

On the Probabilities of Large Deviations of Combinatorial Sums of Independent Random Variables That Satisfy the Linnik Condition

A. N. Frolov

St Petersburg State University, St Petersburg, 199034 Russia

e-mail: Andrei.Frolov@pobox.spbu.ru

Received January 2, 2023; revised January 18, 2023; accepted February 16, 2023

Abstract—New results on the asymptotic behavior of the probabilities of large deviations of combinatorial sums of independent random variables satisfying the Linnik condition are obtained. A zone, where these probabilities are equivalent to the tail of the standard normal law, is found. Such results were previously obtained by the author under Bernstein’s condition. The new results are proved by the truncation method.

Keywords: probabilities of large deviations, combinatorial central limit theorem, combinatorial sum

DOI: 10.1134/S1063454123030044

1. INTRODUCTION

We suppose $\{(X_{nij}), 1 \leq i, j \leq n, n = 2, 3, \dots\}$ is a sequence of matrices of independent random variables and $\{\tilde{\pi}_n = (\pi_n(1), \pi_n(2), \dots, \pi_n(n)), n = 2, 3, \dots\}$ is a sequence of random permutations of the numbers 1, 2, ..., n . Let $\tilde{\pi}_n$ have a uniform distribution on the set of all permutations of 1, 2, ..., n and be independent of (X_{nij}) for any n . We define the combinatorial sum S_n by the relation

$$S_n = \sum_{i=1}^n X_{ni\pi_n(i)}.$$

We note that if the distributions of X_{nij} coincide for all $1 \leq j \leq n$ at all n , then S_n has the same distribution as the sum of independent random variables. Although this case is well studied, we should take it into account when estimating the optimality of the obtained results.

Under certain conditions, the sequence of distributions of normalized combinatorial sums weakly converges to the normal law. Any similar result is called a combinatorial central limit theorem (CLT). Research in this direction started long ago. The combinatorial CLT has been studied by Wald and Wolfowitz [1], Noether [2], Hoeffding [3], Motoo [4], Kolchin and Chistyakov [5]. Later on, nonasymptotic bounds of the type of Berry–Esseen and Esseen inequalities were found. Bolthausen [6], von Bahr [7], Ho and Chen [8], Goldstein [9], Neammanee and Suntornchost [10], Neammanee and Rattanawong [11], Chen, Goldstein and Shao [12], Chen and Fang [13], and Frolov [14, 15] obtained similar results. A.N. Frolov obtained the results for a random number of summands in [16].

Bounds in CLT allow the asymptotics of probabilities of large deviations in logarithmic zones to be found. Usually, in this case, they speak of moderate deviations. We obtained such results for combinatorial sums in [17].

In our work [18], we first obtained results on the asymptotic behavior of probabilities of large deviations of combinatorial sums in power zones. In that work, we assumed that random variables satisfy some analogue of Bernstein’s condition. But for separate particular cases, combinatorial sums do not have independent increments. This makes it difficult to use classical methods of summation theory for independent random variables. In [18], we obtained the bounds of the moment-generating function and its logarithmic derivatives of the normalized combinatorial sum rather than of its particular summands. This is what led to those results.

Bernstein's condition is one of the forms of the existence condition for the exponential moment. A natural problem is to obtain new results on the asymptotics of probabilities of large deviations if it is violated. This is what we discuss in this work. We replace Bernstein's condition with the weaker Linnik condition. To prove the results, we use the truncation method.

2. RESULTS

We suppose $\{(X_{nij}), 1 \leq i, j \leq n, n = 2, 3, \dots\}$ is a sequence of matrices of independent random variables such that

$$\sum_{i=1}^n \mathbf{E}X_{nij} = \sum_{j=1}^n \mathbf{E}X_{nij} = 0 \quad (1)$$

for any n . We suppose $\{\bar{\pi}_n = (\pi_n(1), \pi_n(2), \dots, \pi_n(n)), n = 2, 3, \dots\}$ is a sequence of random permutations of the numbers $1, 2, \dots, n$. We assume that $\bar{\pi}_n$ has a uniform distribution on the set of permutations P_n and is independent of (X_{nij}) for any n .

