (D_3 + 2D_{66})\alpha_{nk}^2\right)}{2D_2 p_{nk} H_j'(p_{nk}b)(\bar{p}_{nk}^2 - p_{nk}^2)} X_n,$$

$$B_n = \frac{(-1)^{n+1} b\sqrt{D_1}\left(D_2 p_{nk}^2 - (D_3 + 2D_{66})\alpha_{nk}^2\right)}{2D_2 \bar{p}_{nk} H_j'(\bar{p}_{nk}b)(\bar{p}_{nk}^2 - p_{nk}^2)} X_n,$$

$$C_n = \frac{(-1)^n a\sqrt{D_2}\left(D_1\bar{q}_{nj}^2 - (D_3 + 2D_{66})\beta_{nj}^2\right)}{2D_1 q_{nj} H_k'(q_{nj}a)(\bar{q}_{nj}^2 - q_{nj}^2)} Y_n,$$

$$D_n = \frac{(-1)^{n+1} a\sqrt{D_2}\left(D_1 q_{nj}^2 - (D_3 + 2D_{66})\beta_{nj}^2\right)}{2D_1 \bar{q}_{nj} H_k'(\bar{q}_{nj}a)(\bar{q}_{nj}^2 - q_{nj}^2)} Y_n$$

the aforementioned conditions are satisfied identically.

The conditions imposed on the moments M_x and M_y yield two functional equations:

$$\frac{b\sqrt{D_1}}{D_2}\sum_{n=1}^{\infty}\frac{X_n}{\bar{p}_{nk}^2 - p_{nk}^2}$$
$$\times\left(\frac{(D_2\bar{p}_{nk}^2 - (D_3 + 2D_{66})\alpha_{nk}^2)(D_{12}p_{nk}^2 - D_1\alpha_{nk}^2)}{p_{nk}}\frac{H_j(p_{nk}y)}{H_j'(p_{nk}b)}\right.$$
$$\left.- \frac{(D_2 p_{nk}^2 - (D_3 + 2D_{66})\alpha_{nk}^2)(D_{12}\bar{p}_{nk}^2 - D_1\alpha_{nk}^2)}{\bar{p}_{nk}}\frac{H_j(\bar{p}_{nk}y)}{H_j'(\bar{p}_{nk}b)}\right)$$
$$= \frac{a\sqrt{D_2}}{D_1}\sum_{m=1}^{\infty}\frac{(-1)^m Y_m}{\bar{q}_{mj}^2 - q_{mj}^2}$$
$$\times\left(\frac{(D_1\bar{q}_{mj}^2 - (D_3 + 2D_{66})\beta_{mj}^2)(D_1 q_{mj}^2 - D_{12}\beta_{mj}^2)}{q_{mj}}\frac{H_k(q_{mj}a)}{H_k'(q_{mj}a)}\right.$$
$$\left.- \frac{(D_1 q_{mj}^2 - (D_3 + 2D_{66})\beta_{mj}^2)(D_1\bar{q}_{mj}^2 - D_{12}\beta_{mj}^2)}{\bar{q}_{mj}}\frac{H_k(\bar{q}_{mj}a)}{H_k'(\bar{q}_{mj}a)}\right)$$
$$\times T_j(\beta_{mj}y), \quad y \in [-b; b],$$

