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MECHANICS

Applying Adjustment Factors in the Rayleigh Method to Calculate the Principal Frequency of the Vibrations of a Shell with a Rectangular Cross Section

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Abstract—The application of adjustment factors in the Rayleigh method to calculate the principal frequency of the vibrations of a shell with a rectangular cross section is considered in this paper. The behavior patterns of the adjustment factors are generalized. The relationship between the adjustment factors and properties of the approximate formulas is analyzed.

Keywords: vibrations, cylindrical shell, Rayleigh method, adjustment factor.

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INTRODUCTION

Vibrations of a cylindrical shell with a rectangular cross section are considered for various fixing options. In the presence of a square section, an approximate analytical solution can be obtained by numerically solving the equation obtained after separating the variables in the Germain–Lagrange equation [1, 2]. For a hinged shell, when the cross section differs little from the square one, it is possible to obtain a solution in the form of an asymptotic expansion [3]. In the general case, for a rectangular section, in addition to asymptotic expansions and the finite element method, it is advisable to use the Rayleigh method, which makes it possible to obtain an approximate formula for the lowest frequency. In [4], the vibration modes are presented in the form of an expansion in a double Fourier series, and the results of calculations are in good agreement with the results of calculations of frequencies by the finite element method. In [5], to solve this problem, an approach was proposed based on the use of the Rayleigh method with correction coefficients that are chosen empirically. When considering some examples of the application of this approach, it was revealed that, despite the fact that, in the general case, the coordinate functions with correction factors do not satisfy all geometric boundary conditions, they can give a smaller error than functions with correction factors that satisfy these conditions. The purpose of this paper is to further study and generalize the regularities of the behavior of the correction coefficients when they are applied in Rayleigh method.

FORMULATION OF THE PROBLEM

Consider a thin cylindrical shell with a rectangular cross section formed by the conjugation of four rectangular plates (Fig. 1).

Let us introduce local rectangular coordinates (x, y) in the plane of the k-th plate (Fig. 2). We will assume that the deformations in the plane of each plate are negligible, the bending moments at the intersection points are equal, and the angles between the mating plates remain straight during bending. These assumptions are equivalent to the following conditions:

$$\begin{aligned} w^{(k)}(0, y) = w^{(k)}(\chi, y) = 0, \\ w_x^{(k)}(\chi, y) = w_x^{(k+1)}(0, y), \\ w_{xx}^{(k)}(\chi, y) = w_{xx}^{(k+1)}(0, y), \end{aligned} \qquad (1)$$

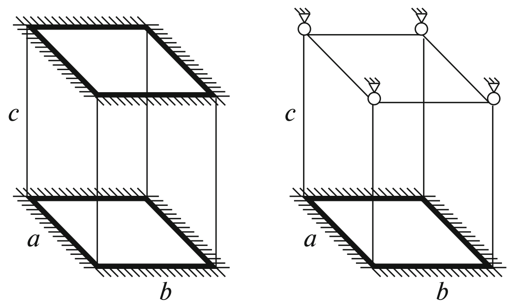

Fig. 1. Cylindrical shell with a rectangular cross section under different boundary conditions (a, b, and c are shell sizes).

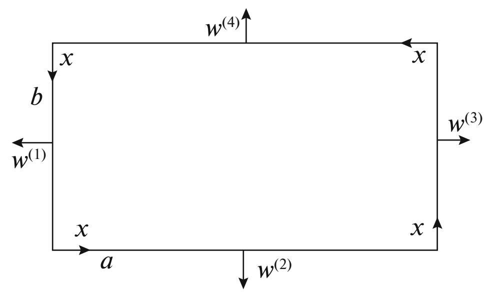

Fig. 2. Local coordinates of the shell's surfaces (view from above).

where $k = \overline{1,4}$ is the plate number, in this case, $k + 1 = 1$ for $k = 4$, $\chi = a$ for even k (or b for odd k); $w^{(k)}(x, y)$ is the deflection of the k-th plate; and $\forall k$ the deflection function satisfies the Lagrange–Germain equation:

$$D\Delta\Delta w^{(k)} - \rho t w^{(k)}\omega^2 = 0, \quad k = \overline{1,4}. \tag{2}$$

Next, the shell frequency is understood as frequency parameter $f = \frac{\omega}{2\pi}\sqrt{\frac{\rho t}{D}}$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $D = \frac{Et^3}{12(1-\nu^2)}$ is the bending stiffness, E is Young's modulus, ν is Poisson's ratio, ρ is the plate mass density, t is the plate thickness, and ω is the natural frequency.

OBTAINING APPROXIMATE SOLUTIONS

As was shown earlier in [3], the plates forming the shell vibrate independently of each other. Taking into account the symmetry, this allows us to consider the vibrations of a plate (a square section) or two mating plates (a rectangular section).