We put

$$S_n = \sum_{i=1}^n X_{ni\pi_n(i)}.$$

It is not difficult to check that

$$\mathbf{E}S_n = 0, \quad \mathbf{D}S_n = \mathbf{E}S_n^2 - (\mathbf{E}S_n)^2 = \frac{1}{n-1} \sum_{i,j=1}^n (\mathbf{E}X_{nij})^2 + \frac{1}{n} \sum_{i,j=1}^n \mathbf{D}X_{nij}.$$

Thus, condition (1) ensures centrality of the combinatorial sums. Replacing $\mathbf{D}X_{nij} = \mathbf{E}X_{nij}^2 - (\mathbf{E}X_{nij})^2$ in the latter formula, we have

$$\mathbf{D}S_n = \frac{1}{n(n-1)} \sum_{i,j=1}^n (\mathbf{E}X_{nij})^2 + \frac{1}{n} \sum_{i,j=1}^n \mathbf{E}X_{nij}^2.$$

If $\mathbf{D}S_n \rightarrow \infty$, the principal part of the variance is the normalized sum of the second moments

$$B_n = \frac{1}{n} \sum_{i,j=1}^n \mathbf{E}X_{nij}^2.$$

Therefore, in what follows, we use $\{B_n\}$ as the norming sequence for S_n .

Further, we assume that the summands have all moments. We put

$$\gamma_n = \max \left\{ \max_{i,j} \frac{\sqrt{n}}{\sqrt{B_n}} \mathbf{E}|X_{nij}|, \max_i \sum_{j=1}^n \frac{\mathbf{E}X_{nij}^2}{B_n}, \max_j \sum_{i=1}^n \frac{\mathbf{E}X_{nij}^2}{B_n}, \sum_{j=1}^n \frac{\mathbf{E}|X_{nij}|^3}{\sqrt{nB_n^{3/2}}} \right\}. \quad (2)$$

We note that $\gamma_n \geq 1$. This follows from the fact that $nB_n = \sum_{i,j=1}^n \mathbf{E}X_{nij}^2 \leq n \max_i \sum_{j=1}^n \mathbf{E}X_{nij}^2$.

The result below was obtained in our work [18].

Theorem 1. *We suppose $\{M_n\}$ is a nondecreasing sequence of positive constants such that for $s = 1, 2, 3$ the inequalities*

$$|\mathbf{E}X_{nij}^k| \leq Dk! M_n^{k-s} \mathbf{E}|X_{nij}|^s \quad (3)$$

hold for all $k \geq s$, all $1 \leq i, j \leq n$, and all $n \geq 2$, where D is the absolute positive constant.

Then, for any sequence of real numbers $\{u_n\}$ such that $u_n \rightarrow \infty$, $u_n^3 = o(\sqrt{n}/\gamma_n)$, and $u_n = o(\sqrt{B_n}/M_n)$ as $n \rightarrow \infty$, the relation

$$\mathbf{P}(S_n \geq u_n \sqrt{B_n}) \sim 1 - \Phi(u_n) \quad \text{as } n \rightarrow \infty \quad (4)$$

holds, where $\Phi(x)$ is the standard normal distribution function.

The condition $u_n^3 = o(\sqrt{n}/\gamma_n)$ is natural for relation (4), which is the exact (not logarithmic) asymptotic of large deviations. If we assume that all X_{nij} are identically distributed, this condition turns into the optimal condition $u_n = o(n^{1/6})$.

Condition (3) is similar to Bernstein’s condition, which is a form of the exponential-moment condition. In [18], we give some variants of this conditions and the examples of random variables that satisfy the hypothesis of Theorem 1. In particular, these include bounded random variables. In the latter case, Spearman’s rank correlation coefficient is an important example of a combinatorial sum.

The theorem below is our principal result. In it, we replace Bernstein’s condition by the weaker Linnik condition, thus expanding the theorem’s application domain. For instance, the Weibull distribution with the parameter α that arises, in particular, as a limit distribution in the theory of extreme order statistics satisfies the Linnik condition and does not satisfy Bernstein’s condition when $\alpha < 1$.

Theorem 2. *We suppose for some $\beta \in (0, 1)$ the inequalities*

$$\mathbf{E}e^{|X_{nij}|^\beta} \leq C\mathbf{E}|X_{nij}|^s \tag{5}$$

hold for $s = 1, 2, 3$, all $1 \leq i, j \leq n$ and all $n \geq 2$, where C is an absolute positive constant.

