===== **MECHANICS** =====

Vibrations of a Plate
with Periodically Changing Parameters

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Abstract—Vibrations of a square plate with periodically changing parameters are considered. The averaged fourth-order partial differential equation for plate deflection *w* is presented. Solution of the problem is obtained with the approximate theory. The approximate results are presented by analytical formulas. Asymptotic averaging (implemented in Wolfram Mathematica) and the finite element method (ANSYS) are used to determine the values of eigenfrequencies. Numerical and asymptotic results are compared.

1. INTRODUCTION

Complicated composite materials have become widespread in modern industry. For instance, in reinforcing building materials, polypropylene fiber, which consists of thin synthetic fibers of different size and diameter, is used. Plastic plates reinforced by carbon fiber with continuous current-carrying channels are widely used in electrical engineering. Many roof materials also are a clear example of plates with variable thickness.

Many problems in the study of the vibrations and stability of reinforced plates are solved either by the finite element method in different software complexes (see, e.g., [1]) or by the boundary element method (see, e.g., [2−4]). Asymptotic solutions are obtained only for several special cases of anisotropic plates and shells in works [5−8]. In the present work, in studying the vibrations of a heterogeneous plate, we apply both asymptotical and numerical methods of solution. To check the reliability of the obtained asymptotic formulas, we compare the analytical and numerical results.

In paper [9], the averaged differential equation was obtained for the deflection of a heterogeneous plate reinforced by parallel fiber strips. In this work, we derive the equation of vibrations of a plate with periodically changing parameters (material properties and thickness) and determine the values of the eigenfrequencies.

2. MAIN EQUATIONS AND ASSUMPTIONS

Consider the square plate with length L and variable thickness h. The plate thickness is small compared to its dimensions in plan view $\left(\dfrac{h}{L} < 0.1\right)$. We regard the middle surface as the reference one and introduce the Cartesian coordinate system $Oxyz$, as shown in Fig. 1.

We write the balance equations of forces and moments [9]:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -\lambda w, \quad \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + Q_x = 0, \tag{1}$$
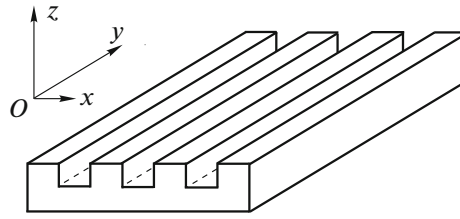
Fig. 1. Plate of variable thickness.

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + Q_y = 0, \quad \lambda = h\rho\omega^2. \tag{2}$$

Here, we have introduced the following designations for bending moments M_x and M_y and torque M_{xy}:

$$M_{xx} = aw_{xx} + bw_{yy}, \quad M_{yy} = bw_{xx} + aw_{yy}, \quad M_{xy} = (a - b)w_{xy}. \tag{3}$$

Value a is called the "stiffness of a unit length of the plate":

$$a = \frac{Eh^3}{12(1 - \mu^2)},$$

where E is the Young modulus, μ is the Poisson coefficient, and $b = \mu \cdot a$.

3. ANALYTICAL SOLUTION TO THE FORMULATED PROBLEM

We apply the method of multiple scales [10, 11]. In addition to variable x, we introduce so-called "quickly varying variable" $\xi = x/\varepsilon$, where ε is the width of the step (width) and each of the unknown functions dependent on variables x and y is formally dependent on variable ξ.

Parameter ε in the case in which the strips have different width is determined by the formula $\varepsilon = \sum_{k=1}^{n} l_k$, where l_k is the dimensionless width (see (24)) of the kth strip.

Note that all values in this work are dimensionless if not stated otherwise. The relation between the dimensional and dimensionless values is introduced in this paper at the point when it is needed.

We represent the asymptotic decomposition for the function describing transverse deflection w as the series

$$w = w_0(x, y) + \sum_{k=1}^{n} (w_k, N_k) \cdot \varepsilon^k, \tag{4}$$

where the parentheses denote the scalar product of vectors and vectors w_k have the form

$$\begin{aligned} w_1 &= (w_{0x}, w_{0y}, 0, 0, 0), \\ w_2 &= (w_{0xx}, w_{0xy}, w_{0yy}, 0, 0). \end{aligned} \tag{5}$$

w_3 is the vector composed of third derivatives of the function w_0 and is not written explicitly, because it is not used below. Vectors N_k are also five-dimensional. Taking into account the chain rule of differentiation

