===== **TO THE MEMORY OF P. E. TOVSTIK** =====

Natural Frequencies of an Inhomogeneous Square Thin Plate

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Abstract—The purpose of the study is to analyze the inhomogeneity effect on the natural transverse vibrations of square thin plates using asymptotic methods, when the thickness and/or stiffness of the plate can be assumed almost constant. The results of calculations by asymptotic formulas, which are determined by the perturbation method, are compared with the numerical results obtained by the finite element method using the COMSOL software package.

1. INTRODUCTION

The structure of the spectrum of natural transverse vibrations of isotropic and homogeneous rectangular plates under various boundary conditions is a well-studied problem. The list of studies on this topic is extensive, and a systematic review of the research results is given in [1].

A large number of studies are also devoted to the vibrations of inhomogeneous rectangular plates. Thus, in [2], the inhomogeneity influence on the vibrations of rectangular thin plates with thickness linearly varying along one direction was studied under different boundary conditions applying the Rayleigh–Ritz method with the use of two-dimensional boundary characteristic orthogonal polynomials. In the plates considered, the inhomogeneity was also associated with linear changes in Young's modulus and density of the material. In [3], it was proposed to solve problems on natural vibrations of rectangular plates of variable thickness under complex boundary conditions using the numerical-analytical spline-collocation method in combination with the discrete-orthogonalization method. In [4], a simple algorithm based on the Ritz method and the expression of the basic displacement function through a polynomial coordinate function that approximately satisfied the essential (geometric) boundary conditions, made it possible to obtain the fundamental natural frequency for a rectangular plate with a bilinearly varying thickness.

Apparently [5] was one of the first studies in which the finite element method was used to investigate the vibrations of rectangular variable thickness plates and to find natural frequencies and vibration modes of rectangular cantilever plates with linearly varying thickness. In [6], natural frequencies of transverse vibrations of rectangular thin variable thickness plates obtained by different methods, such as, the Rayleigh–Ritz method using different shape functions, the optimized Kantorovich method, and the finite element method, were compared for different combinations of boundary conditions. In [7], transverse vibrations of a rectangular plate in different boundary conditions were investigated under the assumption that the thickness is the product of two linear functions $h(x, y) = h_1(x)h_2(y)$. Sequential approximations were found by the Rayleigh–Ritz method using the basis functions satisfying the essential (geometric) boundary conditions.

The perturbation method was used in [8] to analyze bending vibrations of rectangular plates with average slowly varying thickness, and approximate formulas for free vibration frequencies were obtained in explicit form at arbitrary functions of thickness. The mixed boundary grid method (FBGM) was proposed in [9] to analyze free vibrations of variable thickness cantilever plates. The authors obtained the fundamental differential equations for plate bending, which were transformed into integral equations, whose numerical solution was chosen as the Green function to obtain the characteristic equation of natural frequencies.
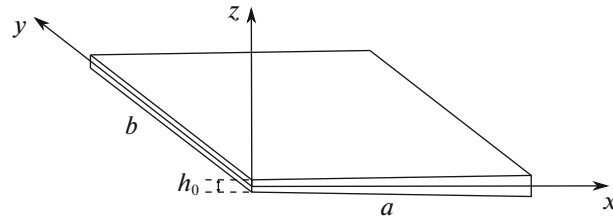
Fig. 1. Rectangular plate with a linearly varying thickness.

Multiple frequencies of square plates with the same boundary conditions at all edges are of special interest. Here, any variation of thickness or stiffness causes a certain loss of symmetry, and we can expect a splitting of multiple frequencies. In [10], the finite element method was used to study the vibrations of a square plate with a parabolically varying thickness and clamped edges. Particular attention was paid to the modes of the plate vibrations that displayed a combination of radial and square symmetries and resulted from the axisymmetric distribution of thickness and the square symmetry of the boundary.

The objective of our study is to obtain asymptotic formulas describing the effect of inhomogeneity of a thin plate thickness or stiffness parameters on its natural frequencies. The algorithm for obtaining such formulas was described, for example, in [11]. The present study continues the research of asymptotic methods of vibrations of inhomogeneous plates of different shapes, which was started by the authors in [12]. In the study, it is assumed that the geometric and physical parameters of the plate are smooth functions of coordinates, constant in one direction and close to constant in the other.

