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MATHEMATICS

Multi-Pass Stable Periodic Points of Diffeomorphism of a Plane with a Homoclinic Orbit

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Abstract—A diffeomorphism of a plane into itself with a fixed hyperbolic point and a non-transversal point homoclinic to it is studied. There are various ways of touching stable and unstable manifolds at the homoclinic point. Periodic points whose trajectories do not leave the neighborhood of the trajectory of a homoclinic point are divided into many types. Periodic points of the same type are called *n*-pass periodic points if their trajectories have *n* turns lying outside a sufficiently small neighborhood of the hyperbolic point. Diffeomorphisms of the plane with a non-transversal homoclinic point were previously analyzed in the studies of Sh. Newhouse, L.P. Shil'nikov, and B.F. Ivanov, where it was assumed that this point is a tangency point of finite order. In these papers, it was shown that infinite sets of stable two-pass and three-pass periodic points can lie in a neighborhood of a homoclinic point. The presence of such sets depends on the properties of the hyperbolic point. In this paper, we assume that a homoclinic point is not a point with a finite-order tangency of a stable and an unstable manifold. It is shown that, for any fixed natural number *n*, the neighborhood of a non-transversal homoclinic point can contain an infinite set of stable *n*-pass periodic points with characteristic exponents bounded away from zero.

Keywords: diffeomorphism, non-transversal homoclinic point, stability, characteristic exponents.
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1. INTRODUCTION

In the paper, we study diffeomorphism of a plane into itself with a fixed hyperbolic point and a non-transversal point homoclinic to it. It is assumed that the homoclinic point is not a point of finite-order tangency. The main goal of this study is to show that, under certain conditions, an arbitrarily small neighborhood of a homoclinic point contains an infinite set of stable multi-pass periodic points with characteristic exponents bounded away from zero. In [1], an example is given for a two-dimensional diffeomorphism with an infinite set of periodic points whose trajectories lie in a bounded set of the plane, and their characteristic exponents are bounded away from zero. It is known that in this case, under small perturbations, the diffeomorphism remains to have arbitrarily many stable periodic points with characteristic exponents bounded away from zero.

The point that lies at the intersection of stable and unstable manifolds of a hyperbolic point is called a homoclinic point. If these manifolds are tangent to each other at the homoclinic point, it is called a non-transversal homoclinic point. A periodic point whose trajectory does not leave the neighborhood of the trajectory of the homoclinic point but has n ($n > 3$) turns lying outside a sufficiently small neighborhood of the hyperbolic point is called a multi-pass or n-pass periodic point.

In [2–6], a neighborhood of a non-transversal homoclinic point was studied. The homoclinic point was assumed to be a point with a finite-order tangency of stable and unstable manifolds.

Let f be self-diffeomorphism of the plane with a hyperbolic fixed point at the origin and a non-transversal point homoclinic to it, and let λ and μ be the eigenvalues of matrix $Df(0)$, where $0 < \lambda < 1 < \mu$. Let

$$\theta = -\frac{\ln\lambda}{\ln\mu}.$$

In [2–6], it was assumed that $\theta > 1$ and it was shown that there exists an unbounded set Θ such that, for $\theta \in \Theta$, a neighborhood of a non-transversal homoclinic point contains infinite sets of two-pass or three-pass stable periodic points; however, it follows from [3] that there exists unbounded set Θ_1 such that, for

$\theta \in \Theta_1$, a neighborhood of a non-transversal homoclinic point contains no stable two-pass periodic points.

The present paper is a continuation of [7, 8]. In these papers, it is assumed that a non-transversal homoclinic point is not a point of finite-order tangency of stable and unstable manifolds. Sufficient conditions are given under which infinite sets of single-pass or two-pass stable periodic points with characteristic exponents bounded away from zero exist in a neighborhood of the homoclinic point. In this paper, we give sufficient conditions in order that, for any $\theta > n$, where n is a natural number ($n > 3$), the neighborhood of the homoclinic point contains an infinite set of n-pass stable periodic points with characteristic exponents bounded away from zero.

