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MATHEMATICS

Limit Theorems for Generalized Perimeters of Random Inscribed Polygons. I

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Abstract—Recently, W. Lao and M. Mayer (2008) developed *U*-max-statistics, where instead of averaging the values of the kernel over various subsets, the maximum of the kernel is considered. Such statistics often appear in stochastic geometry. Their limit distributions are related to the distributions of extreme values. In this paper, we begin to consider the limit theorems for the generalized perimeter (the sum of side powers) of a random inscribed polygon and *U*-max-statistics related to it. We describe extreme values of the generalized perimeter and obtain limit theorems for the cases where the side powers involved in determining the generalized perimeter do not exceed 1.

Keywords: *U*-max-statistics, Poisson approximation, distribution on circumference, generalized perimeter.

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1. INTRODUCTION

In this paper, we consider the limiting behavior of *U*-max-statistics appearing in stochastic geometry. Suppose $\xi_1, \xi_2, \ldots$ is a sequence of independent equally distributed random variables with values in measurable space $(\mathfrak{X}, \mathfrak{A})$, and the real-valued symmetrical Boreal function $h(x_1, \ldots, x_m)$ called an m-degree kernel is defined in space $\mathfrak{X}^m$.

U-max-statistics are determined as follows:

$$H_n = \max_J h(\xi_{i_1}, \ldots, \xi_{i_m}),$$

where $n \geq m$ and the set $J = \{(i_1, \ldots, i_m) : 1 \leq i_1 < \ldots < i_m \leq n\}$ is the set of ordered m-elemental permutations with many indices from the set $\{1, \ldots, n\}$. *U*-min-statistics are determined analogously.

U-max-statistics are introduced independently by Lao and Mayer in their theses [1] and [2]. They developed a method of studying the limiting behavior of *U*-max-statistics using the Poisson approximation from the monograph [3]. Their basic limit theorem from [4] is presented as follows.

Theorem 1. *Consider the U-max-statistics* $H_n = \max_J h(\xi_{i_1}, \ldots, \xi_{i_m})$ *introduced above and determine the following functions for each* $z \in \mathbb{R}$:

$$p_z = \mathbb{P}\{h(\xi_1, \ldots, \xi_m) > z\}, \quad \lambda_{n,z} = \binom{n}{m} p_z,$$

$$\tau_z(r) = \frac{\mathbb{P}\{h(\xi_1, \ldots, \xi_m) > z, h(\xi_{1+m-r}, \xi_{2+m-r}, \ldots, \xi_{2m-r}) > z\}}{p_z};$$

then for all $n \geq m$ *and for all* $z \in \mathbb{R}$ *the following inequality is true*:

$$\left|\mathbb{P}(H_n \leq z) - e^{-\lambda_{n,z}}\right| \leq (1 - e^{-\lambda_{n,z}})\left[p_z\left(\binom{n}{m} - \binom{n-m}{m}\right) + \sum_{r=1}^{m-1}\binom{m}{r}\binom{n-m}{m-r}\tau_z(r)\right].$$

In [5], Silverman and Brown offer the conditions under which the general theorem proved in [4], leads in the limit to the nonsingular Weibull law.

Theorem 2. *In terms of Theorem* 1, *if for a certain sequence of transformations* $z_n : T \to \mathbb{R}$, $T \subset \mathbb{R}$ *for each* $t \in T$ *the conditions*

$$\lim_{n\to\infty} \lambda_{n,z_n(t)} = \lambda_t > 0, \tag{1}$$

$$\lim_{n\to\infty} n^{2m-1} p_{z_n(t)} \tau_{z_n(t)}(m-1) = 0 \tag{2}$$

are fulfilled, the equality $\lim_{n\to\infty}\mathbb{P}(H_n \le z_n(t)) = e^{-\lambda_t}$ *for each* $t \in T$ *holds.*

Remark 1. If $m \ge 2$, condition (2) can be replaced with

$$\lim_{n\to\infty} n^{2m-r}\mathbb{P}\{h(\xi_1,\dots,\xi_m) > z_n(t), h(\xi_{1+m-r},\xi_{2+m-r},\dots,\xi_{2m-r}) > z_n(t)\} = 0 \tag{3}$$

for each $r \in \{1, \dots, m-1\}$.

