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MATHEMATICS

Limit Theorems for the Generalized Perimeters of Random Inscribed Polygons: II

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Abstract—Recently, W. Lao and M. Mayer (2008) developed *U*-max-statistics, where instead of averaging the values of the kernel over various subsets, the maximum of the kernel is considered. Such statistics often appear in stochastic geometry. This is the second part of the work devoted to the study of the generalized perimeter of a random inscribed polygon and the limit behavior of *U*-max-statistics related to it. Here we consider the case where the parameter arising in the definition of a generalized perimeter exceeds 1. The limit theorems in the case of a triangle are formulated and proved.

Keywords: *U*-max-statistics, limit behavior, uniform distribution on circumference, generalized perimeter

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Dedicated to the memory of Prof. Ya.Yu. Nikitin

1. INTRODUCTION

This paper continues the work [1] devoted to the limit behavior of the generalized perimeter of a randomly inscribed polygon. We keep the notation of [1] but the numbering of formulas and propositions starts afresh.

Recall the definition of *U*-max-statistics. Suppose $\xi_1, \xi_2, \ldots$ is a sequence of independent equally distributed random variables, which take values in a certain measurable space $(\mathfrak{X}, \mathfrak{A})$. A real-valued symmetrical Boreal function h defined on $\mathfrak{X}^m$ is called a kernel. *U*-max-statistics mean the random variable

$$H_n = \max_J h(\xi_{i_1}, \ldots, \xi_{i_m}),$$

where $n \geq m$ and the set $J = \{(i_1, \ldots, i_m) : 1 \leq i_1 < \ldots < i_m \leq n\}$ is a set of ordered m-element permutations with the set of indices from the collection $\{1, \ldots, n\}$; and *U*-min statistics are defined analogously.

In [1], the notion of a generalized perimeter is introduced. Assume that d_i ($i = 1, \ldots, m$) are the sides of an m-gon and $h(U_1, \ldots, U_m) = \sum_{i=1}^m d_i$ is its perimeter. We propose to consider the magnitude $h_y(U_1, \ldots, U_m) = \sum_{i=1}^m d_i^y$, $y \in \mathbb{R}$ and call it the generalized perimeter of the polygon.

The first part of [1] is devoted to the case where $y < 1$. The result is the proof of the limit theorems for the minimum generalized perimeter

$$H_{n,m}^y = \min_{1 \leq i_1 < \ldots < i_m \leq n} h_y(U_{i_1}, \ldots, U_{i_m})$$

when $y < 0$ and for the maximum generalized perimeter

$$G_{n,m}^y = \max_{1 \leq i_1 < \ldots < i_m \leq n} h_y(U_{i_1}, \ldots, U_{i_m})$$

when $y \in (0, 1)$. Due to [2], the result can be extended to the case where $y = 1$. This second part of the work is devoted to the study of the case where $y > 1$.

2. THE CASE WHERE $y > 1$

The formulations of the theorems from [1] are similar. It is natural to pose the following question: will an analogous limit theorem be fulfilled for generalized perimeters when $y > 1$? It turns out that they will not be fulfilled. The consistency of the formulas follows from the fact that for $y \in (-\infty, 0) \cup (0, 1]$ a regular m-gon is the point of a strict extremum of the generalized perimeter. However, for $y > 1$ it is not necessarily the case. We can present the following rough estimate on the number of sides (of the m-gon) with which this property is violated.

Proposition 1. *For any fixed* $y > 1$ *when* $m > \pi/\arccos\frac{1}{\sqrt{y}}$, *a regular m-gon is not the maximum point of the generalized perimeter* h_y.