2. BASIC DEFINITIONS AND NOTATION

Let f be a C^1 self-diffeomorphism of the plane with a hyperbolic fixed point at the origin: $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(0) = 0$. Suppose that

$$Df(0) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

where $0 < \lambda < 1 < \mu$.

Fix a natural number $n \geq 4$. Suppose that

$$\theta = -\frac{\ln\lambda}{\ln\mu} > n. \tag{1}$$

As usual, let $W^s(0)$ and $W^u(0)$ be stable and unstable manifolds of the hyperbolic point. It is known that

$$W^s(0) = \left\{z \in \mathbb{R}^2 : \lim_{k\to+\infty} \left\|f^k(z)\right\| = 0\right\},$$

$$W^u(0) = \left\{z \in \mathbb{R}^2 : \lim_{k\to+\infty} \left\|f^{-k}(z)\right\| = 0\right\},$$

where f^k and f^{-k} are the degrees of the diffeomorphisms f and f^{-1}.

Let point w be such that $w \neq 0$, $w \in W^s(0) \cap W^u(0)$; this point is a homoclinic point. Suppose that $W^s(0)$ and $W^u(0)$ are tangent to each other at point w; then, w is a non-transversal homoclinic point. It is clear that

$$\lim_{k\to+\infty} \left\|f^k(w)\right\| = \lim_{k\to+\infty} \left\|f^{-k}(w)\right\| = 0.$$

Assume that f has the following form in some bounded neighborhood V_0 of the origin

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \mu y \end{pmatrix} \tag{2}$$

for $(x, y) \in V_0$.

Let $w_1 = (0, y^0)$ and $w_2 = (x^0, 0)$ be two points of the orbit of homoclinic point w such that $w_1 \in V_0$ and $w_2 \in V_0$. Suppose that, for certain $\bar\lambda$ and $\bar\mu$ such that $\lambda < \bar\lambda < 1$ and $1 < \bar\mu < \mu$, we have inclusion

$$V = \{(x, y) : |x| \leq \bar\lambda^{-1}\left|x^0\right|, |y| \leq \bar\mu\left|y^0\right|\} \subset V_0. \tag{3}$$

Assume that

$$x^0 > 0, \quad y^0 > 0. \tag{4}$$

It follows from the definition of a homoclinic point that there is natural number $\omega > 1$ such that $f^\omega(w_1) = w_2$. We assume that $f^k(w_1) \notin V$, $k = 1, 2, \ldots, \omega - 1$.

Let U be a neighborhood of point w_1 such that $U \subset V$, $f^\omega(U) \subset V$, $f^k(U) \cap V = \emptyset$, $k = 1, 2, \ldots, \omega - 1$, and sets U, $f(U)$, ..., $f^\omega(U)$ do not intersect pairwise.

We call

$$U^0 = V \bigcup f(U) \bigcup \ldots \bigcup f^{\omega-1}(U)$$

the extended neighborhood of the homoclinic point.

Periodic point $u \in U$ is called an n-pass periodic point if its trajectory is located in U^0 and the intersection of its orbit with U consists of n different points.

We denote restriction $f^{\omega}|_U$ by L. It is clear that L is a mapping of class C^1. We write mapping L in coordinates

$$L(x, y) = \begin{pmatrix} x^0 + F_1(x, y - y^0) \\ F_2(x, y - y^0) \end{pmatrix},$$

where $F_1(x, y - y^0)$ and $F_2(x, y - y^0)$ are C^1-functions defined in U such that $F_1(0, 0) = F_2(0, 0) = 0$.

Point w_2 is called a homoclinic point of finite-order tangency if there exists natural number $l > 1$ such that

$$\frac{\partial F_2(0,0)}{\partial y} = \ldots = \frac{\partial^{l-1} F_2(0,0)}{\partial y^{l-1}} = 0, \quad \frac{\partial^{l} F_2(0,0)}{\partial y^{l}} \neq 0. \tag{5}$$

It was assumed in [2–6] that a homoclinic point is a point with a finite order of tangency. In this paper, as in [7, 8], a different way of touching stable and unstable manifolds is studied.