The frequency of a plate (pair of plates) according to Rayleigh method is determined from the ratio

$$f = \frac{1}{2\pi}\sqrt{\frac{\Pi}{T}}, \tag{3}$$

where Π is the potential energy of deformation of the plate (pair of plates) during bending, T is the maximum kinetic energy of the plate (pair of plates), and Π and T depend on the coordinate function characterizing the approximate vibration mode, while it must satisfy the geometric boundary conditions. The closer the form of the coordinate function is to the true waveform, the more accurate the frequency calculation result.

Consider two adjacent shell walls. Let us choose coordinate function W_χ for a wall of width χ. The geometric boundary conditions at the joints of the walls will have the form

$$W_a\big|_{x=0} = W_a\big|_{x=a} = 0, \quad W_b\big|_{x=0} = W_b\big|_{x=b} = 0,$$
$$\left.\frac{\partial W_a}{\partial x}\right|_{x=a} = \left.\frac{\partial W_b}{\partial x}\right|_{x=0}, \quad \left.\frac{\partial W_b}{\partial x}\right|_{x=b} = \left.\frac{\partial W_a}{\partial x}\right|_{x=0}. \tag{4}$$

Then,

$$\Pi = \int_0^c \left(\int_{-\chi_i}^{0} \Pi_{\chi_i} dx + \int_0^{\chi_{i+1}} \Pi_{\chi_{i+1}} dx \right) dy, \tag{5}$$

$$T = \int_0^c \left(\int_{-\chi_i}^{0} T_{\chi_i} dx + \int_0^{\chi_{i+1}} T_{\chi_{i+1}} dx \right) dy, \tag{6}$$

where

$$\Pi_{\chi_i} = D\left(\left(\frac{d^2 W_{\chi_i}}{dx^2} + \frac{d^2 W_{\chi_i}}{dy^2} \right)^2 + 2(1-\nu)\left(\left(\frac{d^2 W_{\chi_i}}{dxdy} \right)^2 - \frac{d^2 W_{\chi_i}}{dx^2}\frac{d^2 W_{\chi_i}}{dy^2} \right) \right), \tag{7}$$

$$T_{\chi_i} = \rho t W_{\chi_i}^2, \tag{8}$$

$$\chi_1 = a, \quad \chi_2 = b. \tag{9}$$

The coordinate function for a wall of width χ can be written in the form

$$W_\chi(\chi, c, x, y) = F_0(\chi) F_1(\chi, x) F_2(c, y), \tag{10}$$

where F_0 is a correction factor and F_1 and F_2 are coordinate functions characterizing vibrations along x and y, respectively. Trigonometric functions or polynomials are conveniently taken as F_1 and F_2.

We will consider the "clamped–clamped" boundary conditions,

$$W_\chi\big|_{y=0} = W_\chi\big|_{y=c} = 0, \quad \left.\frac{\partial W_\chi}{\partial y}\right|_{y=0} = \left.\frac{\partial W_\chi}{\partial y}\right|_{y=c} = 0, \tag{11}$$

and the "clamped–simple support" conditions,

$$W_\chi\big|_{y=0} = W_\chi\big|_{y=c} = 0, \quad \left.\frac{\partial W_\chi}{\partial y}\right|_{y=0} = \left.\frac{\partial^2 W_\chi}{\partial y^2}\right|_{y=c} = 0. \tag{12}$$

Next, we will consider the "clamped–clamped" case, assuming that F_1 and F_2 are trigonometric functions. The reasoning for other variants of functions and boundary conditions can be carried out in a similar way.

We will estimate the accuracy of solutions using relative error $J = \frac{f_R - f_N}{f_N}$, where f_R is the frequency obtained by the Rayleigh method and f_N is the frequency obtained by the finite element method. The calculations were carried out for the following dimensions (change step 1): $a = \overline{1,4}$; $b = \overline{2,4}$; and $c = \overline{2,4}$.

COMPARISON AND ANALYSIS OF SOLUTIONS

Take the coordinate functions

$$W_\chi = \alpha_\chi \sin\frac{\pi x}{\chi}\left(1 - \cos\frac{2\pi y}{c}\right), \tag{13}$$

where $\alpha_\chi = A$ ($\chi = a$), $\alpha_\chi = B$ ($\chi = b$) are some correction factors.

Carrying out calculations by Eqs. (3) and (5)–(8), we obtain that

$$f = \frac{\sqrt{3}\pi}{6}\sqrt{\frac{16A^2a^4b^3 + 8A^2a^2b^3c^2 + 3A^2b^3c^4 + 16B^2a^3b^4 + 8B^2a^3b^2c^2 + 3B^2a^3c^4}{a^3b^3c^4(A^2a + B^2b)}}. \tag{14}$$

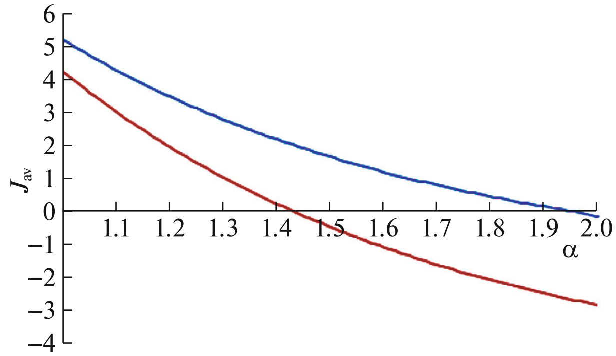

Fig. 3. Change in the average error of the approximate formula depending on α under the boundary conditions "clamped–clamped" (blue line) and "clamped–simple support" (red line).