2. EQUATIONS OF VIBRATIONS OF A THIN SQUARE PLATE WITH VARIABLE PARAMETERS

Natural transverse vibrations of a rectangular thin plate with variable parameters (Fig. 1) are described by the following equations

$$\Delta(D\Delta w) - (1-\nu)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) + \gamma H\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where $w = w(x, y, t)$ is the plate deflection, $D = D(x, y) = \dfrac{EH^3}{12(1-\nu^2)}$ is the cylindrical stiffness; Δ is the Laplace operator; $H = H(x, y)$ is the plate thickness; $E = E(x, y)$, $\nu = \nu(x, y)$, $\gamma = \gamma(x, y)$ are Young's modulus, Poisson's ratio, and the density of the plate material; and x, y, and t are the spatial coordinates and time. In this paper, the plates with $\gamma = $ const and $\nu = $ const are considered. Below, $\nu = 0.3$ in all examples.

Let us separate the variables as

$$w(x, y, t) = W(x, y)\cos(\omega t) \tag{2}$$

and express them in a dimensionless form; then Eq. (1) can be written as follows:

$$\Delta_k^2 W - (1-\nu)\left(\frac{\partial^2 d}{\partial \xi^2}\frac{\partial^2 W}{\partial \eta^2} - 2\frac{\partial^2 d}{\partial \xi\partial \eta}\frac{\partial^2 W}{\partial \xi\partial \eta} + \frac{\partial^2 d}{\partial \eta^2}\frac{\partial^2 W}{\partial \xi^2}\right) + \lambda^4 hW = 0, \tag{3}$$

where

$$\Delta_k^2 = k^2(dW_{\xi\xi}'')_{\xi\xi}'' + (dW_{\eta\eta}'')_{\eta\eta}''/k^2 + (dW_{\xi\xi}'')_{\eta\eta}'' + (dW_{\eta\eta}'')_{\xi\xi}'',$$

$$\xi = x/a, \quad \eta = y/b, \quad k = b/a, \quad d = d(\xi,\eta) = D(\xi,\eta)/D_0,$$

$$h = h(\xi,\eta) = H(\xi,\eta)/h_0, \quad \lambda^4 = \omega^2\gamma h_0 a^2 b^2/D_0, \quad D_0 = \frac{E_0 h_0^3}{12(1-\nu^2)},$$

and h_0 and E_0 are the characteristic values of thickness and Young's modulus.

Next, we consider square plates with $k = 1$ and $\Delta_k = \Delta$.

The variables can be separated only when the parameters change along one coordinate $h(\xi, \eta) = h(\xi)$, $E(\xi, \eta) = E(\xi)$, and the opposite sides of the plate are simply supported along the other coordinate. In this case, the solution can be represented as $W(\xi, \eta) = \sin(m\pi\eta)w(\xi)$, and function $w(\xi)$ is defined by equation

$$\pi^4 m^4 d(\xi)w(\xi) - 2\pi^2 m^2 d(\xi)w''(\xi) + d(\xi)w''''(\xi) - h(\xi)\lambda^4 w(\xi)$$
$$+ d'(\xi)(2w'''(\xi) - 2\pi^2 m^2 w'(\xi)) - d''(\xi)(\pi^2 v m^2 w(\xi) + w''(\xi)) = 0. \tag{4}$$

Equation (3), together with the boundary conditions, forms the boundary value problem on eigenvalues for λ. Below, only the homogeneous boundary conditions at the plate edges ($\xi^* = 0$ or $\xi^* = 1$) of the simply supported (S) or clamped (C) type are considered in the following form:

$$(S): w(\xi^*) = w''(\xi^*) = 0, \quad (C): w(\xi^*) = w'(\xi^*) = 0. \tag{5}$$

In order to compare different approximations of the vibration modes, it is useful to normalize them, whereby normalized solution $W(\xi)$ can be written as

$$W(\xi) = w(\xi)/\|w(\xi)\|, \quad \|W(\xi)\| = 1, \tag{6}$$

and in the linear ε approximation, we have

$$W(\xi) = w_0(\xi)/\|w_0(\xi)\| + \frac{w_1(\xi)\|w_0(\xi)\| - 2w_0(\xi)(w_0(\xi) \cdot w_1(\xi))}{\|w_0(\xi)\|^2}\varepsilon + O(\varepsilon^2). \tag{7}$$

3. NATURAL FREQUENCIES OF A SQUARE PLATE WITH VARIABLE THICKNESS

The perturbation method is used to study the vibration frequencies of a square plate with stiffness and thickness parameters close to constant. We assume that

$$h(\xi) = 1 + \varepsilon h_1(\xi) + \ldots, \quad E(\xi) = 1 + \varepsilon E_1(\xi) + \ldots,$$
$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \ldots, \quad w(\xi) = w_0(r) + \varepsilon w_1(\xi) + \ldots, \quad \varepsilon \ll 1. \tag{8}$$

Upon substituting (8) into Eq. (4) and equating the coefficients at equal powers of ε, we obtain a series of boundary value problems, whose condition for the existence of a solution is the orthogonality of the right-hand sides of the equation to solutions $w_0(\xi)$ [11].