We suppose $\{u_n\}$ is a sequence of real numbers such that $u_n \rightarrow \infty$, $u_n^3 = o(\sqrt{n}/\gamma_n)$, $u_n = o(B_n^{\beta/(2(2-\beta))})$, and $\ln(nc_n + 1) = o(u_n^\beta B_n^{\beta/2})$ as $n \rightarrow \infty$, where

$$c_n = \frac{1}{n} \max_i \sum_{j=1}^n \varphi_{nij} + \frac{1}{n} \max_j \sum_{i=1}^n \varphi_{nij} + \frac{1}{n^2} \sum_{i,j=1}^n \varphi_{nij},$$

$$\varphi_{nij} = \mathbf{E}e^{|X_{nij}|^\beta} I\{|X_{nij}| \geq u_n \sqrt{B_n}\}.$$

Then, relation (4) holds.

Condition (5) holds if for any i, j , and n the random variable X_{nij} can have one of k given distributions.

Moreover, if there exist positive constants A and B such that $\mathbf{E}e^{|X_{nij}|^\beta} \leq A$ and $\mathbf{E}|X_{nij}|^s \geq B$ for all i, j, n , and s , then condition (5) holds.

If X_{nij} have the same distributions for any i, j , and n , then by Theorem 2 relation (4) holds for $u_n = o(n^\alpha)$, where $\alpha = \min\{1/6, \beta/(2(2-\beta))\}$. From Linnik’s work [19], we know that in this case, for $\beta \in (0, 1/2]$, the Linnik condition is optimal to fulfill for relation (4) in the zone $u_n = o(n^{\beta/(2(2-\beta))})$, while for $\beta > 1/2$ it requires additional conditions. Thus, the hypotheses of Theorem 2 cannot be improved in this case.

Remark 1. Theorem 2 still holds for $\beta = 1$. One should replace c_n by zero.

3. PROOFS

We now prove our results.

Proof of Theorem 2. We put $y_n = u_n \sqrt{B_n}$, $\bar{X}_{nij} = X_{nij} I\{|X_{nij}| < y_n\}$, $\tilde{X}_{nij} = X_{nij} I\{|X_{nij}| \geq y_n\}$, and $T_n = \sum_{i=1}^n \bar{X}_{ni\pi_n(i)}$. We suppose $B_0 = \bigcap_{i=1}^n \{X_{ni\pi_n(i)} = \bar{X}_{ni\pi_n(i)}\}$ and $B_1 = \bigcup_{i=1}^n \{X_{ni\pi_n(i)} \neq \bar{X}_{ni\pi_n(i)}\}$. We have

$$\begin{aligned} \mathbf{P}(S_n \geq y_n) &= \mathbf{P}(\{S_n \geq y_n\} \cap B_0) + \mathbf{P}(\{S_n \geq y_n\} \cap B_1) \leq \mathbf{P}(T_n \geq y_n) + \mathbf{P}(B_1) \\ &\leq \mathbf{P}(T_n \geq y_n) + \sum_{i=1}^n \mathbf{P}(|X_{ni\pi_n(i)}| \geq y_n) = \mathbf{P}(T_n \geq y_n) + \frac{1}{n} \sum_{i,j=1}^n \mathbf{P}(|X_{nij}| \geq y_n). \end{aligned}$$

Similarly, we have

$$\mathbf{P}(T_n \geq y_n) \leq \mathbf{P}(S_n \geq y_n) + \frac{1}{n} \sum_{i,j=1}^n \mathbf{P}(|X_{nij}| \geq y_n).$$

Hence, the inequality

$$|\mathbf{P}(S_n \geq y_n) - \mathbf{P}(T_n \geq y_n)| \leq \frac{1}{n} \sum_{i,j=1}^n \mathbf{P}(|X_{nij}| \geq y_n) \tag{6}$$

holds.

