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REVIEW

Isothermal Flows of Micropolar Liquids: Formulation of Problems and Analytical Solutions

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Abstract—Models for flows of a non-Newtonian liquid have been considered within the framework of the micropolar theory. Different forms of constitutive equations and boundary conditions have been compared. Available analytical solutions and possible applications of the micropolar theory have been reviewed. A mechanically substantiated formulation of the problem relevant to the flow of a micropolar liquid in a Brinkman porous medium has been considered. Formulations of the boundary problem have been proposed for a micropolar liquid flowing in a porous cell.

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INTRODUCTION

The study of the functional properties of new materials and the development of novel technologies at the micro- and nanolevel gave substantial impetus to the development of nonclassical hydrodynamic models and the investigation of non-Newtonian liquid flows. It has rather long ago been established that some flow properties cannot be explained in terms of the classical theory of Newtonian liquids. Abnormal variations in the rheological characteristics of liquids subjected to external electromagnetic actions, formation of thin films and boundary layers with properties distinctly depending on a characteristic size, phenomena of a drastic increase in a liquid flow velocity in thin capillaries, and deviations of filtration characteristics of membranes from those expected within the classical theory of Newtonian liquid filtration are among these properties.

Classical hydrodynamics deals with symmetric stress tensor and linear state equations. It means that, a stress tensor is linearly related to a deformation rate tensor, which is also symmetric. Medium particles are considered to be material points, their motion is completely described by a field of linear velocities, and the interaction between the medium particles is of a purely force character.

One of ways in which hydromechanics is developing is associated with the introduction of new rheological models and simulation of medium states by nonlinear stress–deformation rate dependences. Oldroyd's eight-constant model [1] seems to be the most complete nonlinear generalization of the state equation for a viscoelastic medium. It rather adequately describes the behavior of some non-Newtonian liquids, even when its markedly simplified versions are used (because its complete version is difficult to analyze).

Within the framework of another approach, an internal structure of a medium is considered taking into account additional (rotational) degrees of freedom of particles composing a continuum. Together with the proper microrotations of particles, the moment interaction of medium elements and external couple stresses that act on a medium together with external forces are taken into account. As a result, in addition to the tensor of force stresses, a couple-stress tensor arises in the medium, with both tensors being essentially asymmetric. Therefore, the micropolar-liquid theory is sometimes referred to as an "asymmetric theory." Note that the micropolar-liquid theory has a substantial potential as applied to describing colloidal systems, such as suspensions, emulsions, and foams.

In the course of the development of the elasticity theory, Voigt [2] was the first to try to take into account the moments of forces and introduce, in addition to the tensor of force stresses, the couple-stress tensor. However, the complete theory of asymmetric elasticity and a medium with couple stresses and a microstructure was formulated by the Cosserat brothers [3]. Unfortunately, this fundamental work did not receive proper recognition during their lifetimes. This study comprised a generalized theory of mechanics, optics, and electrodynamics; however, numerous material constants and characteristics of the new medium remained unknown. The theory was nonlinear, because it took into account arbitrary deformations, while the mathematical apparatus was rather cumbersome and difficult to use for engineering problems.

The Cosserat model was developed and began to be widely used in the 1960s, after the publication of a number of articles devoted to the asymmetric-elasticity theory followed by the development of the polar-liquid theory. The works have been reviewed and the micropolar-elasticity theory has been systematically described in monograph by Nowacki [4]. A mathematical apparatus for asymmetric hydrodynamics was proposed by Aero et al. [5]; however, Eringen's works have become more popular in this field [6, 7]. Later, he extended this theory and combined it with asymmetric models of elastic media [8, 9]. A rigorous mathematical description of the theory and applications of micropolar liquids was reported by Lukaszewicz in [10]. The theoretical foundations and applications of both the micropolar-elasticity theory and micropolar hydrodynamics have been reviewed in Pabst's work [11] written in the engineering style.

The importance of correct use of the relevant terminology should be noted. The Newtonian-liquid model may be developed in two ways. The first one entails allowance for the force moments acting in a medium and the introduction of a couple-stress tensor in addition to the tensor of force stresses. Such media are referred to as "fluids with couple stresses." The second direction consists in allowance for the existence of an internal microstructure in a medium, with this structure implying rotational motions of microelements, as well as deformations and stresses independent of translational motion. These liquids are referred to as "microstructured" ones and may be considered with no allowance for moments of forces. The most complete models comprise both moments of forces and medium microstructure. Depending on the number of the degrees of freedom that are imparted to medium microparticles, micromorphic, microstretch, and micropolar media are distinguished. Particles composing a micromorphic medium may perform translational and rotational motions and be subjected to diverse deformations. Microstretch media, which consist of particles performing translational and rotational motions, are subjected only to volume deformations; i.e., they "breathe." Micropolar media are composed of absolutely solid particles with three translational and three rotational degrees of freedom. All three of the aforementioned microstructured media were introduced by Eringen in the 1960s, and a hierarchical description and generalization of the theory of such media are given in his two-volume book [8, 9]. A unified description and the comparison between the theories of liquids with couple stresses and microstructured liquids are presented in monograph by Stokes [12]. Corresponding analytical solutions for classical problems, i.e., planar and cylindrical flows and flows around spheres, may be found in the same work.