The main application field of the concept of U-max-statistics is currently related to stochastic geometry: we talk about maximum areas, perimeters, volumes, and similar metric characteristics of figures built by a set of random points on a plane or in spaces of a larger dimension. The form of a geometric figure is determined by the kernel h. Lao and Mayer consider kernels only of small degrees; mostly, they study triangles. Koroleva and Nikitin in [6] pass on to U-max-statistics of a more complex nature. In particular, they consider the maximum perimeter among all perimeters of convex m-sided polygons whose vertices are chosen among n points that are independent and uniformly distributed on a circumference. This study summarizes and develops the results of [6].

2. GENERALIZED PERIMETER OF A POLYGON

Consider the generalization of the concept of a perimeter. Assume that d_i, $i = 1, \dots, m$ are the sides of an inscribed m-sided polygon with vertices $U_1, \dots, U_m$ and suppose $h(U_1, \dots, U_m) = \sum_{i=1}^m d_i$ is its perimeter. We propose to consider the value $h_y(U_1, \dots, U_m) = \sum_{i=1}^m d_i^y$ for $y \in \mathbb{R}$ and call it the generalized perimeter of a polygon.

It is well known that among all the convex inscribed polygons, the largest perimeter (and the square area) has a regular polygon (see, e.g., ([7], Problem 57a)). This was known at least to Legendre [8].

It turns out that this property is retained also for generalized perimeters when $0 < y < 1$, while for $y < 0$, in contrast, on a regular polygon, the minimum of the generalized perimeter is achieved. We have not found evidence of this fact, which is of independent interest, in the literature. Therefore, we present the proof of this fact here.

Lemma 1. *Assume that there exist m points* $V_1, \dots, V_m$ *lying on a unit circumference. Then for* $y < 0$ *the function* $h_y(V_1, \dots, V_m)$ *attains a minimum only at the vertices of a regular m-sided polygon and its minimum value is equal to* $2^y m \sin^y\left(\frac{\pi}{m}\right)$.

Analogously, when $y \in (0, 1)$, *the function* h_y *attains its maximum only at the vertices of a regular m-sided polygon and its maximum value is again equal to* $2^y m \sin^y\left(\frac{\pi}{m}\right)$.

Proof. First, consider the case of negative y. Without detracting from the generality, assume that the points $V_1, \dots, V_m$ are located on the circumference in the assigned order; then the function h_y is presented as $h_y(V_1, \dots, V_m) = \sum_{j=1}^m |V_jV_{j+1}|^y$, where $V_{m+1} = V_1$.

Introduce the notation $\gamma_i = \angle V_iOV_{i+1}$, where $i \in \{1, \dots, m\}$. We have $|V_iV_{i+1}| = 2\sin\left(\frac{\gamma_i}{2}\right)$, where $i \in \{1, \dots, m\}$. It follows that

$$h_y(V_1,\dots,V_m) = 2^y\sum_{i=1}^m\left(\sin\left(\frac{\gamma_i}{2}\right)\right)^y.$$

We can assume that $\gamma_i \in (0, 2\pi)$, otherwise $h_y = \infty$. Consider the function $f(x) = \sin^y x$ for $x \in (0, \pi)$. We find its first two derivatives

$$f'(x) = y\sin^{y-1}x\cos x, \quad f''(x) = y\sin^{y-2}x(y\cos^2 x - 1). \tag{4}$$

It is clear that for negative y this function is strictly convex on $(0, \pi)$. Then by Jensen's inequality for any $y_1, \ldots, y_n \in (0, \pi)$, we obtain

$$\alpha_1 f(y_1) + \ldots + \alpha_n f(y_n) \geq f(\alpha_1 y_1 + \ldots + \alpha_n y_n)$$

provided $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i > 0$. Note that

$$h_y(V_1, \ldots, V_m) = 2^y \sum_{i=1}^{m} \left(\sin\left(\frac{\gamma_i}{2}\right)\right)^y \geq 2^y m f\left(\frac{\gamma_1 + \ldots + \gamma_m}{2m}\right) = 2^y m f\left(\frac{\pi}{m}\right).$$

At the same time, because of the strict convexity, the equality is achieved only in the case where all γ_i are identical. Hence, in the case where $y < 0$, the function $h_y(V_1, \ldots, V_m)$ attains the minimum equal to $2^y m f\left(\frac{\pi}{m}\right) = 2^y m \sin^y\left(\frac{\pi}{m}\right)$ only at the vertices of a regular m-sided polygon.