$$\frac{a\sqrt{D_2}}{D_1}\sum_{n=1}^{\infty}\frac{Y_n}{\bar{q}_{nj}^2 - q_{nj}^2}$$
$$\times\left(\frac{(D_1\bar{q}_{nj}^2 - (D_3 + 2D_{66})\beta_{nj}^2)(D_{12}q_{nj}^2 - D_2\beta_{nj}^2)}{q_{nj}}\right.$$
$$\times\frac{H_k(q_{nj}x)}{H_k'(q_{nj}a)}$$
$$- \frac{(D_1 q_{nj}^2 - (D_3 + 2D_{66})\beta_{nj}^2)(D_{12}\bar{q}_{nj}^2 - D_2\beta_{nj}^2)}{\bar{q}_{nj}}$$
$$\left.\times\frac{H_k(\bar{q}_{nj}x)}{H_k'(\bar{q}_{nj}a)}\right)$$
$$= \frac{b\sqrt{D_1}}{D_2}\sum_{m=1}^{\infty}\frac{(-1)^m X_m}{\bar{p}_{mk}^2 - p_{mk}^2}$$
$$\times\left(\frac{(D_2\bar{p}_{mk}^2 - (D_3 + 2D_{66})\alpha_{mk}^2)(D_2 p_{mk}^2 - D_{12}\alpha_{mk}^2)}{p_{mk}}\right.$$
$$\times\frac{H_j(p_{mk}b)}{H_j'(p_{mk}b)}$$
$$- \frac{(D_2 p_{mk}^2 - (D_3 + 2D_{66})\alpha_{mk}^2)(D_2\bar{p}_{mk}^2 - D_{12}\alpha_{mk}^2)}{\bar{p}_{mk}}$$
$$\left.\times\frac{H_j(\bar{p}_{mk}b)}{H_j'(\bar{p}_{mk}b)}\right)T_k(\alpha_{mk}x), \quad x \in [-a; a].$$

After the hyperbolic functions involved in these equalities are expanded in trigonometric ones,

$$\frac{H_k(qx)}{H_k'(qa)} = \frac{q}{a}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}(2 - \delta_{k0}\delta_{1m})}{\alpha_{mk}^2 + q^2}T_k(\alpha_{mk}x),$$

$$\frac{H_j(py)}{H_j'(pb)} = \frac{p}{b}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}(2 - \delta_{j0}\delta_{1m})}{\beta_{mj}^2 + p^2}T_j(\beta_{mj}y)$$

and the order of summation on the left-hand sides of the equalities is reversed, we obtain an infinite system of linear algebraic equations in the sequences $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$:

$$Y_m = \frac{2 - \delta_{j0}\delta_{1m}}{2a\Delta_m^1\sqrt{D_1 D_2}}$$
$$\times\sum_{n=1}^{\infty}\frac{(4D_{66}^2 + D_1 D_2 - D_3^2)\beta_{mj}^2\alpha_{nk}^2 + D_{12}D_1\Omega^4}{(\alpha_{nk}^2 + q_{mj}^2)(\alpha_{nk}^2 + \bar{q}_{mj}^2)}X_n,$$
$$X_m = \frac{2 - \delta_{k0}\delta_{1m}}{2b\Delta_m^2\sqrt{D_1 D_2}} \qquad (12)$$
$$\times\sum_{n=1}^{\infty}\frac{(4D_{66}^2 + D_1 D_2 - D_3^2)\beta_{nj}^2\alpha_{mk}^2 + D_{12}D_1\Omega^4}{(\beta_{nj}^2 + p_{mk}^2)(\beta_{nj}^2 + \bar{p}_{mk}^2)}Y_n$$
$$(m = 1, 2, \ldots).$$

Here, δ_{mn} is the Kronecker delta,

$$\Delta_m^1 = \frac{H_k(q_{mj}a)}{H_k'(q_{mj}a)}$$
$$\times\frac{(D_1\bar{q}_{mj}^2 - (D_3 + 2D_{66})\beta_{mj}^2)(D_1 q_{mj}^2 - D_{12}\beta_{mj}^2)}{2D_1 q_{mj}(\bar{q}_{mj}^2 - q_{mj}^2)}$$
$$- \frac{H_k(\bar{q}_{mj}a)}{H_k'(\bar{q}_{mj}a)}$$
$$\times\frac{(D_1 q_{mj}^2 - (D_3 + 2D_{66})\beta_{mj}^2)(D_1\bar{q}_{mj}^2 - D_{12}\beta_{mj}^2)}{2D_1\bar{q}_{mj}(\bar{q}_{mj}^2 - q_{mj}^2)},$$
$$\Delta_m^2 = \frac{H_j(p_{mk}b)}{H_j'(p_{mk}b)}$$
$$\times\frac{(D_2\bar{p}_{mk}^2 - (D_3 + 2D_{66})\alpha_{mk}^2)(D_2 p_{mk}^2 - D_{12}\alpha_{mk}^2)}{2D_2 p_{mk}(\bar{p}_{mk}^2 - p_{mk}^2)}$$

$$-\frac{H_j(\bar{p}_{mk}b)}{H_j'(\bar{p}_{mk}b)}$$
$$\times\frac{(D_2p_{mk}^2-(D_3+2D_{66})\alpha_{mk}^2)(D_2\bar{p}_{mk}^2-D_{12}\alpha_{mk}^2)}{2D_2\bar{p}_{mk}(\bar{p}_{mk}^2-p_{mk}^2)}.$$