Let us consider a square plate with variable thickness simply supported at all sides, assuming a constant Young's modulus, $E(\xi) = 1$. The boundary value problem in the zero approximation can be written as

$$\pi^4 m^4 w_0(\xi) - 2\pi^2 m^2 w_0''(\xi) + w_0''''(\xi) - \lambda_0^4 w_0(\xi) = 0,$$
$$w_0(\xi^*) = w_0''(\xi^*) = 0 \quad \text{at} \quad \xi^* = 0,1, \tag{9}$$

and its solution are the following frequencies and modes:

$$\lambda_0 = \sqrt{m^2 + n^2}\pi, \quad w_0(\xi) = \sin(n\pi\xi). \tag{10}$$

Having substituted expressions (10) into first approximation equation

$$\pi^4 m^4 w_1(\xi) - \lambda_0^4 w_1(\xi) - 2\pi^2 m^2 w_1''(\xi) + w_1''''(\xi) = F_1(w_0(\xi), \lambda_0) + F_2(w_0(\xi), \lambda_0)\lambda_1, \tag{11}$$

where

$$F_1 = 3\pi^2 v m^2 w_0(\xi)h_1''(\xi) - 3h_1''(\xi)w_0''(\xi) + 6\pi^2 m^2 h_1'(\xi)w_0'(\xi) - 6w_0'''(\xi)h_1'(\xi)$$
$$- 3\pi^4 m^4 h_1(\xi)w_0(\xi) + 6\pi^2 m^2 h_1(\xi)w_0''(\xi) - 3h_1(\xi)w_0''''(\xi) + \lambda_0^4 h_1(\xi)w_0(\xi),$$
$$F_2 = 4\lambda_0^3 w_0(\xi),$$

we require the orthogonality of its right-hand side to mode $w_0(x)$. Calculating the integrals, we obtain the formula for the first frequency correction,

$$\lambda_1 = -\int_0^1 F_1 w_0(\xi)d\xi \Big/ \int_0^1 F_2 w_0(\xi)d\xi. \tag{12}$$

Let us consider various perturbing functions as examples (see Table 1).

Table 1. Perturbing functions and the corresponding first corrections of frequencies for variable thickness plates

	$h_1(\xi)$	λ_1
1	ξ	$1/4\sqrt{m^2 + n^2}\,\pi$
2	$\xi - 1/2$	0
3	ξ^2	$\dfrac{(2\pi^2 m^4 n^2 - 3m^4 + 4\pi^2 m^2 n^4 - 18m^2 \nu n^2 + 12m^2 n^2 + 2\pi^2 n^6 - 3n^4)}{12\pi n^2 (m^2 + n^2)^{3/2}}$

In the first case (linear increase in thickness at $\varepsilon > 0$ or its decrease at $\varepsilon < 0$), the increase in the plate stiffness is the determining factor. All frequencies increase with the increase in ε, and no splitting of multiple frequencies, i.e., frequencies $\lambda_0^{m,n}$ and $\lambda_0^{n,m}$, occurs in the first approximation.

A linear variation of thickness, while maintaining the plate mass (the second case), has no effect on the frequencies in the first approximation. Similarly, no perturbation in the form of an odd function with respect to the middle of the plate ($\xi = 1/2$) has an effect on the frequencies in the first approximation.

When the thickness varies parabolically (the third case), both the shift and the splitting of the multiple frequencies occur in the first approximation, and the magnitude of the frequency splitting can be written as follows:

$$\delta = \left| \lambda_0^{m,n} - \lambda_0^{n,m} \right| = \left| \frac{(m^2 - n^2)(m^4 + 2(3\nu - 2)m^2 n^2 + n^4)}{4\pi m^2 n^2 (m^2 + n^2)^{3/2}} \right|. \tag{13}$$

If $n \approx m$ at $m \to \infty$, then $\delta = O(1/m)$ at $m \to \infty$. If $n = m + O(1)$, then $\delta = O(1/m^2)$ at $m \to \infty$. In other words, the more that wave numbers m and n differ, the more noticeable is the frequency splitting effect.