Proof. Consider the function $f(x) = (\sin x)^y$. From [1] we know that

$$h_y(U_1,\dots,U_m) = 2^y\sum_{k=1}^{m} f\left(\frac{\pi}{m} + \frac{\alpha_k - \alpha_{k-1}}{2}\right), \quad \text{where} \quad \alpha_k = \angle U_{k+1}OU_1 - \frac{2\pi k}{m};$$

here, it can be considered that the points U_1, ..., U_m are located on the circumference in this order. By the explicit calculations we obtain

$$f''(x) = y(\sin x)^{y-2}(y\cos^2 x - 1); \tag{1}$$

therefore, with these restrictions on m, it is true that $f''\left(\frac{\pi}{m}\right) > 0$. For the expansion of the generalized perimeter we have

$$h_y(U_1,\dots,U_m) = 2^y\sum_{i=1}^{m} f\left(\frac{\pi}{m} + \frac{\alpha_k - \alpha_{k-1}}{2}\right)$$

by Taylor's formula with the remainder in the form of Lagrange; in the neighborhood of points $\frac{\pi}{m}$, the linear term disappears, but with a sufficiently small α_i the positive second term will appear. Therefore, in this case, the regular m-gon will be the point of the local minimum, not maximum.

Thus, we cannot prove the limit theorem for $G^y_{n,m}$ when $y > 1$, because other maximum points whose form is unknown arise. We obtain the following intermediate proposition.

Proposition 2. *Suppose* $f(x) = (\sin x)^y$. *Consider the extremum problem*

$$\begin{cases} \sum_{i=1}^{m} f\left(\frac{\gamma_i}{2}\right) \to \max, \\ \sum_{i=1}^{m} \gamma_i = 2\pi, \\ 0 \le \gamma_1 \le \dots \le \gamma_m \le 2\pi \end{cases} \tag{2}$$

for $y > 1$ *and* $m \ge 3$. *Then for the solutions of this system for* $1 \le k \le m$ *one of the following two conditions holds*:

- $0 = \gamma_1 = \dots = \gamma_{k-1}$, *and the others* $\gamma_k = \dots = \gamma_m = \dfrac{2\pi}{m+1-k} \ge \delta$;
- $0 = \gamma_1 = \dots = \gamma_{k-2}$, $\gamma_{k-1} \in (0, \delta)$, *and the others* $\gamma_k = \dots = \gamma_m \ge \delta$.

Here, $\delta = 2\arccos\frac{1}{\sqrt{y}}$.

Remark 1. Recall that solving this problem is equivalent to finding the inscribed polygon with the maximum generalized perimeter. Thus, we find that it is achieved either on a regular polygon (perhaps, with the number of sides less than m), or on a polygon where all the sides, except one, are equal to each other, but the number of vertices also can be smaller than m. This is consistent with the known results. For example, it is well known that for $y = 2$ the maximum of the generalized perimeter is achieved on the regular triangle, regardless of the number of arguments m of the function h_2 (see, e.g., ([3], problem 11.36)).

Proof. The maximum in this problem is achieved, since we consider the continuous function on the compact set. Suppose the set $\{\gamma_i\}_{i=1}^{m}$ is the solution of extremum problem (2). Let us show that all $\gamma_i \le \pi$. Assume that this condition does not hold. Then there exist γ_i and γ_j such that $\gamma_i < \pi < \gamma_j$. Take $\varepsilon > 0$ such that the inequality $\gamma_i + \varepsilon < \pi < \gamma_j - \varepsilon$ and replace γ_i and γ_j by $\gamma_i + \varepsilon$ and $\gamma_j - \varepsilon$. Then, $\sum_{i=1}^{m} f\left(\frac{\gamma_i}{2}\right)$ increases, and this is in contrast with (2).

Consider some $\gamma_i, \gamma_j \in [\delta, \pi]$ such that $\gamma_i \ne \gamma_j$. Formula (1) shows that $f''\left(\frac{x}{2}\right) < 0$ when $x \in (\delta, \pi)$; therefore, similar to the reasoning of Lemma 1 in [1], the replacement of γ_i and γ_j by $(\gamma_i + \gamma_j)/2$ and $(\gamma_i + \gamma_j)/2$ increases the sum under consideration. Consequently, all γ_i belonging to the interval $[\delta, \pi]$ are equal to each other.