Note that the expressions for Δ_m^1 and Δ_m^2 must be real for any combination of the parameters of the problem. Hence, by virtue of identity (11), the coefficients of system (12) are also real. The nontrivial solution to this system of equations at the natural vibration frequency provides an explicit analytic expression for the eigenmodes of plate vibrations:

$$W_{kj}=\frac{b\sqrt{D_1}}{2D_2}\sum_{n=1}^{\infty}\frac{(-1)^nX_n}{\bar{p}_{nk}^2-p_{nk}^2}$$
$$\times\left(\frac{D_2\bar{p}_{nk}^2-(D_3+2D_{66})\alpha_{nk}^2}{p_{nk}}\frac{H_j(p_{nk}y)}{H_j'(p_{nk}b)}\right.$$
$$\left.-\frac{D_2p_{nk}^2-(D_3+2D_{66})\alpha_{nk}^2}{\bar{p}_{nk}}\frac{H_j(\bar{p}_{nk}y)}{H_j'(\bar{p}_{nk}b)}\right)T_k(\alpha_{nk}x)$$
$$+\frac{a\sqrt{D_2}}{2D_1}\sum_{n=1}^{\infty}\frac{(-1)^nY_n}{\bar{q}_{nj}^2-q_{nj}^2}$$
$$\times\left(\frac{D_1\bar{q}_{nj}^2-(D_3+2D_{66})\beta_{nj}^2}{q_{nj}}\frac{H_k(q_{nj}x)}{H_k'(q_{nj}a)}\right.$$
$$\left.-\frac{D_1q_{nj}^2-(D_3+2D_{66})\beta_{nj}^2}{\bar{q}_{nj}}\frac{H_k(\bar{q}_{nj}x)}{H_k'(\bar{q}_{nj}a)}\right)T_j(\beta_{nj}y).$$

ANALYSIS AND SOLUTION OF THE INFINITE SYSTEM

According to the theory of infinite systems [20], system (12) can be represented in canonical form by applying the change of variables $Z_{2m-1}=Y_m, Z_{2m}=X_m$:

$$Z_m=\sum_{n=1}^{\infty}M_{mn}Z_n\quad(m=1,2,\ldots).\tag{13}$$

An infinite system of linear algebraic equations is called a regular system if, for any row of the system, the sum of the absolute values of coefficients is smaller than unity; if there exists such a constant θ that

$$S_m=\sum_{n=1}^{\infty}|M_{mn}|\le\theta<1,\tag{14}$$

the system is called a fully regular one [20]. When the regularity (full regularity) condition is satisfied, the infinite system can be considered a functional equation in the space of bounded sequences l_∞. For such systems, under certain restrictions on the free terms [20], one can guarantee the existence of a bounded solution. For fully regular systems, one can also guarantee the uniqueness of the bounded solution. If condition (14) is satisfied beginning with a certain number $m>N_R$, the infinite system is called a quasi-regular one and its study can be reduced to analysis of a finite system of order N_R. Evidently, system (12) obtained above cannot be fully regular within the entire frequency range, because, owing to its homogeneity, such a property should lead to the presence of the zero (trivial) solution alone.

To investigate the regularity of system (12), let us use the known values of the series [21]

$$\sum_{n=1}^{\infty}\frac{1}{\alpha_{n0}^2+q^2}=\frac{1}{q^2}+\sum_{n=1}^{\infty}\frac{1}{(\pi n/a)^2+q^2}=\frac{a}{2q}\coth qa+\frac{1}{2q^2},$$

$$\sum_{n=1}^{\infty}\frac{1}{\alpha_{n1}^2+q^2}=\sum_{n=1}^{\infty}\frac{1}{(\pi(n-1/2)/a)^2+q^2}=\frac{a}{2q}\tanh qa.$$