3. FORMULATION OF THEOREMS

Let f be a C^1-self-diffeomorphism of the plane with a hyperbolic fixed point at the origin and a nontransversal point homoclinic to it. Suppose that conditions (2) are satisfied in some bounded neighborhood of the origin. Let $w_1 = (0, y^0)$ and $w_2 = (x^0, 0)$ be two points of the orbit of homoclinic point w such that conditions (3) and (4) hold and extended neighborhood U^0 is defined. Mapping $L = f^{\omega}|_U$ is defined in neighborhood U of point w_1.

Assume that the coordinate functions of mapping L has form

$$\begin{aligned} F_1(x, y - y^0) &= a(y - y^0) + x\varphi_1(x, y - y^0), \\ F_2(x, y - y^0) &= bx + g(y - y^0) + x\varphi_2(x, y - y^0), \end{aligned} \tag{6}$$

where a and b are real numbers such that

$$a < 0, \quad b > 0, \tag{7}$$

and functions g, φ_1, and φ_2 are such that $\varphi_1(0, 0) = \varphi_2(0, 0) = 0$ and $g(0) = \frac{dg(0)}{dy} = 0$. It is clear that φ_1, φ_2, and their first-order derivatives are bounded in U, while function g and its derivative are bounded in a neighborhood of zero.

The nature of tangency of stable and unstable manifolds at point w_2 is specified by the properties of function g. Let us describe the properties of this function using sequences. Let $\sigma = \sigma(k)$ and $\varepsilon = \varepsilon(k)$ be positive vanishing sequences such that, for any k,

$$\sigma(k - 1) - \varepsilon(k - 1) - \sigma(k) - \varepsilon(k) > 0. \tag{8}$$

Let $i_0 = i_0(k)$ be an increasing sequence of natural numbers for which there exists natural number $s \geq n$ such that

$$i_0(k) - i_0(k - 1) > s \tag{9}$$

for any k.

Suppose that, for any k,

$$(\lambda\mu^n)^{i_0(k)} < \varepsilon(k). \tag{10}$$

Let function g be such that there exist sequences with the above properties such that, for any k,

$$g(\sigma(k)) = (y^0 + \Delta(k))\mu^{-i_0(k)}, \tag{11}$$

where $\Delta = \Delta(k)$ is a vanishing sequence of real numbers.

Suppose that there exists real number $\alpha > 1$ such that, for any k, inequality

$$\left|\frac{dg(t)}{dt}\right| < \mu^{-\alpha n i_0(k)} \tag{12}$$

holds for $t \in (\sigma(k) - \varepsilon(k), \sigma(k) + \varepsilon(k))$.

It follows from (11) and (12) that point w_2 is not a point of tangency of finite order. Obviously, conditions (5) are not satisfied in this case.

Let $i_1 = i_1(k)$, $i_2 = i_2(k)$, ..., $i_{n-1} = i_{n-1}(k)$ be increasing sequences of natural numbers such that

$$2 \le i_0(k) - i_m(k) \le s - 1, \quad |i_m(k) - i_l(k)| \ge 1, \tag{13}$$

where $m = 1, 2, ..., n - 1$, $l = 1, 2, ..., n - 1$, and $m \ne l$.

Suppose that function g be such that there exists set of sequences $i_1 = i_1(k)$, $i_2 = i_2(k)$, ..., $i_{n-1} = i_{n-1}(k)$ satisfying conditions (13) such that, for any k, the following system of equations has a solution:

$$\begin{cases} g(\xi_2) - \mu^{-i_2(k)}\xi_3 + a^2b^2\lambda^{i_1(k)+i_{n-1}(k)}\mu^{i_0(k)}\xi_{n-1} \\ = \mu^{-i_2(k)}y^0 - b\lambda^{i_1(k)}(x^0 + a\Delta(k)) - ab^2\lambda^{i_1(k)+i_{n-1}(k)}\mu^{i_0(k)}x^0, \\ g(\xi_3) - \mu^{-i_3(k)}\xi_4 + ab\lambda^{i_2(k)}\xi_2(k) = \mu^{-i_3(k)}y^0 - b\lambda^{i_2(k)}x^0, \\ \dots \\ g(\xi_{n-2}) - \mu^{-i_{n-2}(k)}\xi_{n-1} + ab\lambda^{i_{n-3}(k)}\xi_{n-3} = \mu^{-i_{n-2}(k)}y^0 - b\lambda^{i_{n-3}(k)}x^0, \\ g(\xi_{n-1}) + ab\lambda^{i_{n-2}(k)}\xi_{n-2} = \mu^{-i_{n-1}(k)}(y^0 + \sigma(k)) - b\lambda^{i_{n-2}(k)}x^0. \end{cases} \tag{14}$$