Many studies have been devoted to the simulation of heat-conducting micropolar liquids. Dieppe and Listrov [13] revealed the following thermomechanical effect: in a liquid with a nonuniform temperature field, a relation arises between a heat flux and couple stresses. This leads to a transformation of the equation of moments and the appearance of new terms in the equation of heat conductivity, with these terms being related to microrotations. The theory of heat conductivity of micropolar liquids was also developed by Eringen in 1972 [14]. This theory is rather complex for practical application; therefore, a simplified linear version is, as a rule, used. The problem of heat exchange in micropolar liquids has been considered in detail in monograph by Migun and Prokhorenko [15]. The present review is devoted to the hydrodynamics of isothermal micropolar liquids, for which thermal effects are ignored.

One of the most complex aspects of the development of the theory of microstructured media is the derivation of the state equation. The dependences of stress tensors on deformations or their rates (these dependences are now governed by not only linear, but also angular velocities), are, in the general case, nonlinear. Liquids characterized by linear dependences of stress tensors on measures of deformation rates are commonly referred to as simple microliquids. It has appeared that, within the model of simple micropolar liquids, many diverse problems may be solved analytically in spite of the fact that their theory is rather complex as compared with classical hydrodynamics. In this review, the attention is focused on specifically analytical solutions in terms of the theory of simple micropolar liquids. In the consideration, the accent will be put on the mechanical sense of the used notions and results, which is of importance for the correct use of the models in applications. Since the number of articles and books devoted to this field amounts to several hundred, we did not intend to establish an exact chronology of obtaining all presented solutions with assignment of priorities. Many solutions and their analysis and applications have been considered in reviews [16, 17] and monographs [8–10, 12, 15, 18, 19]. Less attention has, as yet, been focused on flows of micropolar liquids in porous media. In this work, a problem will be formulated concerning a filtration flow of a micropolar liquid in a porous medium and equations and boundary conditions will be derived for the problem of a conjugate flow in a porous cell composed of a solid core, a porous shell, and a region of free flow. In conclusion, main applications of the theory of micropolar liquids will be briefly considered.

MOTION AND STATE EQUATIONS FOR A MICROPOLAR MEDIUM

Before discussing available solutions and formulating new problems, let us pay our attention to the forms of the main equations describing the dynamics of an isothermal micropolar liquid. As in classical hydrodynamics, they are based on the mass conservation law and the Euler laws for an elementary volume of a medium. It should be emphasized that the energy-conservation law and entropy balance may be ignored in the system, provided that the specific internal energy is proportional to the temperature of a liquid, the Fourier law is fulfilled, and all material coefficients (viscosities) are constant [7, 9]. The aforementioned conditions are satisfied for isothermal flows, which will be considered below.

To determine surface forces $\mathbf{T}_n$ and moments of forces $\mathbf{M}_n$ applied in a medium to an elemental area with normal vector $\mathbf{n}$, stress tensor $\hat{\mathbf{t}}$ and couple-stress tensor $\hat{\mathbf{m}}$ are introduced according to the following rules:

$$\mathbf{T_n} = \mathbf{n} \cdot \hat{\mathbf{t}}, \quad \mathbf{M_n} = \mathbf{n} \cdot \hat{\mathbf{m}},$$

here and below, the dot denotes a scalar product. Hence,

$$(\mathbf{T_n})_j = n_i t_{ij}, \quad (\mathbf{M_n})_j = n_i m_{ij}.$$

As a result, we may write the differential form of the laws of motion, namely

continuity equation

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

equation of momentum

$$\rho\dot{\mathbf{v}} = \rho\mathbf{F} + \nabla \cdot \hat{\mathbf{t}}, \tag{2}$$

and equation of moment of momentum

$$\rho\hat{\mathbf{J}} \cdot \dot{\boldsymbol{\omega}} = -\hat{\boldsymbol{\varepsilon}} : \hat{\mathbf{t}} + \rho\mathbf{L} + \nabla \cdot \hat{\mathbf{m}}, \tag{3}$$

where $\mathbf{v}$ and $\boldsymbol{\omega}$ are the vectors of linear and angular velocities, respectively; ρ is the density of a liquid; $\hat{\mathbf{J}}$ is the inertia tensor, which, for an isotropic simple microfluid, has a spherical pattern; $\mathbf{F}$ and $\mathbf{L}$ are the densities of volume forces and moments of forces applied to the medium, respectively; $\hat{\boldsymbol{\varepsilon}}$ is the Levy–Civita tensor; the dot over a symbol denotes the total time derivative; and the colon denotes the convolution over two indices.