The case where $0 < y < 1$ is considered analogously. If $y \in (0, 1)$, then $f''(x) < 0$; therefore, the function is strictly concave. By similar reasons, the function h_y attains its maximum only at the vertices of a regular m-sided polygon.

Suppose the points $U_1, \ldots, U_n$ are independently and uniformly distributed on a unit circumference S_1. Then the following two limit theorems hold; they are the main results of this study.

Theorem 3. *Suppose* $H_{n,m}^y = \min_{1 \leq i_1 < \ldots < i_m \leq n} h_y(U_{i_1}, \ldots, U_{i_m})$. *Then when* $y < 0$, *for any* $t > 0$ *the following equality holds*:

$$\lim_{n\to\infty} \mathbb{P}\left\{ n^{\frac{2m}{m-1}} \left(H_{n,m}^y - m2^y \left(\sin\frac{\pi}{m}\right)^y \right) \leq t \right\} = 1 - e^{-\frac{t^{\frac{m-1}{2}}}{K_1(y)}},$$

where $K_1(y) = m^{\frac{3}{2}} \Gamma\left(\frac{m+1}{2}\right) \left(-2^{y-1} y\pi \left(\sin\frac{\pi}{m}\right)^{y-2} \left(1 - y\cos^2\frac{\pi}{m}\right) \right)^{\frac{m-1}{2}}$.

Theorem 4. *Suppose* $G_{n,m}^y = \max_{1 \leq i_1 < \ldots < i_m \leq n} h_y(U_{i_1}, \ldots, U_{i_m})$. *Then when* $y \in (0, 1)$, *for any* $t > 0$ *the following equality holds*:

$$\lim_{n\to\infty} \mathbb{P}\left\{ n^{\frac{2m}{m-1}} \left(m2^y \left(\sin\frac{\pi}{m}\right)^y - G_{n,m}^y \right) \leq t \right\} = 1 - e^{-\frac{t^{\frac{m-1}{2}}}{K_2(y)}},$$

where $K_2(y) = m^{\frac{3}{2}} \Gamma\left(\frac{m+1}{2}\right) \left(2^{y-1} y\pi \left(\sin\frac{\pi}{m}\right)^{y-2} \left(1 - y\cos^2\frac{\pi}{m}\right) \right)^{\frac{m-1}{2}}$.

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 3 is similar to the corresponding proof in [6] but has several distinctions.

Consider the points $U_1, \ldots, U_m \in S_1$. From Lemma 1 it follows that the minimum of the function h_y is $2^y m \sin^y\left(\frac{\pi}{m}\right)$ and is achieved only at the vertices of a regular m-sided polygon. Introduce the notation $\beta_i = \angle U_1 O U_{i+1}$ (in the counterclockwise direction). Analogously to the work [6], when $z > 2^y m \sin^y\left(\frac{\pi}{m}\right)$, we obtain

$$\mathbb{P}\{h_y(U_1, \ldots, U_m) < z\} = (m-1)!\,\mathbb{P}\{h_y(U_1, \ldots, U_m) < z, \beta_1 < \beta_2 < \ldots < \beta_{m-1}\}. \quad (5)$$

In what follows, we assume that $U_1, \ldots, U_m$ lie on the circumference in this order in the counterclockwise direction.

We call the angles of the form $\angle U_i O U_{i+1}$ central angles. Consider the largest and smallest central angles. Assume that these are the angles $\frac{2\pi}{m} + 2\varphi_1$ and $\frac{2\pi}{m} - 2\varphi_2$ where $\varphi_1, \varphi_2 > 0$.

Lemma 2. *Suppose the inequality* $h_y(U_1, \ldots, U_m) < 2^y m \sin^y \frac{\pi}{m} + s$ *holds. Then there are constants* $C, D > 0$ *depending only on y such that for* $0 < s < D$ *we have* $\varphi_1, \varphi_2 < C\sqrt{s}$. *In other words, with small s, the central angles differ from* $\frac{2\pi}{m}$ *by* $O(\sqrt{s})$.