For all $1 \leq i, j \leq n$, we put $\bar{a}_{nij} = \mathbf{E}\bar{X}_{nij}$, $\bar{\sigma}_{nij}^2 = \mathbf{D}\bar{X}_{nij}$, $\bar{a}_{ni.} = \frac{1}{n} \sum_{j=1}^n \bar{a}_{nij}$, $\bar{a}_{n.j} = \frac{1}{n} \sum_{i=1}^n \bar{a}_{nij}$, $\bar{a}_{n..} = \frac{1}{n^2} \sum_{i,j=1}^n \bar{a}_{nij}$, and $\mu_{nij} = \bar{a}_{ni.} + \bar{a}_{n.j} - \bar{a}_{n..}$. We designate $Y_{nij} = \bar{X}_{nij} - \mu_{nij}$, $R_n = \sum_{i=1}^n Y_{ni\pi_n(i)}$, $\bar{B}_n = \frac{1}{n} \sum_{i,j=1}^n \mathbf{E}Y_{nij}^2$.

First, we show that the relation

$$\mathbf{P}\left(R_n \geq u_n \sqrt{\bar{B}_n}\right) \sim 1 - \Phi(u_n) \quad \text{as } n \rightarrow \infty. \tag{7}$$

holds. To do this, we apply Theorem 1 and use the following result to check its hypotheses.

Lemma 1. *We suppose $y > 0$, $\beta \in (0, 1)$, $\mu \in (-1, 1)$, $\alpha > 0$, and $a \in (0, 1)$. Let X be a random variable and $\bar{X} = XI\{|X| < y\}$. We assume that $\mathbf{E}e^{|\bar{X}|^\beta} \leq \alpha \mathbf{E}|\bar{X} - \mu|^s$ for $s = 1, 2, 3$ and $|\mu| \leq M \ln 2$, where $M = y^{1-\beta}/a$.*

Then,

$$\left|\mathbf{E}(\bar{X} - \mu)^k\right| \leq C(a, \beta) \alpha k! M^{k-s} \mathbf{E}|\bar{X} - \mu|^s$$

for all $k > s$ for $s = 1, 2, 3$, where $C(a, \beta) = 8(2 + 3^{1/\beta}((1-a)\beta)^{-1/\beta})$.

Proof. For all z from the circle $|z| \leq ay^{\beta-1} = 1/M$ and $s = 1, 2, 3$, we have

$$\begin{aligned} \left|\mathbf{E}(\bar{X} - \mu)^s e^{z(\bar{X} - \mu)}\right| &\leq \mathbf{E}|\bar{X} - \mu|^s e^{z(|\bar{X}| + |\mu|)} \leq 2^{s-1} e^{z|\mu|} \mathbf{E}\left(|\bar{X}|^s + |\mu|^s\right) e^{z|\bar{X}|} \\ &\leq 8\mathbf{E}\left(|X|^s + 1\right) e^{z|\bar{X}|^\beta y^{1-\beta}} \leq 8\mathbf{E}|X|^s e^{a|X|^\beta} + 8\mathbf{E}e^{a|\bar{X}|^\beta} \\ &\leq 8\mathbf{E}e^{|\bar{X}|^\beta} \sup_{x \geq 0} x^s e^{(a-1)x^\beta} + 8\mathbf{E}e^{|\bar{X}|^\beta} I\{|X| < y\} + 8\mathbf{P}\{|X| \geq y\} \\ &\leq 8\left(\frac{s^{1/\beta}}{((1-a)\beta)^{1/\beta}} + 2\right) \mathbf{E}e^{|\bar{X}|^\beta} \leq C(a, \beta) \alpha \mathbf{E}|\bar{X} - \mu|^s. \end{aligned}$$

By the Cauchy inequalities for the coefficients of expansion of the analytical function $\mathbf{E}(\bar{X} - \mu)^s e^{z(\bar{X} - \mu)}$, we obtain what is required. □

We show that condition (3) is fulfilled for Y_{nij} with $M_n = y_n^{1-\beta}/a$. Given condition (5), it is sufficient to show that $|\mu_{nij}| \leq M_n \ln 2$ and $\mathbf{E}|X_{nij}|^s \leq \alpha \mathbf{E}|\bar{X}_{nij} - \mu_{nij}|^s$ for $s = 1, 2, 3$.

The functions xe^{-x^β} , $x^2e^{-x^\beta}$, and $x^3e^{-x^\beta}$ decrease for $x \geq x_0 > 0$. Further, we take that $y_n > x_0$.