Let us discuss the derivation of the state equation for a micropolar medium at a qualitative level focusing our attention, primarily, on the mechanical interpretation of the used values. A rigorous mathematical derivation may be found in [7–9, 12, 20]. Within the theory of micropolar elasticity, this problem has been accurately described in Nowacki's book [4].

In order to write equations of state, it is necessary to realize which values may serve as the measures of deformation of a medium element and deformations of which types may occur in a medium with the properties under consideration. As a limiting case, it is desirable to obtain the Navier-Stokes law—$\hat{\mathbf{t}} = -p\hat{\mathbf{G}} + \hat{\boldsymbol{\tau}}$ used in the classical theory. Here, p is the hydrostatic pressure and $\hat{\mathbf{G}}$ is a metric tensor. Tangential-stress tensor $\hat{\boldsymbol{\tau}}$ is determined by the expression $\hat{\boldsymbol{\tau}} = \lambda(\mathrm{tr}\hat{\mathbf{D}})\hat{\mathbf{G}} + 2\mu\hat{\mathbf{D}}$, where μ and λ are the first and second viscosity coefficients of a liquid, respectively; $\hat{\mathbf{D}}$ is the tensor of the rates of deformations, with the components of this tensor in the Cartesian coordinates having the form of $\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$. As it has been shown in terms of classical thermodynamics, the deformation-rate tensor completely determines the deformation of a medium element. The diagonal and off-diagonal components of $\hat{\mathbf{D}}$ are responsible for variations in the volume and shape of the element, respectively. The linear relation to the stress tensor yields expressions for the normal and shear stresses of the medium element, respectively. At the same time, antisymmetric tensor $\hat{\mathbf{W}} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)$, which is referred to as the "vorticity tensor," determines the rotation of the medium element as a solid body, because the vector associated with it is equal to $\frac{1}{2}\mathrm{curl}\mathbf{v}$. In the presence of an independent rotational motion of medium elements with angular velocity $\boldsymbol{\omega}(\mathbf{x})$, tensor $\hat{\mathbf{W}}$ acquires a correction associated with this vector and takes the form of $\hat{\mathbf{W}} - \hat{\boldsymbol{\varepsilon}} \cdot \boldsymbol{\omega}$. Note that the sum of tensors $\hat{\mathbf{D}}$ and $\hat{\mathbf{W}}$ represents transposed velocity gradient $(\nabla\mathbf{v})^T$, while the components of the velocity gradient are determined as $(\nabla\mathbf{v})_{ij} = \partial v_i/\partial x_j$. Then, tensor $\hat{\boldsymbol{\gamma}} = (\nabla\mathbf{v})^T - \hat{\boldsymbol{\varepsilon}} \cdot \boldsymbol{\omega}$ characterizes the rates of the volume, shear, and rotational deformations of a micropolar medium. However, these are not all the possible types of deformation to which an element of a micropolar continuum may be subjected. Owing to the dependence of the angular velocity $\boldsymbol{\omega}(\mathbf{x})$ on the spatial coordinate, the medium element may undergo twisting and bending. The rates of these two types of deformation, which are inherent in micropolar media and absent in nonpolar liquids, are characterized by curvature–twist-rate tensor $\hat{\boldsymbol{\chi}} \equiv (\nabla\boldsymbol{\omega})^T, (\nabla\boldsymbol{\omega})_{ij} = \partial\omega_i/\partial x_j$, introduced in [20]. Tensors $\hat{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\chi}}$ determine the rates of all possible types of deformation in a micropolar liquid. Then, they should, in a linear manner, be related to tensors $\hat{\boldsymbol{\tau}}$ and $\hat{\mathbf{m}}$.

In the Eringen theory, the expression for the stress tensor is presented in the form of linear addends to the classical terms:

$$\hat{\boldsymbol{\tau}} = \lambda(\mathrm{tr}\hat{\mathbf{D}})\hat{\mathbf{G}} + 2\mu\hat{\mathbf{D}} + 2\mu_0(\hat{\boldsymbol{\gamma}} - \hat{\mathbf{D}}) + 2\mu_1(\hat{\boldsymbol{\gamma}}^T - \hat{\mathbf{D}}).$$