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{1}{\varepsilon}\frac{\partial f}{\partial \xi}, \quad \frac{d^2 f}{dx^2} = \left(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial \xi}\right)^2 \cdot f,$$

we write the following expressions:

$$\begin{aligned} w_{xx} &= \varepsilon^{-1}(w_1, N_{1\xi,\xi}) + (w_2, I_1 + 2N_{1,\xi} + N_{2,\xi\xi}) \\ &\quad + \varepsilon(w_3, N_1 + 2N_{2,\xi} + N_{2,\xi\xi}) + \varepsilon^2(w_4, N_2 + 2N_{3,\xi} + N_{4,\xi\xi}), \\ w_{yy} &= (w_2, I_3) + \varepsilon(w_3, AN_1) + \varepsilon^2(w_4, AN_2) + \dots, \\ w_{xy} &= (w_2, I_2) + \varepsilon(w_3, B(N_{2\xi} + N_1)) + \varepsilon^2(w_4, B(N_2 + N_{3,\xi})) + \dots. \end{aligned} \tag{6}$$

Here, A and B are matrices of 0 and 1, I_k is the five-dimensional unit vector, and other coordinates are zeros.

$$M_{xx} = a\varepsilon^{-1} \cdot (w_1, N_{1,\xi\xi}) + \varepsilon^0 \cdot (w_2, a(I_1 + 2N_{1,\xi} + N_{2,\xi\xi}) + bI_3)$$

$$+ \varepsilon(w_3, (a(N_1 + 2N_{2,\xi} + N_{3,\xi\xi}) + b(w_3, AN_1)) + \varepsilon^2(w_4, (a(N_2 + 2N_{3,\xi} + N_{4,\xi\xi}) + bAN_2)).$$

We write the expansion of Q_x and Q_y in ε:

$$Q_x = -\varepsilon^{-2} \cdot \frac{\partial M_{0xx}}{\partial \xi} - \varepsilon^{-1} \cdot \left(\frac{\partial M_{1xx}}{\partial \xi} + \frac{\partial M_{0xx}}{\partial x} \right) - \varepsilon^0 \cdot \left(\frac{\partial M_{2xx}}{\partial \xi} + \frac{\partial M_{1xx}}{\partial x} + \frac{\partial M_{0xy}}{\partial y} \right),$$

$$Q_y = -\varepsilon^{-1} \cdot \left(\frac{\partial M_{0xy}}{\partial \xi} + \frac{\partial M_{0yy}}{\partial y} \right) - \varepsilon^0 \cdot \left(\frac{\partial M_{1xy}}{\partial \xi} + \frac{\partial M_{1yy}}{\partial y} + \frac{\partial M_{0xy}}{\partial x} \right) - \varepsilon \cdot \left(\frac{\partial M_{2xy}}{\partial \xi} + \frac{\partial M_{2yy}}{\partial y} + \frac{\partial M_{1xy}}{\partial x} \right).$$

We substitute the above-mentioned expressions into Eqs. (1)–(3) and arrive at

$$\text{at } \varepsilon^{-3}: \quad \frac{\partial Q_{0x}}{\partial \xi} = 0, \tag{7}$$

$$\text{at } \varepsilon^{-2}: \quad \frac{\partial Q_{1x}}{\partial \xi} + \frac{\partial Q_{0x}}{\partial x} = 0, \tag{8}$$

$$\text{at } \varepsilon^{-1}: \quad \frac{\partial Q_{2x}}{\partial \xi} + \frac{\partial Q_{1x}}{\partial x} + \frac{\partial Q_{0y}}{\partial y} = 0, \tag{9}$$

$$\text{at } \varepsilon^0: \quad \frac{\partial Q_{3x}}{\partial \xi} + \frac{\partial Q_{2x}}{\partial x} + \frac{\partial Q_{1y}}{\partial y} + \ldots = 0. \tag{10}$$

We assume that N_k are periodic functions; therefore, averaging of Eq. (10) yields

$$\int_0^1 \left(\frac{\partial Q_{2x}}{\partial x} + \frac{\partial Q_{1y}}{\partial y} + \ldots \right) d\xi = 0,$$

and if we take into account that Q_{2x} and Q_{1y}, expressed through the moments, contain the derivatives with respect to ξ, then we obtain the following equation:

$$\int_0^1 \left(\frac{\partial^2 M_{1xx}}{\partial x^2} + \frac{\partial^2 M_{1yy}}{\partial y^2} + 2\frac{\partial^2 M_{0xy}}{\partial x \partial y} + \ldots - q \right) d\xi = 0. \tag{11}$$

To satisfy Eq. (7), we need to set $N_1 = 0$ (which it is not hard to prove). We then write Eq. (8) as

$$\frac{\partial^2 M_{1xx}}{\partial \xi^2} = 0.$$

We denote the expression $a(I_1 + N_{2\xi\xi}) + bI_3 = C$ and, using $\int_0^1 N_{2\xi\xi} d\xi = 0$, finally obtain

$$C = \frac{I_3 \int_0^1 \mu d\xi + I_1 \cdot (l_1 + l_2)}{\int_0^1 a^{-1} d\xi}.$$