In order to construct the first approximation solution, we should substitute the value of λ_1 into Eq. (11) and solve it using three of the four boundary conditions of the (S) type. The fourth condition will be fulfilled automatically. The analytical expressions for the first approximations of vibration modes ($w_1(\xi)$) and the second corrections of frequencies (λ_2) were found by using the Mathematica 11.3 software package.

Since the formulas are cumbersome, we give expressions only for the vibration modes with fundamental frequency ($m = n = 1$) of a plate with a parabolically varying thickness:

$$w_1(\xi) = \frac{2}{\pi}\left(-\pi(\xi + 1)\sin(\pi\xi) + \left(-\pi^2(\xi - 1)\xi - 1\right)\cos(\pi\xi) \right.$$
$$\left. + \cosh\left(\frac{\sqrt{3}\pi}{2}\right)\sinh\left(\frac{1}{2}\sqrt{3}\pi(1 - 2\xi)\right)\right). \tag{14}$$

For this plate, the abovementioned modes normalized to formulas (7) are shown in Fig. 2.
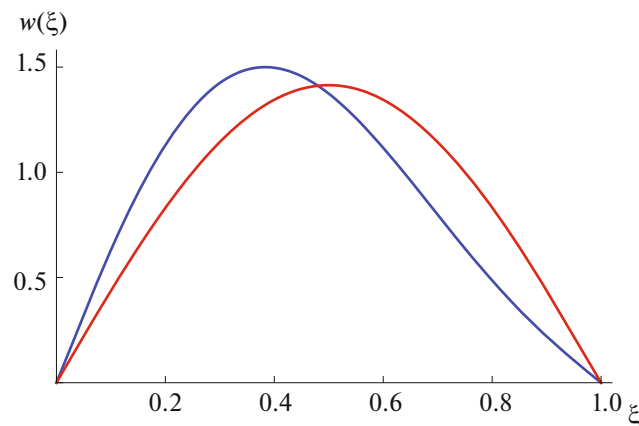

Fig. 2. The first vibration mode of a constant thickness plate (red line) and of a plate with a parabolically varying thickness (blue line) at $\varepsilon = 0.6$.
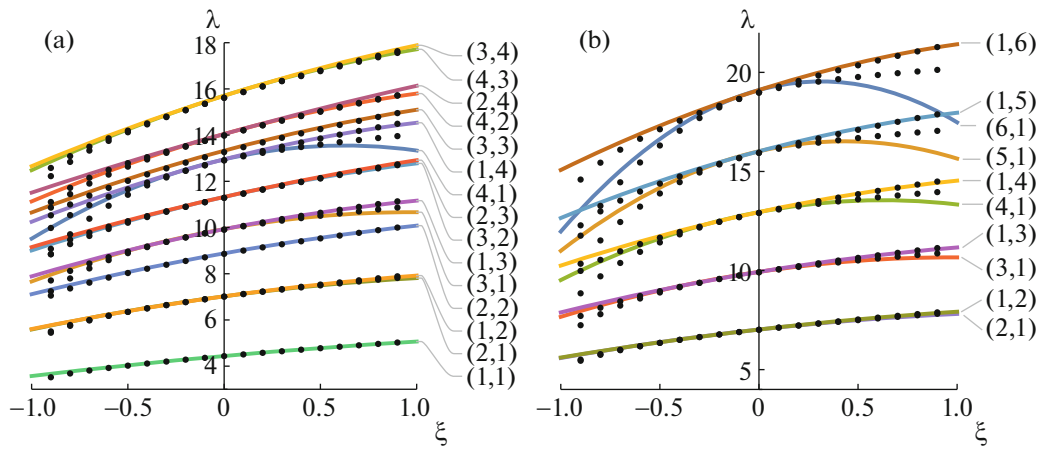
Fig. 3. Dependence of lower frequencies (a) and multiple frequencies of the $\lambda^{m,1}$ and $\lambda^{1,m}$ type (b) on small parameter ε for a plate with a parabolically varying thickness.