Combining these formulas and using the above notations, we derive

$$\sum_{n=1}^{\infty}\frac{1}{\alpha_{nk}^2+q^2}=\frac{a}{2q}\frac{H_k(qa)}{H_k'(qa)}+\frac{1-k}{2q^2}.\tag{15}$$

In a similar way, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{\beta_{nj}^2+p^2}=\frac{b}{2p}\frac{H_j(pb)}{H_j'(pb)}+\frac{1-j}{2p^2}.\tag{16}$$

Since the coefficients of system (12) are alternating for only the first rows and columns of the infinite matrix, the series taken under regularity conditions (14) can be exactly calculated using Eqs. (15) and (16):

$$S_{2m-1}=\sum_{n=1}^{N}(|M_{2m-1,n}|-M_{2m-1,n})+\frac{2-\delta_{j0}\delta_{1m}}{4a\sqrt{D_1D_2}|\Delta_m^1|}$$
$$\times\left\{(D_1D_2+4D_{66}^2-D_3^2)\beta_{mj}^2\left(\frac{a}{\bar{q}_{mj}}\frac{H_k(\bar{q}_{mj}a)}{H_k'(\bar{q}_{mj}a)}+\frac{1-k}{\bar{q}_{mj}^2}\right)\right.$$
$$-\frac{(D_1D_2+4D_{66}^2-D_3^2)\beta_{mj}^2q_{mj}^2-D_{12}D_1\Omega^4}{\bar{q}_{mj}^2-q_{mj}^2}$$
$$\left.\times\left(\frac{a}{q_{mj}}\frac{H_k(q_{mj}a)}{H_k'(q_{mj}a)}-\frac{a}{\bar{q}_{mj}}\frac{H_k(\bar{q}_{mj}a)}{H_k'(\bar{q}_{mj}a)}+\frac{1-k}{q_{mj}^2\bar{q}_{mj}^2}(\bar{q}_{mj}^2-q_{mj}^2)\right)\right\},$$

$$S_{2m}=\sum_{n=1}^{N}(|M_{2m,n}|-M_{2m,n})+\frac{2-\delta_{k0}\delta_{1m}}{4b\sqrt{D_1D_2}|\Delta_m^2|}$$
$$\times\left\{(D_1D_2+4D_{66}^2-D_3^2)\alpha_{mk}^2\left(\frac{b}{\bar{p}_{mk}}\frac{H_j(\bar{p}_{mk}b)}{H_j'(\bar{p}_{mk}b)}+\frac{1-j}{2\bar{p}_{mk}^2}\right)\right.$$
$$-\frac{(D_1D_2+4D_{66}^2-D_3^2)\alpha_{mk}^2p_{mk}^2-D_{12}D_1\Omega^4}{\bar{p}_{mk}^2-p_{mk}^2}$$

$$\times\left(\frac{b}{p_{mk}}\frac{H_j(p_{mk}b)}{H_j'(p_{mk}b)}-\frac{b}{\overline{p}_{mk}}\frac{H_j(\overline{p}_{mk}b)}{H_j'(\overline{p}_{mk}b)}+\frac{1-j}{p_{mk}^2\overline{p}_{mk}^2}(\overline{p}_{mk}^2-p_{mk}^2)\right)\Bigg\}.$$

Here, the number $N = N(\Omega)$ is chosen so as to provide a positive value of M_{mn} ($n > N$). Considering the asymptotics for $m\to\infty$ and taking into account the relations

$$\Delta_m^1=\frac{D_1D_2+4D_{66}\sqrt{D_1D_2}-D_{12}^2}{2\sqrt{2}\sqrt{D_2(D_3+\sqrt{D_1D_2})}}\beta_{mj}+O\left(\frac{1}{\beta_{mj}}\right),$$
$$\Delta_m^2=\frac{D_1D_2+4D_{66}\sqrt{D_1D_2}-D_{12}^2}{2\sqrt{2}\sqrt{D_1(D_3+\sqrt{D_1D_2})}}\alpha_{mk}+O\left(\frac{1}{\alpha_{mk}}\right),$$

we find that the even and odd sums tend to the same constant limit:

$$\lim_{m\to\infty}S_{2m-1}=\lim_{m\to\infty}S_{2m}=\frac{D_1D_2-D_{12}^2-4D_{66}D_{12}}{D_1D_2-D_{12}^2+4D_{66}\sqrt{D_1D_2}}=\theta<1. \tag{17}$$

Formula (17) suggests that there always exists a number N_R beginning with which series under regularity conditions become smaller than unity; i.e., system (12) is a quasi-regular one.