Assume that there are two angles $\gamma_i, \gamma_j \in (0, \delta)$. Without loss of generality suppose that $f'\left(\frac{\gamma_i}{2}\right) \ge f'\left(\frac{\gamma_j}{2}\right)$. Take $\varepsilon > 0$ such that $\gamma_i + \varepsilon$ and $\gamma_j - \varepsilon$ also belong to the interval $[0, \delta]$. Using the reasoning of Lemma 1 of [1], we can write two formulas

$$f\left(\frac{\gamma_i + \varepsilon}{2}\right) = f\left(\frac{\gamma_i}{2}\right) + f'\left(\frac{\gamma_i}{2}\right)\frac{\varepsilon}{2} + \frac{1}{2} f''(\varphi_1)\frac{\varepsilon^2}{4}, \tag{3}$$

$$f\left(\frac{\gamma_j - \varepsilon}{2}\right) = f\left(\frac{\gamma_j}{2}\right) - f'\left(\frac{\gamma_j}{2}\right)\frac{\varepsilon}{2} + \frac{1}{2} f''(\varphi_2)\frac{\varepsilon^2}{4}, \tag{4}$$

where $\varphi_1, \varphi_2 \in \left(0, \frac{\delta}{2}\right)$. Because due to (1) we have $f''(\varphi_1)$, $f''(\varphi_2) > 0$ and also $f'\left(\frac{\gamma_i}{2}\right) \ge f'\left(\frac{\gamma_j}{2}\right)$, after the summation of (3) and (4), together with the replacement of γ_i and γ_j by $\gamma_i + \varepsilon$ and $\gamma_j - \varepsilon$, the value of the maximized sum will increase. This is also in contrast with (2).

Thus, for any solution of this problem, all angles γ_i larger than δ are equal to each other, and there is not more than one angle γ_i from the interval $(0, \delta)$.

The subsequent study of U-max-statistics related to a generalized perimeter depends on the structure of maximum points of the considered extremum problem. From Proposition 1 it follows that for $y > 1$, a regular m-gon is not always the maximum point; here, the maximum can be attained on more complex systems of points.

However, even in the case where the maximum is reached on a regular k-gon for $k < m$ (e.g., in the case where $y = 2$), the same reasoning cannot be done. The difficulty lies in the fact that the derivatives at points 0 and $\frac{2\pi}{k}$ generally do not match and the linear term is retained. Therefore, we supplement Proposition 1 with another proposition related to the case of the coincidence of the original random points.

Proposition 3. *When* $y > 1$ *and* $m > 1 + \pi \Big/ \left(\arccos\frac{1}{\sqrt{y}}\right)$, *the generalized perimeter attains the maximum when some* U_i *match.*

Proof. This simple proposition immediately follows from Proposition 2. In fact, if all $\gamma_i \ne 0$, then either they are all equal to $\frac{2\pi}{m}$, or $\gamma_i < \delta$ and for the rest of γ_i we have $\delta \le \gamma_2 = \ldots = \gamma_m = \frac{2\pi - \gamma_1}{m - 1} < \frac{2\pi}{m - 1}$. However, under these conditions the inequality $\frac{2\pi}{m - 1} < \delta$ holds, and this leads us to a contradiction.

We can consider another problem: fix m and try to find out for which $y > 1$ a regular m-gon is still a point of the maximum; and for which, it is not. Rough estimates on y are contained in Propositions 1 and 3 of this section. In the case where $m = 3$, we can move slightly further ahead.

3. THE CASE OF A TRIANGLE

For a triangle, the complete investigation of the case where $y \in [1, 2]$ is possible. The limit theorem for U-max-statistics is presented here as follows.

Theorem A. *Suppose $U_1, \ldots, U_n$ are independent and uniformly distributed points on a unit circumference S_1 with the center at point O. Assume that h_y is a generalized perimeter defined above. Introduce the notation*

$$G^y_{3,n} = \max_{1\le i<j<k\le n} h_y(U_i, U_j, U_k).$$

Then, provided $y \in [1, 2]$, for any $t > 0$ the following equality holds:

$$\lim_{n\to\infty} \mathbb{P}\{n^3(3^{\frac{y+2}{2}} - G^y_{3,n}) \le t\} = 1 - e^{-\frac{t}{K_2(y)}},$$

where $K_2(y) = 3^{\frac{y+1}{2}}\pi y\left(2 - \frac{y}{2}\right)$.