Let $\xi_2 = \xi_2(k)$, $\xi_3 = \xi_3(k)$, ..., $\xi_{n-1} = \xi_{n-1}(k)$ be a solution of system (14). Sequence $\xi_1 = \xi_1(k)$ is defined as follows

$$\xi_1(k) = \Delta(k) + \mu^{i_0(k)}\lambda^{i_{n-1}(k)}b(x^0 + a\xi_{n-1}(k)).$$

Theorem 1. *Let f be a self-diffeomorphism of the plane with a fixed hyperbolic point at the origin and a non-transversal point homoclinic to it. Let conditions* (1)–(4), (6)–(12) *hold. Suppose that that there exists a set of increasing sequences of natural numbers $i_1 = i_1(k)$, $i_2 = i_2(k)$, ..., $i_{n-1} = i_{n-1}(k)$ satisfying conditions* (13) *such that system of equations* (14) *is solvable. Assume that the following inequalities hold for any k*:

$$\left|g(\xi_1(k)) - (y^0 + \xi_2(k))\mu^{-i_1(k)} + \lambda^{i_0(k)}b(x^0 + a\sigma(k))\right| < \varepsilon(k)\mu^{-(n-1)i_0(k)}. \tag{15}$$

Then, any neighborhood of homoclinic point w_1 contains a countable set of n-pass stable periodic points the characteristic exponents of which are bounded away from zero.

Note that in order to meet conditions (15), it is necessary that $\xi_1(k) \notin (\sigma(k) - \varepsilon(k), \sigma(k) + \varepsilon(k))$. Otherwise, inequalities (15) contradict (11) and (12).

Since $g(t)$ is a C^1-function of one variable defined in a neighborhood of zero that satisfies (11) and (12), there exist sequences $\tau_1 = \tau_1(k)$ and $\tau_2 = \tau_2(k)$ such that $(\tau_1(k), \tau_2(k)) \subset (\sigma(k) + \varepsilon(k), \sigma(k-1) - \varepsilon(k-1))$, and

$$\frac{dg(t)}{dt} > \mu^{-(\theta+1)(n-2)^{-1}i_0(k)} \tag{16}$$

for $t \in (\tau_1(k), \tau_2(k))$.

Theorem 2. *Let $g(t)$ be a C^1-function of one variable defined in a neighborhood of zero and satisfying* (8)–(12). *Suppose that, for any k, inclusions*

$$[y^0\mu^{-i_0(k)+1}, y^0\mu^{-i_0(k)+s}] \subset (g(\tau_1(k)), g(\tau_2(k))), \tag{17}$$

hold, where sequences $\tau_1(k)$ and $\tau_2(k)$ are defined by conditions (16). *Then, for any set of increasing natural sequences $i_1 = i_1(k)$, $i_2 = i_2(k)$, ..., $i_{n-1} = i_{n-1}(k)$ satisfying conditions* (13), *system of equations* (14) *is solvable.*

4. PROOF OF THEOREM 1

Let k be a sufficiently large natural number. Let σ, ε, Δ, i_0, i_1, ..., i_{n-1}, ξ_1, ξ_2, ..., ξ_{n-1} be elements of the corresponding sequences with sufficiently large k. In the proof of the theorem, index k of the sequences is omitted.