Given condition (10), for all i , we have

$$n|\bar{a}_{ni.}| \leq \left|\sum_{j=1}^n \mathbf{E}\bar{X}_{nij}\right| = \left|\sum_{j=1}^n \mathbf{E}\tilde{X}_{nij}\right| \leq \sum_{j=1}^n \mathbf{E}|X_{nij}| I\{|X_{nij}| \geq y_n\} \leq y_n e^{-y_n^\beta} \sum_{j=1}^n \Phi_{nij}. \tag{8}$$

Similarly, we have

$$n|\bar{a}_{n.j}| \leq y_n e^{-y_n^\beta} \sum_{i=1}^n \Phi_{nij} \quad \text{for all } j \text{ and } n^2|\bar{a}_{n..}| \leq y_n e^{-y_n^\beta} \sum_{i,j=1}^n \Phi_{nij}. \tag{9}$$

Hence,

$$\max_{i,j} |\mu_{nij}| \leq c_n y_n e^{-y_n^\beta} = \varepsilon_n = o(1). \tag{10}$$

Further, for $s = 1, 2, 3$ and all i and j , the inequalities

$$\begin{aligned} \mathbf{E}|\tilde{X}_{nij}|^s &= \mathbf{E}|X_{nij}|^s I\{|X_{nij}| \geq y_n\} \leq y_n^s e^{-y_n^\beta} \varphi_{nij} \\ &\leq y_n^3 e^{-y_n^\beta} C \mathbf{E}|X_{nij}|^s \leq 0.05 \mathbf{E}|X_{nij}|^s, \\ |\mu_{nij}|^s &\leq 0.05 C^{-1} \leq 0.05 C^{-1} \mathbf{E}e^{|X_{nij}|^\beta} \leq 0.05 \mathbf{E}|X_{nij}|^s \end{aligned} \tag{11}$$

hold for all sufficiently large n . Therefore,

$$\mathbf{E}|X_{nij}|^s \leq 3^{s-1} (\mathbf{E}|\bar{X}_{nij} - \mu_{nij}|^s + \mathbf{E}|\tilde{X}_{nij}|^s + |\mu_{nij}|^s) \leq 9 \mathbf{E}|\bar{X}_{nij} - \mu_{nij}|^s + 0.9 \mathbf{E}|X_{nij}|^s.$$

Hence, $\mathbf{E}|X_{nij}|^s \leq 90 \mathbf{E}|\bar{X}_{nij} - \mu_{nij}|^s$. Moreover, $\mathbf{E}|\bar{X}_{nij} - \mu_{nij}|^s \leq 4 (\mathbf{E}|\bar{X}_{nij}|^s + |\mu_{nij}|^s) \leq 5 \mathbf{E}|X_{nij}|^s$. Thus, for $s = 1, 2, 3$ and all i and j , the inequalities

$$0.2 \mathbf{E}|\bar{X}_{nij} - \mu_{nij}|^s \leq \mathbf{E}|X_{nij}|^s \leq 90 \mathbf{E}|\bar{X}_{nij} - \mu_{nij}|^s \tag{12}$$

hold for all rather large n . Hence, by Lemma 1, condition (3) holds for Y_{nij} with $M_n = y_n^{1-\beta}/a$.

Now, we estimate the difference of the variances B_n and \bar{B}_n . We have

$$\begin{aligned} B_n - \bar{B}_n &= \frac{1}{n} \sum_{i,j=1}^n (\mathbf{E}X_{nij}^2 - \mathbf{E}Y_{nij}^2) = \frac{1}{n} \sum_{i,j=1}^n (\mathbf{E}X_{nij}^2 - \mathbf{E}(\bar{X}_{nij} - \mu_{nij})^2) \\ &= \frac{1}{n} \sum_{i,j=1}^n (\mathbf{E}\tilde{X}_{nij}^2 - 2\mu_{nij} \mathbf{E}\bar{X}_{nij} + \mu_{nij}^2) \\ &= \frac{1}{n} \sum_{i,j=1}^n \mathbf{E}\tilde{X}_{nij}^2 - \frac{2}{n} \sum_{i,j=1}^n \mu_{nij} \bar{a}_{nij} + \frac{1}{n} \sum_{i,j=1}^n \mu_{nij}^2. \end{aligned}$$