With the substitution

$$Z_m=\sum_{l=1}^{N_R}\xi_m^lZ_l\quad(m>N_R) \tag{18}$$

the infinite system is reduced to a set of fully regular infinite systems of equations in $\left\{\xi_m^l\right\}_{m=N_R+1}^{\infty}$ ($l = 1, 2, ..., N_R$) with the same matrix:

$$\xi_m^l=\sum_{n=N_R+1}^{\infty}M_{mn}\xi_n^l+M_{ml}\quad(m=N_R+1,N_R+2,\ldots). \tag{19}$$

From the boundedness of the free terms of these systems, it follows that each of them has a unique bounded solution. Hence, the problem of the existence of a bounded solution to the initial quasi-regular system (12) appears to be equivalent to the problem of the existence of a solution to the finite system of equations in the first unknowns $\{Z_m\}_{m=1}^{N_R}$. This system is obtained by substituting Eq. (18) in Eq. (13) for $m = 1, 2, ..., N_R$:

$$Z_m=\sum_{n=1}^{N_R}Q_{mn}Z_n, \tag{20}$$

where $Q_{mn}=M_{mn}+\sum_{l=N_R+1}^{\infty}M_{ml}\xi_l^n$.

Thus, the zero value of the determinant of finite system (20) yields a dispersion equation for determining the natural frequencies of the plate:

$$\det\|\delta_{mn}-Q_{mn}\|=0.$$

To construct an efficient algorithm for solving systems (19), we analytically determine the asymptotics of their solutions. For this purpose, we change the variables:

$$\xi_{2m-1}^l=D_1^{\frac{\lambda}{4}+\frac{1}{2}}\beta_{mj}^{-(2+\lambda)}y_m^l,\quad \xi_{2m}^l=D_2^{\frac{\lambda}{4}+\frac{1}{2}}\alpha_{mk}^{-(2+\lambda)}x_m^l.$$

Here, λ is determined from the condition that the transformed systems

$$y_m^l=\frac{\beta_{mj}^{2+\lambda}}{aD_1\Delta_m^1}\left(\frac{D_2}{D_1}\right)^{\lambda/4}\times\sum_{n=N_r+1}^{\infty}\frac{(4D_{66}^2+D_1D_2-D_3^2)\beta_{mj}^2\alpha_{nk}^{-\lambda}+D_{12}D_1\Omega^4\alpha_{nk}^{-2-\lambda}}{(\alpha_{nk}^2+q_{mj}^2)(\alpha_{nk}^2+\overline{q}_{mj}^2)}x_n^l+\frac{M_{2m-1,l}\beta_{mj}^{2+\lambda}}{D_1^{1/2+\lambda/4}},$$
$$x_m^l=\frac{\alpha_{mk}^{2+\lambda}}{b\Delta_m^2}\left(\frac{D_1}{D_2}\right)^{\lambda/4}\times\sum_{n=N_r+1}^{\infty}\frac{(4D_{66}^2+D_1D_2-D_3^2)\alpha_{mk}^2\beta_{nj}^{-\lambda}+D_{12}D_1\Omega^4\beta_{nj}^{-2-\lambda}}{(\beta_{nj}^2+p_{mk}^2)(\beta_{nj}^2+\overline{p}_{mk}^2)}y_n^l+\frac{M_{2m,l}\alpha_{mk}^{2+\lambda}}{D_2^{1/2+\lambda/4}},\quad(2N_r=N_R;\quad m=N_r+1,N_r+2,\ldots) \tag{21}$$

satisfy the generalization of the Koyalovich asymptotic expression law [9].