For any k, we define $x_0 = \lambda^{i_{n-1}}(x^0 + a\xi_{n-1})$, $x_1 = \lambda^{i_0}(x^0 + a\sigma)$, and $x_m = \lambda^{i_{m-1}}(x^0 + a\xi_{m-1})$, where $m = 2, 3, ..., n-1$. Let us define sets

$$U_0 = \{|x - x_0| \le \lambda^{i_0}(|a| + 1)\varepsilon, \left|y - (y^0 + \sigma)\right| \le \varepsilon\},$$

$$U_m = \{|x - x_m| \le \lambda^{i_0}(|a| + 1)\varepsilon, \left|y - (y^0 + \xi_m)\right| \le \varepsilon\mu^{-(i_m + i_{m+1} + \ldots + i_{n-1})}\},$$

where $m = 1, 2, ..., n-1$. We consider that $U_m \subset U$ for $m = 0, 1, ..., n-1$.

We show that inclusions $f^{i_m}L(U_m) \subset U_{m+1}$ (where $m = 0, 1, ..., n-2$) and $f^{i_{n-1}}L(U_{n-1}) \subset U_0$ hold.

Let $(x, y) \in U_0$. Clearly, $x = x_0 + u_0$, $y = y^0 + \sigma + v_0$, where $|u_0| \le \lambda^{i_0}(|a| + 1)\varepsilon$ and $|v_0| \le \varepsilon$. Define

$$\begin{pmatrix} \bar{x}_0 \\ \bar{y}_0 \end{pmatrix} = f^{i_0}L\begin{pmatrix} x \\ y \end{pmatrix}.$$

From conditions (2) and (6), we have

$$\bar{x}_0 = \lambda^{i_0}[x^0 + a(\sigma + v_0) + (x_0 + u_0)\varphi_1(x_0 + u_0, \sigma + v_0)],$$

$$\bar{y}_0 = \mu^{i_0}[b(x_0 + u_0) + g(\sigma) + g(\sigma + v_0) - g(\sigma) + (x_0 + u_0)\varphi_2(x_0 + u_0, \sigma + v_0)],$$

whence, in view of conditions (11), we obtain

$$|\bar{x}_0 - x_1| \le \lambda^{i_0}[|a|\varepsilon + (|x_0| + \lambda^{i_0}(|a| + 1)\varepsilon)|\varphi_1(x_0 + u_0, \sigma + v_0)|],$$

$$\left|\bar{y}_0 - (y^0 + \xi_1)\right| \le \mu^{i_0}[b\lambda^{i_0}(|a| + 1)\varepsilon + |g(\sigma + v_0) - g(\sigma)| + (|x_0| + \lambda^{i_0}(|a| + 1)\varepsilon)|\varphi_2(x_0 + u, \sigma + v)|].$$

From conditions (12), we have

$$|g(\sigma + v_0) - g(\sigma)| \le \varepsilon\mu^{-n\alpha i_0},$$

from which, taking into account the properties of functions φ_1 and φ_2 and conditions (10), we get

$$|\bar{x}_0 - x_1| \le \lambda^{i_0}(|a| + 1)\varepsilon,$$

$$\left|\bar{y}_0 - (y^0 + \xi_1)\right| \le \varepsilon\mu^{-(i_1 + i_2 + \ldots + i_{n-1})}.$$

Inclusion $f^{i_0}L(U_0) \subset U_1$ is proven.

Let $(x, y) \in U_1$. Clearly, $x = x_1 + u_1$, $y = y^0 + \xi_1 + v_1$, where $|u_1| \le \lambda^{i_0}(|a| + 1)\varepsilon$ and $|v_1| \le \varepsilon\mu^{-(i_1 + i_2 + \ldots + i_{n-1})}$. Define

$$\begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix} = f^{i_1}L\begin{pmatrix} x \\ y \end{pmatrix}.$$

From conditions (2) and (6), we obtain

$$\bar{x}_1 = \lambda^{i_1}[x^0 + a(\xi_1 + v_1) + (x_1 + u_1)\varphi_1(x_1 + u_1, \xi_1 + v_1)],$$

$$\bar{y}_1 = \mu^{i_1}[b(x_1 + u_1) + g(\xi_1) + g(\xi_1 + v_1) - g(\xi_1) + (x_1 + u_1)\varphi_2(x_1 + u_1, \xi_1 + v_1)].$$