Further,

$$\begin{aligned} \frac{1}{n} \sum_{i,j=1}^n \mu_{nij} \bar{a}_{nij} &= \frac{1}{n} \sum_{i,j=1}^n \bar{a}_{ni} \bar{a}_{nj} + \frac{1}{n} \sum_{i,j=1}^n \bar{a}_{n,j} \bar{a}_{nij} - \bar{a}_{n..} \frac{1}{n} \sum_{i,j=1}^n \bar{a}_{nij} \\ &= \sum_{i=1}^n \bar{a}_{ni}^2 + \sum_{j=1}^n \bar{a}_{n,j}^2 - n \bar{a}_{n..}^2. \end{aligned}$$

By relations (8)–(11), we have

$$\begin{aligned} \sum_{i=1}^n \bar{a}_{ni}^2 &\leq n \varepsilon_n^2, \quad \sum_{j=1}^n \bar{a}_{n,j}^2 \leq n \varepsilon_n^2, \quad n \bar{a}_{n..}^2 \leq n \varepsilon_n^2, \quad \sum_{i,j=1}^n \mu_{nij}^2 \leq n^2 \varepsilon_n^2, \\ \sum_{i,j=1}^n \mathbf{E}\tilde{X}_{nij}^2 &\leq y_n n^2 \varepsilon_n. \end{aligned} \tag{13}$$

Hence,

$$|B_n - \bar{B}_n| \leq 8 y_n n \varepsilon_n.$$

Moreover,

$$\begin{aligned} y_n n \varepsilon_n (1 - \Phi(u_n))^{-1} &= \exp \left\{ -y_n^\beta + 2 \ln y_n + \ln(nc_n) + \frac{u_n^2}{2} + \ln(\sqrt{2\pi} u_n) \right\} \\ &= \exp \left\{ -y_n^\beta + o(u_n^\beta B_n^{\beta/2}) \right\} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{14}$$

In particular, $B_n \sim \bar{B}_n$. Given inequalities (12), we can conclude that the variable $\bar{\gamma}$ specified by formula (2) with X_{nij} replaced by Y_{nij} satisfies the relation $\gamma_n = O(\bar{\gamma}_n)$ as $n \rightarrow \infty$.

By Theorem 1, relation (7) holds for any sequence of real numbers $\{u_n\}$ such that $u_n \rightarrow \infty$, $u_n^3 = o(\sqrt{n/\bar{\gamma}_n}) = o(\sqrt{n/\gamma_n})$, and $u_n = o(\sqrt{\bar{B}_n/M_n}) = o(B_n^{\beta/(2(2-\beta))})$ as $n \rightarrow \infty$.

Further, we have

$$\mathbf{P}(T_n \geq y_n) = \mathbf{P}(R_n + n\bar{a}_{n..} \geq y_n) = \mathbf{P}(R_n \geq v_n \sqrt{\bar{B}_n}),$$

where

$$v_n = \frac{y_n - n\bar{a}_{n..}}{\sqrt{\bar{B}_n}} = \frac{u_n \sqrt{\bar{B}_n + O(y_n n \varepsilon_n)} + O(n \varepsilon_n)}{\sqrt{\bar{B}_n}} \sim u_n \quad \text{as } n \rightarrow \infty.$$

Since

$$\frac{v_n^2}{2} = \frac{u_n^2 (\bar{B}_n + O(y_n n \varepsilon_n)) + O(u_n \sqrt{\bar{B}_n} n \varepsilon_n)}{2\bar{B}_n} = \frac{u_n^2}{2} + o(1) \quad \text{as } n \rightarrow \infty,$$

we can conclude that $1 - \Phi(v_n) \sim 1 - \Phi(u_n)$ as $n \rightarrow \infty$. Given (7), we obtain

$$\mathbf{P}(T_n \geq y_n) = \mathbf{P}(R_n \geq v_n \sqrt{\bar{B}_n}) \sim 1 - \Phi(u_n) \quad \text{as } n \rightarrow \infty.$$

It follows from inequalities (6) and (13) and relation (14) that

$$\mathbf{P}(S_n \geq y_n) = \mathbf{P}(T_n \geq y_n) + O(y_n n \varepsilon_n) = (1 - \Phi(u_n))(1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

The theorem is proved completely. □

Proof of Remark 1. If $\beta = 1$, then by Lemma 1 with $\bar{X} = X$ and $\mu = 0$ (truncation and centering are not required in this case) condition (3) holds with $M_n = 1/a$. Remark 1 follows from Theorem 1. The conditions on c_n are superfluous in this case. □

ACKNOWLEDGMENTS

I thank the reviewers for a number of useful remarks that helped improve the text of the work.