From the condition that the free terms of systems (21) are bounded, we find that $\lambda\in[0;1)$. Omitting the mathematical proof for brevity, we should by noted that, under the conditions of the aforementioned theorem, the coefficients of system (21) satisfy the estimates for any index $\lambda\in[0;1)$. However, a necessary condition for the existence of a common nonzero limit of the solutions to each of the systems (21)

$$\lim_{m\to\infty}y_m^l=\lim_{m\to\infty}x_m^l=K_l>0$$

is the fact that these systems should remain regular but should no longer satisfy the full regularity condition; i.e., under the regularity conditions, the series should tend to unity from below. This yields the equation for calculating λ.

Indeed, using the Euler–Maclaurin formulas to calculate the value of the series

$$S_N^j(z)=\sum_{n=N+1}^{\infty}\frac{(n-1+j/2)^{-\lambda}}{(n-1+j/2)^2+z^2}=\frac{(N+j/2)^{-1-\lambda}}{1+\lambda}\,{}_2F_1\left(1;\frac{1+\lambda}{2};\frac{3+\lambda}{2};-\left(\frac{z}{N+j/2}\right)^2\right)$$

Table 1. Elastic parameters of materials

Material	$E_1 \times 10^{-5}$, kgf/cm^2	$G \times 10^{-5}$, kgf/cm^2	ν_{12}	ν_{21}
M_1—glass	0.7	0.28	0.25	0.25
M_2—glass-cloth-base laminate KAST-V	2.0	0.40	0.20	0.11
M_3—epoxy glass	0.61	0.12	0.23	0.09

$$+\frac{(N+j/2)^{-\lambda}}{2((N+j/2)^2+z^2)}+\frac{\lambda(N+j/2)^{-\lambda-1}}{12((N+j/2)^2+z^2)}+\frac{(N+j/2)^{1-\lambda}}{6((N+j/2)^2+z^2)^2}-\frac{(N+j/2)^{1-\lambda}}{720((N+j/2)^2+z^2)}\times\left(\frac{\lambda(\lambda+1)(\lambda+2)}{(N+j/2)^4}+\frac{6\lambda^2}{((N+j/2)^2+z^2)(N+j/2)^2}-\frac{24(1-\lambda)}{((N+j/2)^2+z^2)^2}\right)$$

and considering its asymptotics at $z \to \infty$:

$$S_N^j(z) = \frac{\pi}{2\cos\frac{\pi\lambda}{2}}\frac{1}{z^{1+\lambda}} + O\left(\frac{1}{z^2}\right),$$

we find that, under regularity conditions (14) for systems (21), the series tend from below to the following value:

$$f(\lambda) = \frac{\sqrt{2}\sqrt[4]{D_1D_2}}{\sqrt{\sqrt{D_1D_2}-D_3}}\frac{D_1D_2 - D_{12}^2 - 4D_{66}D_{12}}{D_1D_2 - D_{12}^2 + 4D_{66}\sqrt{D_1D_2}} \times \frac{\sin\left(\frac{\lambda+1}{2}\arctan\sqrt{\frac{D_1D_2}{D_3^2}-1}\right)}{\cos\frac{\pi\lambda}{2}}.$$

Hence, the equality

$$f(\lambda) = 1 \tag{22}$$

yields the desired equation for determining the index λ.

Table 1 shows the elastic constants for glass and glass-reinforced plastic. The solution to Eq. (22) for these materials is illustrated by Table 2.

The known power law describing the decrease of the solutions to systems (21) makes it possible, with the use of Eq. (18), to determine the following form of the principal term of the asymptotics for the nontrivial solution to the initial infinite system:

$$Y_m = \frac{KD_1^{\frac{\lambda}{4}+\frac{1}{2}}}{\beta_{mj}^{2+\lambda}}, \quad X_m = \frac{KD_2^{\frac{\lambda}{4}+\frac{1}{2}}}{\alpha_{mk}^{2+\lambda}} \quad (m \to \infty)$$

Then, the solution to initial infinite system (12) is reduced to determining a nontrivial solution to the finite system of equations in the first unknowns $Y_1, X_1, Y_2, X_2, \ldots, Y_N, X_N$ and limiting constant K. Thus, it is possible to determine the entire sequence of unknown coefficients in the general solution W_{kj}, which makes it possible to determine the analytic solution to problem (1)–(3) stated above.