From conditions (15), we have

$$|\bar{x}_1 - x_2| \le \lambda^{i_1}[|a|\varepsilon\mu^{-(i_1 + i_2 + \ldots + i_{n-1})} + (|x_1| + \lambda^{i_0}(|a| + 1)\varepsilon)|\varphi_1(x_1 + u_1, \xi_1 + v_1)|],$$

$$\left|\bar{y}_1 - (y^0 + \xi_2)\right| \le \mu^{i_1}[b\lambda^{i_0}(|a| + 1)\varepsilon + |g(\xi_1 - v_1) - g(\xi_1)| + \varepsilon\mu^{-i_0(n-1)}]$$

$$+ \mu^{i_1}(|x_1| + \lambda^{i_0}(|a| + 1)\varepsilon)|\varphi_2(x_1 + u_1, \xi_1 + v_1)|.$$

It follows from the properties of functions g that

$$|g(\xi_1 + v_1) - g(\xi_1)| \le 0.5\varepsilon\mu^{-(i_1+i_2+\ldots+i_{n-1})};$$

hence, we obtain

$$|\bar{x}_1 - x_1| \le \lambda^{i_0}(|a| + 1)\varepsilon,$$

$$|\bar{y}_1 - (y^0 + \xi_1)| \le \varepsilon\mu^{-(i_2+\ldots+i_{n-1})}.$$

Inclusion $f^{i_1}L(U_1) \subset U_2$ is proven.

Inclusions $f^{i_m}L(U_m) \subset U_{m+1}$, where $m = 2, 3, \ldots, n - 1$, and $f^{i_{n-1}}L(U_{n-1}) \subset U_0$ can be proven in a similar way taking conditions (14) into account; consequently, we obtain $f^{i_{n-1}}L\ldots f^{i_0}L(U_0) \subset U_0$.

It follows from the previous reasoning that there exists point $z_0 \in U_0$, $z_0 = (x_0^*, y_0^*)$ such that $f^{i_{n-1}}L\ldots f^{i_0}L(z_0) = z_0$ and U_0 contains a periodic point of the original diffeomorphism. Let $z_m = (x_m^*, y_m^*)$, $z_m \in U_m$ be points from the orbit of the periodic point such that $z_m = f^{i_{m-1}}L\ldots f^{i_0}L(z_0)$, $m = 1, 2, \ldots, n - 1$.

Define

$$\Xi = Df^{i_{n-1}}L\ldots f^{i_0}L(z_0) = Df^{i_{n-1}}L(z_{n-1})\ldots Df^{i_0}L(z_0).$$

In order to prove the stability of points z_0, we estimate the eigenvalues of this matrix. It is clear that

$$Df^{i_m}L(z_m) = \begin{pmatrix} \lambda^{i_m}\dfrac{\partial x\varphi_1(x, y - y^0)}{\partial x} & \lambda^{i_m}\left(a + \dfrac{\partial x\varphi_1(x, y - y^0)}{\partial y}\right) \\ \mu^{i_m}\left(b + \dfrac{\partial x\varphi_2(x, y - y^0)}{\partial x}\right) & \mu^{i_m}\left(\dfrac{dg(y - y^0)}{dy} + \dfrac{\partial x\varphi_2(x, y - y^0)}{\partial y}\right)\end{pmatrix}_{\substack{x = x_m^* \\ y = y_m^*}},$$

where $m = 0, 1, 2, \ldots, n - 1$.

It is easy to see that

$$\operatorname{Det}\Xi = (-ab)^n(\lambda\mu)^{ni_0}A, \tag{18}$$

where quantity A depends on k but is bounded for all k.