FUNDING

This work is supported by the Russian Science Foundation (grant no. 23-21-00078). <https://rscf.ru/project/23-21-00078/>

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

REFERENCES

1. A. Wald and J. Wolfowitz, "Statistical tests based on permutations of observations," *Ann. Math. Stat.* **15**, 358–372 (1944).
2. G. E. Noether, "On a theorem by Wald and Wolfowitz," *Ann. Math. Stat.* **20**, 455–458 (1949).
3. W. Hoeffding, "A combinatorial central limit theorem," *Ann. Math. Stat.* **22**, 558–566 (1951).
4. M. Motoo, "On Hoeffding's combinatorial central limit theorem," *Ann. Inst. Stat. Math.* **8**, 145–154 (1957).
5. V. F. Kolchin and V. P. Chistyakov, "On a combinatorial limit theorem," *Theory Probab. Its Appl.* **18**, 728–739 (1973).
<https://doi.org/10.1137/1118093>
6. E. Bolthausen, "An estimate of the remainder in a combinatorial central limit theorem," *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **66**, 379–386 (1984).
7. B. von Bahr, "Remainder term estimate in a combinatorial central limit theorem," *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **35**, 131–139 (1976).
8. S. T. Ho and L. H. Y. Chen, "An Lp bounds for the remainder in a combinatorial central limit theorem," *Ann. Probab.* **6**, 231–249 (1978).
9. L. Goldstein, "Berry–Esseen bounds for combinatorial central limit theorems and pattern occurrences, using zero and size biasing," *J. Appl. Probab.* **42**, 661–683 (2005).

10. K. Neammanee and J. Suntornchost, “A uniform bound on a combinatorial central limit theorem,” *Stoch. Anal. Appl.* **3**, 559–578 (2005).
11. K. Neammanee and P. Rattanawong, “A constant on a uniform bound of a combinatorial central limit theorem,” *J. Math. Res.* **1**, 91–103 (2009).
12. L. H. Y. Chen, L. Goldstein, and Q.-M. Shao, *Normal Approximation by Stein’s Method* (Springer-Verlag, Berlin, 2011).
13. L. H. Y. Chen and X. Fang, “On the error bound in a combinatorial central limit theorem,” *Bernoulli* **21**, 335–359 (2015).
14. A. N. Frolov, “Esseen type bounds of the remainder in a combinatorial CLT,” *J. Stat. Plann. Inference* **149**, 90–97 (2014).
15. A. N. Frolov, “Bounds of the remainder in a combinatorial central limit theorem,” *Stat. Probab. Lett.* **105**, 37–46 (2015).
16. A. N. Frolov, “On the probabilities of moderate deviations for combinatorial sums,” *Vestn. St. Petersburg Univ.: Math.* **48**, 23–28 (2015).
<https://doi.org/10.3103/S1063454115010045>
17. A. N. Frolov, “On Esseen type inequalities for combinatorial random sums,” *Commun. Stat. - Theory Meth.* **46**, 5932–5940 (2017).
18. A. N. Frolov, “On large deviations for combinatorial sums,” *J. Stat. Plann. Inference* **217**, 24–32 (2022).
19. Yu. V. Linnik, “Limit theorems for sums of independent random variables. I,” *Teor. Veroyatn. Ee Primen.* **6**, 145–163 (1961); Yu. V. Linnik, “Limit theorems for sums of independent random variables. II,” *Teor. Veroyatn. Ee Primen.* **6**, 377–391 (1961); Yu. V. Linnik, “Limit theorems for sums of independent random variables. III,” *Teor. Veroyatn. Ee Primen.* **7**, 121–134 (1962).

Translated by M. Talacheva