Table 2. Index λ in asymptotics of solution

Material	λ
M_1—glass	0.809211
M_2—glass-cloth-base laminate KAST-V	0.581107
M_3—epoxy glass	0.614004

Table 3. First natural frequencies $\mu = 4\Omega^2$ for quadratic isotropic plate at $\nu = 0.3$

N	$N = 10$	[4]
1	13.4682	13.4728
2	19.5960	19.5961
3	24.2702	24.2702
4	34.8008	34.8011
5	61.0949	61.0932
6	63.6868	63.6870
7	69.2653	69.5020
8	77.1724	77.5897
9	105.461	105.463
10	117.108	117.109

NUMERICAL RESULTS

The proposed approach was implemented by a computer program to calculate the natural frequencies and construct the eigenmodes of rectangular orthotropic plates. The particular case of an isotropic plate was considered:

$$\nu = \nu_{12} = \nu_{21}; \quad E = E_1; \quad G = \frac{E}{2(1+\nu)}.$$

First, the results reported in the literature were compared to those obtained by the Rayleigh–Ritz method [4]. Table 3 presents the frequency parameter $\mu = 4\Omega^2$ for convenience. Almost all the values coincide except for the seventh and eighth modes, which are skew-symmetric in both coordinates. In this case, the difference is about 0.5%. Presumably, this is

Table 4. First natural frequencies Ω for quadratic plate

n	M_1	Symmetry	M_2	Symmetry	M_3	Symmetry
1	1.8645	(1, 1)	1.5916	(1, 1)	1.5832	(1, 1)
2	2.2434	(0, 0)	2.0213	(0, 0)	1.8792	(0, 0)
3	2.4503	(0, 0)	2.3683	(0, 0)	2.3653	(0, 0)
4	2.9834	(1, 0 /(0, 1))	2.5628	(1, 0)	2.4872	(1, 0)
5	3.9144	(1, 0 /(0, 1))	2.7436	(0, 1)	2.7349	(0, 1)
6	4.0316	(0, 0)	3.3787	(0, 1)	3.1388	(0, 1)
7	4.2093	(1, 1)	3.5314	(0, 0)	3.4892	(0, 0)
8	4.3988	(1, 1)	3.6937	(1, 1)	3.5140	(1, 1)
9	5.1780	(1, 0 /(0, 1))	3.9211	(1, 0)	3.9210	(1, 0)
10	5.4316	(0, 0)	4.1467	(1, 1)	4.1395	(1, 1)

related to the weaker convergence of the Rayleigh–Ritz method in the chosen odd functions.

Table 4 shows the first ten natural frequencies for square plates made of glass and glass-reinforced plastic (see Table 1). The figure shows the corresponding Chladni figures, i.e., the nodal lines of the eigenmodes. For the square plate made from the isotropic material M_1, the eigenvalues that correspond to modes symmetric about one of the coordinates and skew-symmetric about the other coordinate belong to two eigenmodes $W_{01}(x, y)$ and $W_{10}(x, y)$. Therefore, for these natural frequencies, the figure shows two Chladni figures: the first of them corresponds to $W_{01}(x, y)$ and the second to $W_{01}(x, y) - W_{10}(x, y)$. Note that it is the second figure that is most often observed in experiments. From Table 4 and the figure, it follows that, in all the examples, the fundamental frequency corresponds to the modes that are skew-symmetric about both of the coordinates. The results obtained for two orthotropic materials M_2 and M_3 are closer to each other, as compared to the results for the material M_1. This manifests itself in the closeness of natural frequencies and in the similarity of the Chladni figures. The maximal difference between the isotropic and orthotropic materials is observed for symmetric modes, in particular, for the second and third modes. However, for all the three materials, the symmetry types are similar and the difference between the Chladni figures observed for the isotropic material M_1 and those observed for the two orthotropic materials M_2 and M_3 is considerable.

CONCLUSIONS

Thus, the proposed algorithm for constructing natural frequencies and vibration modes of a rectangular orthotropic plate makes it possible to solve the above-stated problem with the required accuracy on the basis of the well-known asymptotic law describing the decrease of coefficients in the general solution.

Comparison of the calculated natural frequencies with the known values shows their full coincidence. In the examples presented above, the difference in the Chladni figures obtained for the isotropic and orthothropic plates most clearly manifests itself for the symmetric modes, in particular, for the second and third modes.

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