Proof. The proof of this theorem is ideologically the same as that of Theorems 3 and 4 from [1]. It is necessary to prove analogs of Lemmas 1 and 2 form these theorems; then, the remainder of the proof will be very similar, and the constants in Theorem A are obtained from the constants in Theorem 4 from [1] by the substitution $m = 3$. The analog of Lemma 1 claiming that for $y \in [1, 2]$ the maximum of the function h_y is achieved only on a regular triangle (and the maximum value is $3^{\frac{y+2}{2}}$) has been known for a long time. Perhaps, for the first time this analog appeared in Hille's work [4].

To prove the analog of Lemma 2, it is necessary to estimate the difference of the central angles from $\frac{2\pi}{3}$. For convenience, denote them by $2\psi_1$, $2\psi_2$, and $2\psi_3$. Then, we have

$$h_y(U_1, U_2, U_3) = h_y(2\psi_1, 2\psi_2, 2\psi_3) = 2^y \sum_{1\le i\le 3} \sin^y \psi_i = 2^y \sum_{1\le i\le 3} f(\psi_i). \tag{5}$$

We introduced the function h_y as a function of three points taken on a unit circumference; however, it is also determined by three central angles. Therefore, we can present the perimeter h_y as a function of non-negative angles giving in total 2π. The following lemma holds.

Lemma B. *Assume that the following condition holds:*

$$h_y(2\psi_1, 2\psi_2, 2\psi_3) > 3^{\frac{y+2}{2}} - s. \tag{6}$$

Then there are the constants C, $D > 0$ depending only on y such that for $0 < s < D$ the following inequality holds:

$$\max_{1\le i\le 3}\left|\psi_i - \frac{\pi}{3}\right| < C\sqrt{s}.$$

In other words, for small s, the central angles differ from $\frac{2\pi}{3}$ by $O(\sqrt{s})$.

Proof. We rest on formula (5). From formula (1) it follows that $f''(x) < 0$ for $y \in [1, 2]$ and $x \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$. We fix some small $\varepsilon \in \left(0, \frac{\pi}{8}\right)$, which we will choose later. The function $f''(x)$ is continuous on the interval $\left[\frac{\pi}{4} + \varepsilon, \frac{3\pi}{4} - \varepsilon\right]$ and negative; hence, there exists $\Delta(\varepsilon) > 0$ such that

$$f''(x) < -\Delta(\varepsilon) < 0 \quad \text{when} \quad x \in \left[\frac{\pi}{4} + \varepsilon, \frac{3\pi}{4} - \varepsilon\right].$$

We prove the supporting statement.

Assertion. *Assume that condition* (6) *holds. Suppose that in the set of angles* ψ_1, ψ_2, *and* ψ_3, *there are* ψ_i *and* $\psi_j \in \left[\frac{\pi}{4}+\varepsilon, \frac{3\pi}{4}-\varepsilon\right]$. *Then*, $|\psi_i - \psi_j| < C(\varepsilon, y)\sqrt{s}$, *where the constant* $C(\varepsilon, y)$ *depends only on* ε *and* y.

Proof. We replace the central angles $2\psi_i$ and $2\psi_j$ by $\psi_i + \psi_j$ and $\psi_i + \psi_j$. Analogously to the reasoning of Lemma 2 from [1], we obtain

$$s > 2^y\left(2\sin^y\left(\frac{\psi_i+\psi_j}{2}\right) - (\sin\psi_i)^y - (\sin\psi_j)^y\right) = -2^{y-3}(\psi_i-\psi_j)^2(f''(\varphi_1)+f''(\varphi_2)), \tag{7}$$

where $\varphi_1, \varphi_2 \in \left[\frac{\pi}{4}+\varepsilon, \frac{3\pi}{4}-\varepsilon\right]$. This brings us to the relation $4s > 2^y\Delta(\varepsilon)(\psi_i - \psi_j)^2$; hence,

$$|\psi_i - \psi_j| < C(\varepsilon, y)\sqrt{s} = O(\sqrt{s}), \quad \text{where} \quad C(\varepsilon, y) = \frac{2}{\sqrt{2^y\Delta(\varepsilon)}} > 0.$$

Consider all possible cases of the relation of the angles ψ_1, ψ_2, and ψ_3. We demonstrate that constants D and ε can be taken in such a way that condition (6) can be fulfilled only in Case 2; then the central angles will differ from $\frac{2\pi}{m}$ by not more than $2C(\varepsilon, y)\sqrt{s}$. The constant $C(\varepsilon, y)$ is taken from the assertion made above. Without loss of generality, we consider that $\psi_3 \geq \psi_2 \geq \psi_1$.