We introduce notation

$$\varphi_2^{(m)} = \varphi_2(x_m^*, y_m^* - y^0),$$

$$g_m = \frac{dg(y_m^* - y^0)}{dy},$$

where $m = 0, 1, 2, \ldots, n - 1$. The following equalities define matrices Ψ and Φ_m for $m = 0, 1, 2, \ldots, n - 1$:

$$(\lambda\mu)^{i_0}\Phi_m = Df^{i_m}L(z_m) - \begin{pmatrix} 0 & \lambda^{i_m}a \\ \mu^{i_m}(b + \varphi_2^{(m)}) & \mu^{i_m}g_m \end{pmatrix},$$

$$(\lambda\mu^n)^{i_0}\Psi = \Xi - \prod_{m=0}^{n-1}\begin{pmatrix} 0 & \lambda^{i_{n-1-m}}a \\ \mu^{i_{n-1-m}}(b + \varphi_2^{(n-1-m)}) & \mu^{i_{n-1-m}}g_{n-1-m}\end{pmatrix}.$$

Elements of the matrices Ψ and Φ_m, where $m = 0, 1, 2, \ldots, n - 1$, depend on k but is bounded for all k.

It is easy to show by induction on l for $l = 1, 2, \ldots, n - 1$ that

$$\prod_{m=0}^{l}\begin{pmatrix} 0 & \lambda^{i_{l-m}}a \\ \mu^{i_{l-m}}(b + \varphi_2^{(l-m)}) & \mu^{i_{l-m}}g_{l-m}\end{pmatrix} = \begin{pmatrix} \lambda^{i_0}\mu^{li_0}\beta_{11}(l) & \lambda^{i_0}\mu^{li_0}\beta_{12}(l) \\ \mu^{(l+1)i_0}\beta_{21}(l) & \lambda^{i_0}\mu^{li_0}\beta_{22}(l) + \mu^{i_0+i_1+\ldots+i_l}g_0g_1\ldots g_l\end{pmatrix},$$

where $\beta_{11}(l)$, $\beta_{12}(l)$, $\beta_{21}(l)$, $\beta_{22}(l)$, $l = 1, 2, \ldots, n - 1$, depend on k but is bounded for all k.

From (12) we have $|g_0| < \mu^{-\alpha n i_0}$; consequently, there exist a quantity B independent of k such that

$$\mathrm{Tr}\,\Xi \le B\mu^{-\gamma i_0},$$

where $\mathrm{Tr}\Xi$ is the trace of the matrix Ξ and $\gamma = \min[\theta - n, n(\alpha - 1)]$.

Let ρ_1 and ρ_2 be the eigenvalues of the matrix Ξ. It is known that

$$\rho_1\rho_2 = \mathrm{Det}\,\Xi,$$

$$\rho_1 + \rho_2 = \mathrm{Tr}\,\Xi.$$

Let us show that there exist $C > 0$ and k_0 such that, for any $k > k_0$ and $m = 1, 2$, we have the inequalities

$$|\rho_m| \le C\mu^{-\gamma i_0}. \tag{19}$$

Suppose that (19) do not hold, then, for any $C > B$, there exists a sequence of indices k such that

$$|\rho_1| > C\mu^{-\gamma i_0};$$

therefore,

$$|\rho_2| \ge |\rho_1| - |\mathrm{Tr}\,\Xi| > (C - B)\mu^{-\gamma i_0},$$

hence,

$$|\mathrm{Det}\,\Xi| \ge C(C - B)\mu^{-2\gamma i_0}.$$

The last inequality contradicts equalities (18). Inequalities (19) are proven.

Characteristic exponents ν_m, $m = 1, 2$, of periodic points $z_0 \in U_0$ are defined as follows:

$$\nu_m = (i_0 + i_1 + \ldots + i_{n-1} + n\omega)^{-1}\ln|\rho_m|,$$

where $m = 1, 2$.

It follows from conditions (13), (19) that, for sufficiently large k,

$$\nu_m \le -\gamma \ln \mu (2n)^{-1},$$

where $m = 1, 2$.

The last inequalities prove Theorem 1.

5. PROOF OF THEOREM 2

Let k be a sufficiently large natural number. In the proof of the theorem, the index k of the sequences $\sigma, \varepsilon, \Delta, i_0, i_1, \ldots, i_{n-1}, \tau_1$, and τ_2 is omitted. It is assumed that $\xi_2, \xi_3, \ldots, \xi_{n-1}$ are real variables.