When $A \gg B$ ($B \gg A$), the frequency according to formula (14) will tend to the frequency of a shell with a square cross section with side a (b). This is also true if we set $A = 0$ or $B = 0$.

In the case of a rectangular section with $A = 1$ and $B = \frac{b}{a}$, functions (13) satisfy all the geometric boundary conditions. The error of this solution varies from 2.1 to 14.9%, reaching a minimum at $a = b$. However, with a different choice of A and B, it is also possible to obtain a more accurate solution. Additional assumptions should be made to clarify the possible choices. We will assume that, in the case of a rectangular section, the values of the derivative of the deflection in the corner when approaching from different sides are not equal to each other, that is, that the angle between the walls ceases to be right.

Based on this, we will require that the coordinate functions satisfy all the geometric boundary conditions at the lower and upper edges, while at the lateral edges of the shell only the conditions for the absence of deflection at the line of intersection of the walls need to be satisfied. As will be shown below, functions satisfying such weakened conditions make it possible to obtain approximate formulas in this problem that are not inferior in accuracy to formulas using coordinate functions that satisfy all geometric boundary conditions.

In particular, correction factors of the form χ^{α}, where $\alpha \geq 1$, ensure that the above assumptions are fulfilled. Substituting them into formula (14), we obtain that

$$f = \frac{\sqrt{3}\pi}{6}\sqrt{\frac{16a^{2\alpha+4}b^3 + 8a^{2\alpha+2}b^3c^2 + 3a^{2\alpha}b^3c^4 + 16a^3b^{2\alpha+4} + 8a^3b^{2\alpha+2}c^2 + 3a^3b^{2\alpha}c^4}{a^3b^3c^4(a^{2\alpha+1} + b^{2\alpha+1})}}. \quad (15)$$

With increasing α, the frequency value calculated by (15) decreases. When $\alpha \gg 1$, it tends to the frequency of a shell with a square cross section, the side of which is equal to the maximum of a and b. The error of this solution is minimal for $a = b$. For fixed a and c, the error in the formulas decreases with increasing b for $a < b$ and increases for $a \geq b$. For fixed a and b, with increasing c, the error in the case of a rectangular section increases. For a square section, the error when varying a increases for $a < c$ and decreases for $a \geq c$.

Based on the behavior of the error, for any predetermined accuracy, correction factors can be selected that ensure this accuracy in the approximate formula. Depending on α, the resulting formula can be both relatively simple and rather cumbersome, which was shown in [5].

For "clamped–simple support" boundary conditions, the results are similar. At $A = B = 1$, the error varies from 0.1 to 15.7%. In this case, the error decreases with increasing α faster than under the "clamped–clamped" conditions (Fig. 3).

It follows from the definition of the Rayleigh method that it always gives an upper bound for the frequency. Therefore, a solution will be considered correct if $f_R > f_N$ for it. It follows from the results obtained that $\forall\alpha\ \exists a_0$, b_0, and c_0 such that $\forall a \leq a_0$, $b \leq b_0$, $c \leq c_0$: $f_R \leq f_N$, which indicates the presence of a relationship between α and the shell size. As α grows, this property first manifests itself for a pronounced rectangular section $\left(\frac{b}{a} \geq 3\right)$; the deviation of the section from the square one, at which the solution ceases to be correct, then decreases. In the considered range of sizes, the solution is everywhere correct for $\alpha \leq 1.37$ ("clamped–clamped") and $\alpha \leq 1.29$ ("clamped–simple support").

It can also be noted that the error in the formula is partly due to the presence of high degrees of a, b, and c in the denominator, and it grows most rapidly when one of the sides of the section tends to zero. With increasing α, the contribution of this error to the formula decreases.

CONCLUSIONS

From the results obtained, we can conclude that the use of correction coefficients χ^α in the Rayleigh method can lead to a more accurate solution, despite the fact that the coordinate functions with such correction coefficients in the general case do not satisfy the condition of equality of angles. The accuracy of the solution is related to the size of the shells, and the closeness of the section to the square plays a role. For large α, the approximate frequency values can be less than their values found by the finite element method in most of the considered size range. At the same time, for small α, a sufficient level of accuracy is provided for applications.

ADDITIONAL INFORMATION

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