Case 1: $\frac{\pi}{2} \geq \psi_3$; $\frac{\pi}{4} + \varepsilon \geq \psi_2 \geq \psi_1$.

As previously stated, we want to show that with appropriate restrictions on ε and D, condition (6) is not met. This will demonstrate the impossibility of this case under the conditions of Lemma B.

We use the condition $\sum_{1\leq i\leq 3}\psi_i = \pi$. Together with the condition that defines Case 1, the following restrictions on the angles are obtained:

$$\begin{cases} \frac{\pi}{2} \geq \psi_3 \geq \frac{\pi}{2} - 2\varepsilon, \\ \frac{\pi}{4} + \varepsilon \geq \psi_1, \quad \psi_2 \geq \frac{\pi}{4} - \varepsilon. \end{cases}$$

It is easy to understand that with small ε, the value of $h_y(\psi_1, \psi_2, \psi_3)$ slightly differs from $h_y\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right)$. In fact, each of the sinuses can be expanded on Taylor's formula with the remainder in the form of Lagrange in the neighborhood of a nearby point ($\frac{\pi}{2}$ or $\frac{\pi}{4}$). We obtain the following equality:

$$h_y(\psi_1, \psi_2, \psi_3) \leq h_y\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right) + 2\left(\left|f'\left(\frac{\pi}{4}\right)\right| + \left|f'\left(\frac{\pi}{2}\right)\right|\right)\varepsilon + 3M\varepsilon^2, \tag{8}$$

where

$$M = \max_{x\in\left[\frac{\pi}{8},\pi\right]}|f''(x)|.$$

We want to take estimates from above for D and ε such that the following inequality holds:

$$s + 2\left(\left|f'\left(\frac{\pi}{4}\right)\right| + \left|f'\left(\frac{\pi}{2}\right)\right|\right)\varepsilon + 3M\varepsilon^2 < 3^{\frac{y+2}{2}} - h_y\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right). \tag{9}$$

Combined with (8), inequality (9) entails the noncompliance of (6) and the inability of this case.

We present the conditions for the fulfillment of (9). For this purpose, we require that the following two inequalities hold:

$$D < \left(3^{\frac{y+2}{2}} - h_y\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right)\right)/2,$$

$$2\left(\left|f'\left(\frac{\pi}{4}\right)\right| + \left|f'\left(\frac{\pi}{2}\right)\right|\right)\varepsilon + \varepsilon^2 M < \left(3^{\frac{y+2}{2}} - h_y\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right)\right)/2.$$

By the summation of these two inequalities, we obtain (9). As noted above, the maximum of the function h_y equal to $3^{\frac{y+2}{2}}$ is achieved only if all central angles are equal to $\frac{2\pi}{3}$. This means that there are positive expressions on the right-hand side of the inequalities and the required inequalities are correct. This is equal to the following restrictions on ε and D:

$$D < \left(3^{\frac{y+2}{2}} - h_y\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right)\right)/2,$$
$$M\varepsilon < \left[-\left(\left|f'\left(\frac{\pi}{4}\right)\right| + \left|f'\left(\frac{\pi}{2}\right)\right|\right) + \sqrt{\left(\left|f'\left(\frac{\pi}{4}\right)\right| + \left|f'\left(\frac{\pi}{2}\right)\right|\right)^2 + M\left(3^{\frac{y+2}{2}} - h_y\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right)\right)/2}\right]. \quad (10)$$

Thus, this case is also not possible with the appropriate restrictions on D and ε.

Case 2: $\frac{\pi}{2} \geq \psi_3 \geq \psi_2 \geq \psi_1 \geq \frac{\pi}{4} + \varepsilon$.