Proof. Consider another polygon, which has the same central angles but these two specified angles are adjacent to each other. Then the lengths of the sides of the polygon and values of the function h_y do not vary, and the following inequality holds:

$$2^y m \sin^y\left(\frac{\pi}{m}\right) \le h_y(U_1,\ldots,U_m) < 2^y m \sin^y\left(\frac{\pi}{m}\right) + s.$$

Replace the central angles $\frac{2\pi}{m} + 2\varphi_1$ and $\frac{2\pi}{m} - 2\varphi_2$ by two identical angles $\frac{2\pi}{m} + \varphi_1 - \varphi_2$. With this replacement, the function value could be reduced no more than by s; therefore,

$$2^y\left(\sin^y\left(\frac{\pi}{m} + \varphi_1\right) + \sin^y\left(\frac{\pi}{m} - \varphi_2\right) - 2\sin^y\left(\frac{\pi}{m} + \frac{\varphi_1 - \varphi_2}{2}\right)\right) < s. \tag{6}$$

Consider one of the terms. The Taylor expansion with the remainder in the Lagrange form gives

$$\sin^y\left(\frac{\pi}{m} \pm \varphi_i\right) = \sin^y\left(\frac{\pi}{m} + \frac{\varphi_1 - \varphi_2}{2}\right)$$
$$\pm\, y\left(\sin\left(\frac{\pi}{m} + \frac{\varphi_1 - \varphi_2}{2}\right)\right)^{y-1} \cos\left(\frac{\pi}{m} + \frac{\varphi_1 - \varphi_2}{2}\right)\left(\frac{\varphi_1 + \varphi_2}{2}\right)$$
$$+ \frac{1}{2} y(\sin\theta_i)^{y-2}(y\cos^2\theta_i - 1)\left(\frac{\varphi_1 + \varphi_2}{2}\right)^2,$$

where $i \in \{1, 2\}$ and $\theta_i \in \left[\frac{\pi}{m} - \varphi_2, \frac{\pi}{m} + \varphi_1\right]$. Consequently, due to (6) the following inequality holds:

$$s > 2^y\left(\sin^y\left(\frac{\pi}{m} + \varphi_1\right) + \sin^y\left(\frac{\pi}{m} - \varphi_2\right) - 2\sin^y\left(\frac{\pi}{m} + \frac{\varphi_1 - \varphi_2}{2}\right)\right)$$
$$= 2^{y-2}\left(\frac{1}{2} y(\sin\theta_1)^{y-2}(y\cos^2\theta_1 - 1) + \frac{1}{2} y(\sin\theta_2)^{y-2}(y\cos^2\theta_2 - 1)\right)(\varphi_1 + \varphi_2)^2.$$

Note that when $y < 1$ and $x \in (0, \pi)$ we have $(\sin x)^{y-2} \ge 1$ and $y\cos^2 x - 1 < 0$, along with the estimate $|y\cos^2 x - 1| > \min(|y - 1|, 1)$. Hence, for $y < 0$ the constant in $(\varphi_1 + \varphi_2)^2$ can be estimated from below as $C_1(y) > 0$. Consequently, $C_1(y)\,(\varphi_1 + \varphi_2)^2 < s$ and then $\varphi_1 + \varphi_2 < \frac{1}{\sqrt{C_1(y)}}\sqrt{s}$, and this proves Lemma 2.

In what follows, we need the following proposition.

Lemma 3. *In terms of Theorem* 1, *it is true that*

$$\lim_{s\to 0} s^{-\frac{m-1}{2}} \mathbb{P}\left\{h_y(U_1,\ldots,U_m) < 2^y m \sin^y \frac{\pi}{m} + s\right\} = \frac{\Gamma(m+1)}{K_1(y)}.$$

Proof. Introduce the notation $\alpha_k = \beta_k - \frac{2\pi k}{m}$, $\alpha_0 = 0$, and $\alpha_m = 0$. Recall that β_i is uniformly and independently distributed on the interval $[0, 2\pi]$. Hence, α_k are independent and uniformly distributed on the intervals $\left[-\frac{2\pi k}{m}, 2\pi - \frac{2\pi k}{m}\right]$, respectively. In the new designations, the function h_y is presented as follows:

$$h_y(U_1,\ldots,U_m) = 2^y \sum_{k=1}^{m}\left(\sin\left(\frac{\pi}{m} + \frac{\alpha_k - \alpha_{k-1}}{2}\right)\right)^y.$$