Here, l_1 and l_2 are the dimensionless widths of the first and second strips of the plate, respectively. Hence,

$$M_{1xx} = (w_2, a(I_1 + N_{2,\xi\xi}) + bI_3) = (C, w_2), \tag{12}$$

$$M_{1yy} = (w_2, b(I_1 + N_{2,\xi\xi}) + aI_3) = (w_2, b(C - bI_3)/a + aI_3), \tag{13}$$

$$M_{0xy} = (w_2, I_2). \tag{14}$$

Table 1. Properties of materials

Material	Young modulus, E, 10^{11} N/m^2	Density of material, ρ, kg/m^3	Poisson coefficient, μ
Steel	1.93	8030	0.29
Titanium	1.02	4850	0.30

Thus, with allowance for expressions (12)−(14), Eq. (11) becomes

$$Kw_{0xxxx} + Lw_{0xxyy} + Mw_{0yyyy} = -\lambda_0 w. \tag{15}$$

Equation (15) is the fourth-order averaged partial differential equation for plate deflection w. If we write constant vector C as

$$C = C_1 \cdot I_1 + C_3 \cdot I_3,$$

then the coefficients of the averaged equation of vibrations (15) take the following form:

$$K = \int_0^1 C_1 d\xi, \quad L = \int_0^1 \left(\frac{b}{a} \cdot C_1 + C_3 + 2 \cdot (a - b) \right) d\xi, \tag{16}$$

$$M = \int_0^1 \left(\frac{b}{a} \cdot C_3 - \frac{b^2}{a} + a \right) d\xi, \quad \lambda_0 = \int_0^1 \lambda d\xi. \tag{17}$$

Equation (15) taking (16)−(17) into account was solved by the Bubnov−Galerkin method, and the zeroth approximation in the case of fixed boundaries was

$$w_0(x, y) = A \cdot x^2 (x - 1)^2 \cdot y^2 (y - 1)^2, \tag{18}$$

or, in the case of hinged support of the plate boundaries,

$$w_0(x, y) = A \cdot \sin(x \cdot \pi) \sin(y \cdot \pi). \tag{19}$$

Note that expressions (18) and (19) are written for dimensionless variables x and y. The coupling between dimensionless coordinates x and y and dimensional ones $\hat{x}$ and $\hat{y}$ is performed by the formulas $x = \hat{x}/L$ and $y = \hat{y}/L$.

The value for the first eigenfrequency is obtained by multiplying Eq. (15) with the first eigenmode with subsequent integration over the plate area. Finally, the formula for computing the frequency parameter λ becomes

—for the fixed boundaries of the plate,

$$\lambda = (504 \cdot K \cdot b^4 + 144 \cdot L \cdot a^2 \cdot b^2 + 504 \cdot M \cdot a^4)/(a^4 \cdot b^4), \tag{20}$$

—for the hinged support of the boundaries,

$$\lambda = \pi^4 (K \cdot b^4 + L \cdot a^2 \cdot b^2 + M \cdot a^4)/(a^4 \cdot b^4). \tag{21}$$

To transform dimensionless frequency parameter λ to the standard units of frequencies of periodic processes (in hertz), we use the formula

$$\omega = (\lambda/\varrho_{av})^{0.5}/(2 \cdot \pi). \tag{22}$$

In expression (22), we have introduced denotation ϱ_{av}, which is the mean density per unit area. The relation between mean density ϱ_{av} with bulk densities of strips ρ_1 and ρ_2 (kg/m^3) is given in formula (23) obtained by integrating the inertia term with respect to ξ.

$$\varrho_{av} = \rho_1 \cdot h_1 \cdot l_1 + \rho_2 \cdot h_2 \cdot l_2. \tag{23}$$

Dimensionless widths of the strips l_1 and l_2 are related to dimensional widths $\hat{l}_1$ and $\hat{l}_2$ by relation (24) so that $l_1 + l_2 = 1$:

$$l_1 = \frac{\hat{l}_1}{(\hat{l}_1 + \hat{l}_2)}, \quad l_2 = \frac{\hat{l}_2}{(\hat{l}_1 + \hat{l}_2)}. \tag{24}$$
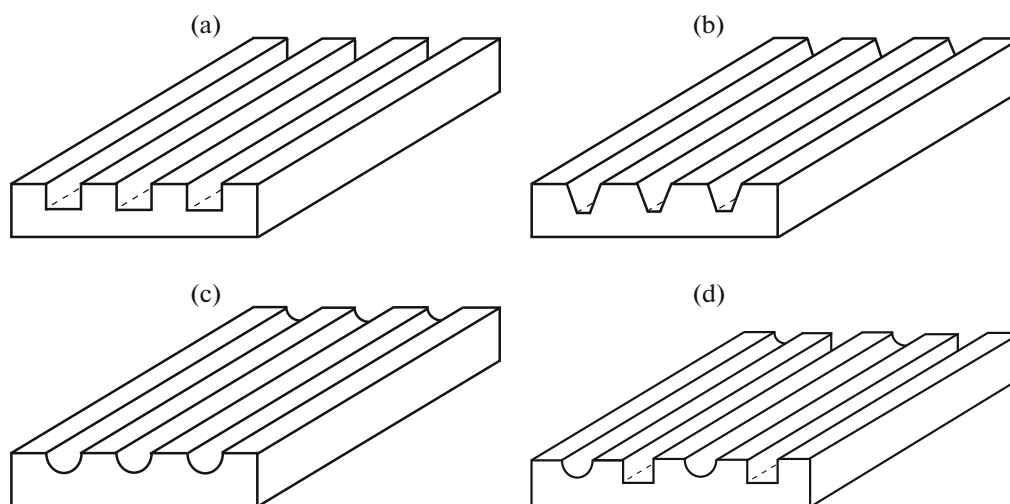
Fig. 2. Variants of reinforced plates.

4. NUMERICAL SOLUTION TO THE FORMULATED PROBLEM

The algorithms and programs developed on the basis of analytical formulas allow calculating different forms of reinforced plates. In particular, we present some variants of such plates in Fig. 2.

In Table 1, we give the values of constant coefficients determining the properties of chosen materials.

To demonstrate the reliability of the obtained formulas and the possibilities of their further use in studying vibrations of reinforced plates, we performed the following example calculations. We considered square plates having the shape depicted in Fig. 2a and consisting of several strips (see the first column of Table 2). In its turn, each strip was composed of two ministrips with different properties of material. Thus, the entire plate is represented as periodically repeating sequence of strips.

In the experiment represented in Table 2, we considered the square plate with a side of 1 m. It was approximated by a model divided into 1764 (42 × 42) shell elements. In the ANSYS 14 software package, we created a mathematical model of a plate heterogeneous over the thickness. The program was written in the APDL language using handbooks [12, 13]. The thickness of the first ministrip was 0.01 m, the thickness of the second one was 0.005 m, and there were a total of seven composite strips.

In the fourth column of Table 2 we present the values of the plate eigenfrequencies obtained by asymptotic formulas (20) and (21). The procedure for substituting expressions (12)−(14) into (15) and the solu-

Table 2. Eigenfrequencies of reinforced plate

| Boundary conditions	Material	Widths of ministrips, m	Asymptotic formulas, Hz (20), (21)	Finite element method, Hz ANSYS
Fixed	Steel	$l_1 = 2/42, l_2 = 4/42$	57.265	56.830
Hinged	Steel	$l_1 = 2/42, l_2 = 4/42$	31.864	32.020
Fixed	Steel	$l_1 = 3/42, l_2 = 3/42$	62.356	61.653
Hinged	Steel	$l_1 = 3/42, l_2 = 3/42$	34.721	34.804
Fixed	Titanium	$l_1 = 2/42, l_2 = 4/42$	53.659	53.236
Hinged	Titanium	$l_1 = 2/42, l_2 = 4/42$	29.834	29.977
Fixed	Titanium	$l_1 = 3/42, l_2 = 3/42$	58.424	57.746
Hinged	Titanium	$l_1 = 3/42, l_2 = 3/42$	32.503	32.577
Fixed	Titanium	$l_1 = 4/42, l_2 = 2/42$	62.821	62.083
Hinged	Titanium	$l_1 = 4/42, l_2 = 2/42$	34.985	34.960
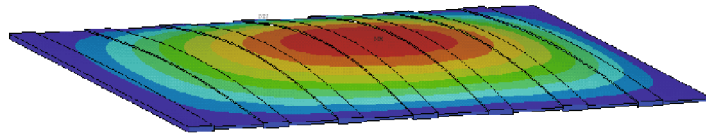
Fig. 3. First eigenmode of inhomogeneous plate.

tion to this equation by the Bubnov–Galerkin method was conducted in Mathematica 8. In the last column, we present the values of eigenfrequencies of the plate obtained by the numerical finite element method in ANSYS [1].

In Fig. 3, we plot the first eigenmode of a plate that is heterogeneous over thickness.

5. CONCLUSIONS

The advantage of the method of averaging over other analytical methods consists in the fact that it allows obtaining the equations of the averaged medium and formulating the problems for it. For instance, for a plate with inserts or periodically varying parameters, the system of complicated differential equations describing the vibrations or stability were replaced with smoothed, averaged equations for the plate only. The developed algorithms and programs based on analytical formulas allow different types of heterogeneous plates to be computed. Analysis of all experiments on studying the vibrations of plates with varying parameters shows reliability of the proposed formulas. In our research, we compared the analytical results and the numerical results obtained by the finite element method using the ANSYS software package. The relative error of calculations is no larger than 3%.

ADDITIONAL INFORMATION

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