It is easy to see that $\psi_1, \psi_2, \psi_3 \in \left[\frac{\pi}{4} + \varepsilon, \frac{3\pi}{4} - \varepsilon\right]$; therefore, due to the assertion made above (in the case where (6) holds), the inequality $|\psi_i - \psi_j| < C(\varepsilon, y)\sqrt{s}$ for any $i, j \in \{1, 2, 3\}$ is true. Together with the condition $\sum_{1\leq i\leq 3} \psi_i = \pi$, we obtain that $\left|\frac{\pi}{3} - \psi_i\right| < C(\varepsilon, y)\sqrt{s}$ for any $i \in \{1, 2, 3\}$, and this is what we wanted to prove.

Case 3: $\frac{\pi}{2} \geq \psi_3 \geq \psi_2 \geq \psi_1 \geq \frac{\pi}{4} + \varepsilon$.

We show that the restrictions on D and ε can be placed in such a way that condition (6) here is also not possible.

The angles $\psi_2, \psi_3 \in \left[\frac{\pi}{4} + \varepsilon, \frac{\pi}{2}\right]$; therefore, from formula (7) it is clear the that when we replace the central angles $2\psi_2$ and $2\psi_3$ by two angles $\psi_2 + \psi_3$ and $\psi_2 + \psi_3$, the value of the function h_y will not decrease. This means that if condition (6) holds, then after such a replacement, it cannot stop being fulfilled. Hence, we can consider the case where the two central angles are equal to 2γ and the third angle is equal to $2(\pi - 2\gamma) < \frac{\pi}{2} + 2\varepsilon$. If we prove that with sufficiently small D and ε such a case is impossible, we will prove that condition (6) here also does not hold.

Note that the condition $0 \leq \pi - 2\gamma < \frac{\pi}{4} + \varepsilon$ means that $\gamma \in \left(\frac{3\pi}{8} - \frac{\varepsilon}{2}, \frac{\pi}{2}\right]$. We consider only $\varepsilon < \frac{\pi}{24}$. Then, $\gamma \in \left[\frac{17\pi}{48}, \frac{\pi}{2}\right]$. We find the maximum of the function

$$g(\gamma) = h(2\pi - 4\gamma, 2\gamma, 2\gamma) = 2^y(2f(\gamma) + f(\pi - 2\gamma)) \quad \text{at} \quad \gamma \in \left[\frac{17\pi}{48}, \frac{\pi}{2}\right].$$

The points of the local extremum of the function g that are not the ends of the interval must satisfy the relation

$$g'(x) = 2^y(2f'(x) - 2f'(\pi - 2x)) = 0.$$

This is equivalent to the condition $f'(x) = f'(\pi - 2x)$.

Proposition 4. *With* $y \in [1, 2]$, *the equation* $f'(x) = f'(\pi - 2x)$ *has only one root on the interval* $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$, *which is equal to* $\frac{\pi}{3}$.

Proof. We already know that $f'(x) = y(\sin x)^{y-1}\cos x$. We put this formula into the required equation. We obtain that $(\sin x)^{y-1}\cos x = (\sin 2x)^{y-1}\cos(\pi - 2x)$. This is equivalent to the equation

$$(\sin x)^{y-1}\cos x = -(2\sin x\cos x)^{y-1}\cos 2x.$$

We move everything on one side and obtain

$$r(x) := \cos 2x(\cos x)^{y-2} = -\frac{1}{2^{y-1}}.$$

It is clear that $x = \frac{\pi}{3}$ is the solution to this equation. It is also seen that for $y \in [1, 2]$ the function $r(x)$ is negative and strictly decreases on the interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$; hence, the solution of this equation is unique.

Since $\frac{\pi}{3} \notin \left(\frac{17\pi}{48}, \frac{\pi}{2}\right)$, the function g is monotonic on the interval $\left[\frac{17\pi}{48}, \frac{\pi}{2}\right]$ and its maximum value is $\max\left(h_y\left(\frac{14\pi}{48}, \frac{17\pi}{48}, \frac{17\pi}{48}\right), h_y\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)\right)$.