If for small s the condition $h_y(U_1, \ldots, U_m) < 2^y m \sin^y \frac{\pi}{m} + s$ holds, then by Lemma 2 we have $\alpha_k = O(\sqrt{s})$. This allows us to expand each term by the Taylor formula with the remainder in the form of Lagrange at the point $\frac{\pi}{m}$. We use (4), together with the third-derivative formula for $\sin^y(x)$. To shorten the writing, we put $\Delta_k := \frac{\alpha_k - \alpha_{k-1}}{2}$. We obtain

$$2^y \sum_{k=1}^{m} \sin^y\left(\frac{\pi}{m} + \Delta_k\right) = 2^y m \sin^y\left(\frac{\pi}{m}\right) + 2^y y\left(\sin\left(\frac{\pi}{m}\right)\right)^{y-1} \cos\left(\frac{\pi}{m}\right) \sum_{k=1}^{m} \Delta_k$$

$$+ 2^{y-1} y\left(\sin\left(\frac{\pi}{m}\right)\right)^{y-2}\left(y\cos^2\left(\frac{\pi}{m}\right) - 1\right)\sum_{k=1}^{m}(\Delta_k)^2$$

$$+ \sum_{k=1}^{m} \frac{2^{y-1}}{3} y(\sin(\eta_k))^{y-3}\cos(\eta_k)(y^2\cos^2(\eta_k) - 3y + 2)(\Delta_k)^3,$$

where $\eta_k \in \left[\frac{\pi}{m} - 2C\sqrt{s}, \frac{\pi}{m} + 2C\sqrt{s}\right]$ by Lemma 2.

Since the linear term disappears, the condition $h(U_1, \dots, U_m) < 2^y m \sin^y \frac{\pi}{m} + s$ is equivalent to the relation

$$2^{y-1} y\left(\sin\left(\frac{\pi}{m}\right)\right)^{y-2}\left(y\cos^2\left(\frac{\pi}{m}\right) - 1\right)\sum_{k=1}^{m}(\Delta_k)^2$$

$$+ \sum_{k=1}^{m} \frac{2^{y}}{6} y(\sin(\eta_k))^{y-3}\cos(\eta_k)(y^2\cos^2(\eta_k) - 3y + 2)(\Delta_k)^3 < s.$$

In turn, this is equivalent to the inequality

$$4\sum_{k=1}^{m}(\Delta_k)^2 + \frac{1}{D}\sum_{k=1}^{m}\frac{2^y}{6} y(\sin(\eta_k))^{y-3}\cos(\eta_k)(y^2\cos^2(\eta_k) - 3y + 2)(\Delta_k)^3 < \frac{s}{D}, \tag{7}$$

where

$$D = 2^{y-3} y\left(\sin\left(\frac{\pi}{m}\right)\right)^{y-2}\left(y\cos^2\left(\frac{\pi}{m}\right) - 1\right). \tag{8}$$

Consider the coefficient in the kth summand of the third order

$$2^y \frac{1}{6} y(\sin(\eta_k))^{y-3}\cos(\eta_k)(y^2\cos^2(\eta_k) - 3y + 2).$$

Because $\left|\eta_k - \frac{\pi}{m}\right| < 2C\sqrt{s}$, for small s this coefficient in modulus is smaller than

$$\frac{2^y}{6} y\left(\sin\left(\frac{\pi}{2m}\right)\right)^{y-3} \cdot 1 \cdot \max(|y^2 - 3y + 2|, |-3y + 2|) := T;$$

therefore, the left-hand side in (7) can be estimated as follows:

$$4\sum_{k=1}^{m}(\Delta_k)^2 + \frac{1}{D}\sum_{k=1}^{m}\frac{2^y}{6} y(\sin(\eta_k))^{y-3}\cos(\eta_k)(y^2\cos^2(\eta_k) - 3y + 2)(\Delta_k)^3$$

$$\le 4\sum_{k=1}^{m}(\Delta_k)^2 + \frac{T}{D}\sum_{k=1}^{m}(|\Delta_k|)^3 \le 4\sum_{k=1}^{m}(\Delta_k)^2 + \frac{T}{D}\max|\Delta_k|\sum_{k=1}^{m}(\Delta_k)^2$$

$$\le \left(1 + \frac{TC}{2D}\sqrt{s}\right)\sum_{k=1}^{m}(\alpha_k - \alpha_{k-1})^2.$$