Let $P = \{(\xi_2, \xi_3, \ldots, \xi_{n-1}), \xi_m \in (\tau_1, \tau_2), m = 2, 3, \ldots, n-1\}$. On these sets, for any k, we define mapping $G: P \to \mathbb{R}^{n-2}$, or in coordinate form

$$G(\xi_2, \xi_3, \ldots, \xi_{n-1}) = \begin{pmatrix} G_2(\xi_2, \xi_3, \ldots, \xi_{n-1}) \\ G_3(\xi_2, \xi_3, \ldots, \xi_{n-1}) \\ \ldots \\ G_{n-1}(\xi_2, \xi_3, \ldots, \xi_{n-1}) \end{pmatrix},$$

where

$$G_2(\xi_2, \xi_3, \ldots, \xi_{n-1}) = g(\xi_2) - \mu^{-i_2}\xi_3 + a^2b^2\lambda^{i_1 + i_{n-1}}\mu^{i_0}\xi_{n-1},$$

$$G_3(\xi_2, \xi_3, \ldots, \xi_{n-1}) = g(\xi_3) - \mu^{-i_3}\xi_4 + ab\lambda^{i_2}\xi_2,$$

$$\ldots$$

$$G_{n-2}(\xi_2, \xi_3, \ldots, \xi_{n-1}) = g(\xi_{n-2}) - \mu^{-i_{n-2}}\xi_{n-1} + ab\lambda^{i_{n-3}}\xi_{n-3},$$

$$G_{n-1}(\xi_2, \xi_3, \ldots, \xi_{n-1}) = g(\xi_{n-1}) + ab\lambda^{i_{n-2}}\xi_{n-2}.$$

Let H_{n-2} be the determinant of order $n-2$ of form

$$H_{n-2} = \mathrm{Det}\begin{pmatrix} \frac{dg(\xi_2)}{d\xi_2} & -\mu^{-i_2} & 0 & \dots & 0 & 0 \\ ab\lambda^{i_2} & \frac{dg(\xi_3)}{d\xi_3} & -\mu^{-i_3} & \dots & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & \frac{dg(\xi_{n-2})}{d\xi_{n-2}} & -\mu^{-i_{n-2}} \\ 0 & 0 & 0 & \dots & ab\lambda^{i_{n-2}} & \frac{dg(\xi_{n-1})}{d\xi_{n-1}} \end{pmatrix}.$$

It is clear that

$$H_1 = \frac{dg(\xi_2)}{d\xi_2},$$

$$H_2 = \frac{dg(\xi_2)}{d\xi_2}\frac{dg(\xi_3)}{d\xi_3} + ab(\lambda(\mu)^{-1})^{i_2}.$$

Using the properties of the determinant, we can easily show that

$$H_{n-2} = \frac{dg(\xi_{n-1})}{d\xi_{n-1}} H_{n-3} + ab(\lambda(\mu)^{-1})^{i_{n-2}} H_{n-4},$$

$$\mathrm{Det}\, DG(\xi_2, \xi_3, \dots, \xi_{n-1}) = H_{n-2} + (-1)^{n-1}(ab)^{n-1}\lambda^{i_1+i_2+\dots+i_{n-1}}\mu^{i_0}.$$

It follows from the last formulas and conditions (16) and (17) that

$$\mathrm{Det}\, DG(\xi_2, \xi_3, \dots, \xi_{n-1}) > 0$$

for any $(\xi_2, \xi_3, \dots, \xi_{n-1}) \in P$. It is easy to see that the mapping G is one-to-one on P.

From conditions (13) for sufficiently large k and any $m = 2, 3, \dots, n-1$, we have

$$|G_m(\xi_2, \xi_3, \dots, \xi_{n-1}) - g(\xi_m)| < y^0\mu^{-i_0}\left(\mu^{\frac{1}{2}} - 1\right).$$

It follows from these inequalities and conditions (17) that the following inclusions hold for sufficiently large k:

$$\left\{(\xi_2, \xi_3, \dots, \xi_{n-1}), \xi_m \in \left[y^0\mu^{-i_0+\frac{3}{2}}, y^0\mu^{-i_0+s-\frac{1}{2}}\right],\ m = 2, 3, \dots, n-1\right\} \subset G(P).$$

The last inclusions imply the existence of a solution to system (14).

Theorem 2 is proven.

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