The dependence of the lower frequencies of the plate with a parabolically varying thickness on small parameter ε is shown in Fig. 3a, where (m, n) are the wave numbers corresponding to the frequencies. In the specified range of ε variation, all frequencies increase monotonically with the increase in ε, except the frequencies of $\lambda^{m,1}$ type at large values of m. These frequencies reach their maximum at $\varepsilon = O(1/m^2)$, with the maximum point rapidly tending to zero with the increase in m (Fig. 3b). In Fig. 3, solid lines correspond to the frequency values calculated by the asymptotic formulas, and dots correspond to the values obtained using the COMSOL Multiphysics 5.4 finite-element software package. At small values of ε, the asymptotic results are in good agreement with numerical ones. However, with the increase in the wave number values, the applicability range of the asymptotic formulas narrows.

4. NATURAL FREQUENCIES OF A SQUARE PLATE WITH VARIABLE STIFFNESS

The variable stiffness effect on the natural frequencies of the plate is investigated in a similar way. We assume that $h(\xi) = 1$, and Young's modulus $E(\xi)$ is variable. The values of the first corrections of the frequencies for same types of perturbations as those considered earlier are given in Table 2.

The influence associated with the stiffness variation is qualitatively close to the influence of the thickness variation, but is less significant. As an example, we consider the effect of a linear variation of the plate stiffness on the vibration frequencies (see Fig. 4).

When $E(\xi) = \xi$, the lower frequencies grow with the increase in ε and split poorly. When $E(\xi) = \xi - 1/2$, when the mean stiffness of the plate is constant, all frequencies reach their maximum at $\xi = 0$. The greatest splitting is observed for frequencies of the $\lambda^{m,1}$ and $\lambda^{1,m}$ type at $m > 1$.

5. CONCLUSIONS

It is found that natural frequencies of a plate are split only in the second approximation if the stiffness/thickness variations are linear with respect to ε and a coordinate. However, if the parameter varies

Table 2. Perturbing functions and the corresponding first corrections of frequencies for variable stiffness plates

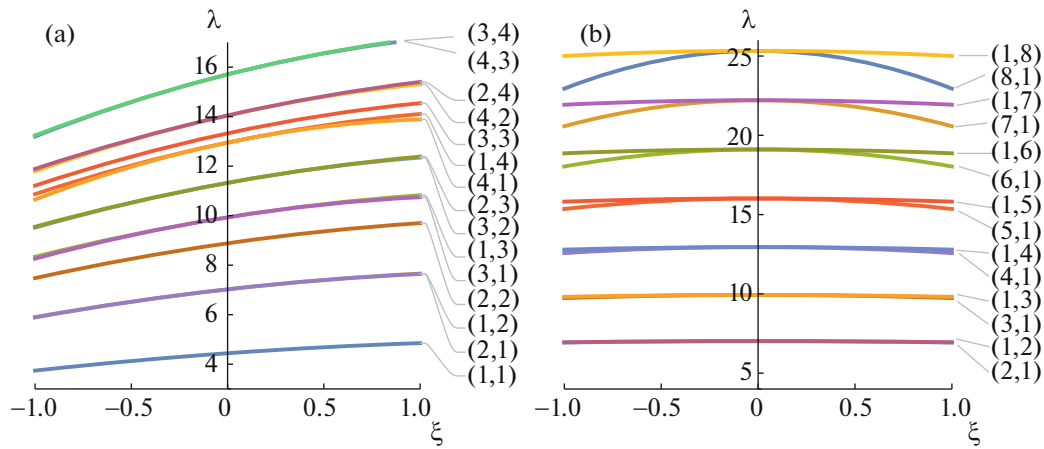|	$E_1(\xi)$	λ_1
1	ξ	$\dfrac{1}{8} n\sqrt{m^2 + n^2}\,\pi$
2	$\xi - 1/2$	0
3	ξ^2	$\dfrac{m^4(2\pi^2 n^2 - 3) + 2m^2 n^2(-6\nu + 2\pi^2 n^2 + 3) + n^4(2\pi^2 n^2 - 3)}{24\pi n^2 (m^2 + n^2)^{3/2}}$

Fig. 4. Dependence of lower frequencies on small parameter ε for a plate with stiffness $E(\xi) = \xi$ (a) and dependence of multiple frequencies of the $\lambda^{m,1}$ and $\lambda^{1,m}$ type on small parameter ε for a plate with stiffness $E(\xi) = \xi - 1/2$ (b).

nonlinearly, for example, parabolically along the coordinate, the multiple natural frequencies split in the first approximation. In addition, the asymptotic formulas make it possible to determine, which of the two multiple frequencies corresponding to wave numbers n and m varies faster with the variations of the small parameter.

FUNDING

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