Thus, we have

$$\mathbb{P}\left\{h_y(U_1, \dots, U_m) < 2^y m \sin^y \frac{\pi}{m} + s, \beta_1 < \beta_2 < \dots < \beta_{m-1}\right\}$$

$$\geq \mathbb{P}\left\{\sum_{k=1}^{m}(\alpha_k-\alpha_{k-1})^2 < \frac{s}{D\left(1+\frac{TC}{2D}\sqrt{s}\right)},\ |\alpha_k|<2C\sqrt{s}\right\}.$$

Analogously, we can obtain the inequality

$$\mathbb{P}\left\{h_y(U_1,\ldots,U_m) < 2^y m\sin^y\frac{\pi}{m}+s,\ \beta_1<\beta_2<\ldots<\beta_{m-1}\right\}$$

$$\leq \mathbb{P}\left\{\sum_{k=1}^{m}(\alpha_k-\alpha_{k-1})^2 < \frac{s}{D\left(1-\frac{DC}{2D}\sqrt{s}\right)},\ |\alpha_k|<2C\sqrt{s}\right\}.$$

Consider the quadratic form $Q(\alpha)=\frac{1}{2}\sum_{k=1}^{m}(\alpha_k-\alpha_{k-1})^2=\sum_{k=1}^{m-1}\alpha_k^2-\sum_{k=2}^{m-1}\alpha_k\alpha_{k-1}$. Applying the reasoning from [6] and with the help of orthogonal transformation replace this quadratic form by the quadratic form $W(Y)=\sum_{k=1}^{m-1}\lambda_k Y_k^2$, where $\lambda_k = 1-\cos\frac{\pi k}{m}$, $k = 1, \ldots, m-1$. It is clear that all $\lambda_k > 0$; hence, the condition $W(Y) < s$ implies the condition $Y_k = O(\sqrt{s})$. Therefore, $\alpha_k = O(\sqrt{s})$. It follows that there exists a constant L depending only on the quadratic form Q such that $\alpha_k < L\sqrt{s}$. We can consider that $L \leq 2C$; hence, the condition of the smallness of α_k can be removed from under the probability sign. Transforming the previous relations, we obtain the inequality

$$\mathbb{P}\left\{Q(\alpha) < \frac{s}{2D\left(1+\frac{TC}{2D}\sqrt{s}\right)}\right\}$$

$$\leq \mathbb{P}\left\{h_y(U_1,\ldots,U_m) < 2^y m\sin^y\frac{\pi}{m}+s,\ \beta_1<\beta_2<\ldots<\beta_{m-1}\right\} \tag{9}$$

$$\leq \mathbb{P}\left\{Q(\alpha) < \frac{s}{2D\left(1-\frac{TC}{2D}\sqrt{s}\right)}\right\}.$$

We estimate the left-hand and right-hand sides of the inequality using the following lemma. The proof of this lemma is analogous to the reasoning from ([6], p. 104).

Lemma 4. *For any positive* $F > \delta > 0$ *and* $0 < s < Z(\delta)$ *the following equality holds*:

$$\mathbb{P}\left\{Q(\alpha) < \frac{s}{F}\right\} = \frac{s^{\frac{m-1}{2}}}{(2\pi F)^{\frac{m-1}{2}} m\sqrt{m}\,\Gamma\left(\frac{m+1}{2}\right)},$$

where $Z(\delta)$ *is a certain positive constant depending only on* δ.