This value depends only on y and due to the uniqueness of the maximum it is smaller than $3^{\frac{y+2}{2}}$. This means that we can impose the following conditions on D and ε:

$$D < 3^{\frac{y+2}{2}} - \max\left(h_y\left(\frac{14\pi}{48}, \frac{17\pi}{48}, \frac{17\pi}{48}\right), h_y\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)\right), \quad \varepsilon < \frac{\pi}{24}. \tag{11}$$

Under these restrictions, we have $3^{\frac{y+2}{2}} - s > \max_{\gamma \in \left[\frac{17\pi}{48}, \frac{\pi}{2}\right]} g(\gamma) \geq h_y(2\psi_1, 2\psi_2, 2\psi_3)$, and this is in contrast with condition (6). This completes the consideration of Case 3.

Case 4: $\psi_3 > \frac{\pi}{2} > \psi_2 \geq \psi_1$.

Assume that inequality (6) holds. Because the function $f(x)$ monotonically increases when $x \in \left[0, \frac{\pi}{2}\right]$ and decreases when $x \in \left[\frac{\pi}{2}, \pi\right]$, when replacing the angles ψ_3 and ψ_2 by the angles $\frac{\pi}{2}$ and $\psi_2 + \psi_3 - \frac{\pi}{2}$, the value of the function h_y will not decrease, and inequality (6) holds. Denote the new angles by $\varphi_1 \leq \varphi_2 \leq \varphi_3 = \frac{\pi}{2}$. Now all the angles φ_i do not exceed $\frac{\pi}{2}$ and this situation is related to one of the previous three cases. It has previously been shown that under conditions (10) and (11), inequality (6) cannot be fulfilled in Cases 1 and 3. Then Case 2 must be fulfilled. According to the previously proven, in Case 2 the inequality $\left|\frac{\pi}{3} - \varphi_i\right| < C(\varepsilon, y)\sqrt{s}$ for all $i \in \{1, 2, 3\}$ is true, including the case were $\varphi_3 = \frac{\pi}{2}$. Hence, the following inequality must be met:

$$\frac{\pi}{6} = \left|\frac{\pi}{3} - \varphi_3\right| < C(\varepsilon, y)\sqrt{s} < C(\varepsilon, y)\sqrt{D}.$$

However, this inequality is violated by imposing the following restriction on D:

$$D < \left(\frac{\pi}{6C(\varepsilon, y)}\right)^2. \tag{12}$$

Thus, with the fulfillment of (10), (11), and (12), inequality (6) in Case 4 cannot be met.

We have considered various configurations of positive angles ψ_1, ψ_2, and ψ_3 provided that $\sum_{i=1}^{3} \psi_i = \pi$. Consider inequalities whose fulfillment we wanted. Initially we stated that $\varepsilon \in \left(0, \frac{\pi}{8}\right)$ and the fulfillment of inequalities (10), (11), and (12) is also necessary. Not that all inequalities on ε are strict and are fulfilled when $\varepsilon = 0$, and include only constants that depend on y and ε; hence, there exists a suitable ε depending only on y. All inequalities are strict as well, hold for $D = 0$, and contain only constants depending on ε and y. Therefore, for the ε chosen earlier, there exist some $D > 0$ that satisfies inequalities (10), (11), and (12). With these ε and D, the fulfillment of (6) is impossible in Cases 1, 3, and 4, and in Case 2 entails the conclusion of Lemma B.

Thus, we have proved the analogs of Lemmas 1 and 2 from the first part of the study. The final part of the proof or Theorem A repeats the proof of Theorems 3 and 4 from [1].

Remark 2. From Proposition 3 it follows that for $y > 4$, the maximum of the generalized perimeter of a triangle will be achieved on a configuration that represents a pair of diametrically opposed points; the third point will coincide with one of them. What happens when $y \in (2, 4]$ remains an open question.

CONCLUSIONS

In this series of two papers, we consider the limit behavior of the extremum values of a random generalized perimeter. We studied this problem in the cases where an extremum in its determined analog is achieved only on a regular polygon. In the cases where this is managed to be proven (namely, when $y \leq 1$ for a polygon and when $y \in [1, 2]$ for a triangle), the limit theorems for the corresponding U-max-statistics are obtained. The subsequent study of the limit behavior of a generalized perimeter depends on the structure of polygons on which its extremum is attained.

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