We put in formula (9) the result of Lemma 4 and obtain the two-sided inequality

$$\frac{s^{\frac{m-1}{2}}}{\left(4\pi D\left(1+\frac{TC}{2D}\sqrt{s}\right)\right)^{\frac{m-1}{2}}\Gamma\left(\frac{m+1}{2}\right)\sqrt{m}}$$

$$\leq \mathbb{P}\left\{h_y(U_1,\ldots,U_m) < 2^y m\sin^y\frac{\pi}{m}+s, \beta_1<\beta_2<\ldots<\beta_{m-1}\right\}$$

$$\leq \frac{s^{\frac{m-1}{2}}}{\left(4\pi D\left(1-\frac{TC}{2D}\sqrt{s}\right)\right)^{\frac{m-1}{2}}\Gamma\left(\frac{m+1}{2}\right)\sqrt{m}}.$$

Remove the conditions of ordering angles using condition (5) and also divide by $s^{\frac{m-1}{2}}$. We obtain the new two-sided inequality

$$\frac{(m-1)!}{\left(4\pi D\left(1+\frac{TC}{2D}\sqrt{s}\right)\right)^{\frac{m-1}{2}}\Gamma\left(\frac{m+1}{2}\right)\sqrt{m}} \le s^{-\frac{m-1}{2}}\mathbb{P}\left\{h_y(U_1,\dots,U_m) < 2^y m\sin^y\frac{\pi}{m}+s\right\}$$

$$\le \frac{(m-1)!}{\left(4\pi D\left(1-\frac{TC}{2D}\sqrt{s}\right)\right)^{\frac{m-1}{2}}\Gamma\left(\frac{m+1}{2}\right)\sqrt{m}}.$$

Let s tend to 0. It is clear that the upper and lower estimates join each other; hence,

$$\lim_{s\to 0} s^{-\frac{m-1}{2}}\mathbb{P}\left\{h_y(U_1,\dots,U_m) < 2^y m\sin^y\frac{\pi}{m}+s\right\} = \frac{(m-1)!}{(4\pi D)^{\frac{m-1}{2}}\Gamma\left(\frac{m+1}{2}\right)\sqrt{m}}.$$

By substituting the value of D from (8), we obtain the result of Lemma 3.

Consider for $t > 0$ the function $z_n = 2^y m\sin^y\frac{\pi}{m} + tn^{-\frac{2m}{m-1}}$. Then

$$\lambda_{n,z_n(t)} = \binom{n}{m}\mathbb{P}\left\{h_y(U_1,\dots,U_m) < 2^y m\sin^y\frac{\pi}{m}+tn^{-\frac{2m}{m-1}}\right\}.$$

Suppose $s = tn^{-\frac{2m}{m-1}}$; then $n^m s^{\frac{m-1}{2}} = t^{\frac{m-1}{2}}$. Check condition (1) from Theorem 2:

$$\lim_{n\to\infty}\lambda_{n,z_n(t)} = \lim_{n\to\infty}\frac{n!}{m!(n-m)!}\mathbb{P}\left\{h_y(U_1,\dots,U_m) < 2^y m\sin^y\frac{\pi}{m}+s\right\}$$

$$= \frac{1}{m!}\lim_{n\to\infty}\frac{n!}{n^m(n-m)!}n^m s^{\frac{m-1}{2}} s^{-\frac{m-1}{2}}\mathbb{P}\left\{h_y(U_1,\dots,U_m) < 2^y m\sin^y\frac{\pi}{m}+s\right\}$$

$$= \frac{1}{m!}t^{\frac{m-1}{2}}\lim_{n\to\infty}\left(tn^{-\frac{2m}{m-1}}\right)^{-\frac{m-1}{2}}\mathbb{P}\left\{h_y(U_1,\dots,U_m) < 2^y m\sin^y\frac{\pi}{m}+tn^{-\frac{2m}{m-1}}\right\}$$

$$= \frac{1}{m!}t^{\frac{m-1}{2}}\frac{\Gamma(m+1)}{K_1(y)} = \frac{t^{\frac{m-1}{2}}}{K_1(y)} =: \lambda_t > 0.$$

The last limiting transition was performed due to Lemma 3. We prove that condition (3) also holds.

Lemma 5. *For any* $r \in \{1, \dots, m-1\}$, *the following relation holds*:

$$\lim_{n\to\infty} n^{2m-r}\mathbb{P}\{h_y(U_1,\dots,U_m) < z_n(t),\ h_y(U_{1+m-r},\dots,U_{2m-r}) < z_n(t)\} = 0.$$

Proof. Introduce the notation: $\beta_i = \angle U_1 O U_{i+1}$ when $i \in \{1, \dots, 2m-r-1\}$ and $\gamma_i = \angle U_{m-r+1} O U_{i+1}$ when $i \in \{m-r+1, \dots, 2m-r-1\}$. It is clear that when $i \ge m$, we have $\gamma_i = (\beta_i - \beta_{m-r}) \bmod 2\pi$. Suppose

$I = (i_1, \ldots, i_{m-1})$ and $J = (j_1, \ldots, j_{m-1})$ are two permutations of $m-1$ elements. Introduce the event $Q = \{\beta_{i_1} < \ldots < \beta_{i_{m-1}}, \gamma_{m-r+j_1} < \ldots < \gamma_{m-r+j_{m-1}}\}$ and estimate the probability

$$\mathbb{P}\{[(h_y(U_1,\ldots,U_m) < z_n(t)) \cap (h_y(U_{1+m-r},\ldots,U_{2m-r}) < z_n(t))] \cap Q\}. \tag{10}$$

By Lemma 2 we have $\left|\beta_{i_k} - \frac{2\pi k}{m}\right| < 2C\sqrt{s}$ for $i < m$. Analogously we obtain $\left|\gamma_{m-r+j_k} - \frac{2\pi k}{m}\right| < 2C\sqrt{s}$. Denote by l the number of the position of the element $m-r$ in the permutation I and by t_i, the number of the position of γ_i in the permutation J. Then for $i \geq m$ we obtain the inequality

$$\left|\beta_i - \frac{2\pi(l+t_i)}{m}\right| \leq \left|\beta_i - \beta_{m-r} - \frac{2\pi t_i}{m}\right| + \left|\beta_{m-r} - \frac{2\pi l}{m}\right| < 4C\sqrt{s}.$$

Therefore, probability (10), due to the distribution of β_i, can be estimated as $\left(\frac{4C\sqrt{s}}{2\pi}\right)^{m-1}\left(\frac{8C\sqrt{s}}{2\pi}\right)^{m-r}$.

Summing this estimate by all permutations and substituting $s = tn^{-\frac{2m}{m-1}}$, we obtain the inequality

$$n^{2m-r}\mathbb{P}\{h_y(U_1,\ldots,U_m) < z_n(t),\ h_y(U_{1+m-r},\ldots,U_{2m-r}) < z_n(t)\}$$

$$\leq n^{2m-r}((m-1)!)^2\left(\frac{4C\sqrt{tn^{-\frac{2m}{m-1}}}}{2\pi}\right)^{m-1}\left(\frac{8C\sqrt{tn^{-\frac{2m}{m-1}}}}{2\pi}\right)^{m-r} = O\left(n^{-\frac{m-r}{m-1}}\right) = o(1).$$

Return to the proof of Theorem 3. We can use Theorem 2, because we proved that its terms are met. Then $\lim_{n\to\infty}\mathbb{P}(H^y_{n,m} \geq z_n(t)) = e^{-\lambda_t}$ for any $t \in T$. Consequently,

$$\lim_{n\to\infty}\mathbb{P}\left(H^y_{n,m} \geq 2^y m \sin^y\frac{\pi}{m} + tn^{-\frac{2m}{m-1}}\right) = e^{-\frac{t^{\frac{m-1}{2}}}{K_1(y)}}.$$

Hence, for any $t > 0$ it is true that

$$\lim_{n\to\infty}\mathbb{P}\left\{n^{\frac{2m}{m-1}}\left(H^y_{n,m} - 2^y m \sin^y\frac{\pi}{m}\right) \leq t\right\} = 1 - e^{-\frac{t^{\frac{m-1}{2}}}{K_1(y)}}.$$

Theorem 3 is completely proved.

Proof of Theorem 4 is similar to the proof of Theorem 3 and therefore is omitted. The condition $y \in (0, 1)$ guarantees that the function $(\sin x)^y$ has the strictly negative second derivative on $[0, \pi]$ (it can be considered that at the ends of the segment the first two derivatives are determined by continuity); therefore, the strict convexity is replaced by the strict concavity. Then keep the reasoning (here, we change the signs in the probabilities, because now we consider the maximum, not the minimum).

Remark 2. These arguments are not applicable for $y = 1$, because the second derivative of the function $\sin^y x$ is not separated from 0 on $[0, \pi]$. However, [6] presents a method for the proof of Lemmas 1 and 2 without this fact. Therefore, the obtained constant is applicable also for $y = 1$.

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