The introduction of the transposed tensors is necessary to ensure isotropy of the written function: the term proportional to $\mathrm{tr}(\hat{\boldsymbol{\gamma}} - \hat{\mathbf{D}})$ is absent, because it is equal to zero. In addition to the classical viscosities of a Newtonian liquid λ and μ, Eringen introduced the new viscosities μ_0 and μ_1. After a rearrangement of the terms, the stress tensor was represented as

$$\hat{\boldsymbol{\tau}} = \lambda(\mathrm{tr}\hat{\mathbf{D}})\hat{\mathbf{G}} + 2(\mu + \mu_1 - \mu_0)\hat{\mathbf{D}} + 2(\mu_0 - \mu_1)\hat{\gamma},$$

while the combinations of the viscosity coefficients were denoted as follows:

$$\kappa_v = 2(\mu_0 - \mu_1); \;\; \mu_v = \mu - \kappa_v/2 = \mu + \mu_1 - \mu_0.$$

Coefficient κ_v has been called the "dynamic microrotation viscosity," while the term "viscosity" has been retained for coefficient μ_v. As a result, the stress tensor for an incompressible liquid for which, $\mathrm{tr}\hat{\mathbf{D}} = \nabla \cdot \mathbf{v} = 0$ takes the following form:

$$\hat{\mathbf{t}} = -p\hat{\mathbf{G}} + 2\mu_v\hat{\mathbf{D}} + \kappa_v\hat{\gamma}. \quad (4)$$

In Cartesian coordinates, its components are written as follows:

$$t_{ij} = -p\delta_{ij} + \mu_v\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \kappa_v\left(\frac{\partial v_j}{\partial x_i} - \varepsilon_{ijk}\omega_k\right).$$

In this form, the stress tensor is used for micropolar liquids in the majority of studies.

Let us pay attention to the following important circumstance: viscosity coefficient μ_v is, in the general case, unequal to the dynamic viscosity of a Newtonian liquid. They coincide with each other only in the case of $\kappa_v = 0$, which corresponds to the classical limiting case and transforms the expression for the stress tensor into the classical equation.

The couple stress tensor has been expressed in Eringen's works via the curvature–twist rate tensor as

$$\hat{\mathbf{m}} = \alpha(\mathrm{tr}\hat{\boldsymbol{\chi}})\hat{\mathbf{G}} + \beta\hat{\boldsymbol{\chi}}^T + \gamma\hat{\boldsymbol{\chi}}, \quad (5)$$

or, in Cartesian coordinates, as

$$m_{ij} = \alpha\left(\frac{\partial\omega_k}{\partial x_k}\right)\delta_{ij} + \beta\frac{\partial\omega_i}{\partial x_j} + \gamma\frac{\partial\omega_j}{\partial x_i},$$

where coefficients α, β and γ, termed "angular viscosity coefficients," have been introduced.

After the expressions derived for stress tensors are substituted into Eqs. (2) and (3), they, with allowance for Eq. (1), acquire the following form:

$$\begin{gathered}\nabla \cdot \mathbf{v} = 0,\\ \rho\dot{\mathbf{v}} = \rho\mathbf{F} - \nabla p + (\mu_v + \kappa_v)\Delta\mathbf{v} + \kappa_v\nabla\times\boldsymbol{\omega},\\ \rho\hat{\mathbf{J}}\cdot\dot{\boldsymbol{\omega}} = \rho\mathbf{L} + (\alpha + \beta)\nabla\nabla\cdot\boldsymbol{\omega}\\ +\,\gamma\Delta\boldsymbol{\omega} + \kappa_v\nabla\times\mathbf{v} - 2\kappa_v\boldsymbol{\omega}.\end{gathered} \quad (6)$$

In this form, the set of equations for liquid motion was derived by Eringen; therefore, we shall refer to it as the Eringen set of equations. At $\kappa_v = 0$, the first and second equations of set (6) form the Navier–Stokes set of equations for a nonpolar liquid, with μ_v reflecting the dynamic viscosity of this liquid. When set (6) is used at $\kappa_v \neq 0$, the necessity, as a rule, arises to compare the obtained solutions with their classical analogs. Therewith, it should be taken into account that viscosity coefficient μ_v is unequal to dynamic viscosity μ of a Newtonian liquid. Otherwise, the quantitative estimates and theoretical conclusions would be distorted. Eringen himself has noted this circumstance in his book [9]. In [12], Stokes proposed to redefine viscosity coefficients in a manner such that the passage to the model of a nonpolar liquid could be performed formally. The possibility to take into account the relation between these dependences using simple replacement $\mu_v = \mu - \kappa_v/2$ in an obtained solution has been noted in [21].

When deriving state equations for the theory of micropolar elasticity, Nowacki [4] proposed to derive an explicit dependence on symmetric and antisymmetric components of deformation tensors. Realizing this idea for liquids and deformation-rate tensors, we obtain

$$\hat{\mathbf{t}} = (-p + \lambda\mathrm{tr}\hat{\gamma})\hat{\mathbf{G}} + 2\mu\hat{\gamma}^{(S)} + 2\kappa\hat{\gamma}^{(A)}, \quad (7)$$

$$\hat{\mathbf{m}} = \alpha(\mathrm{tr}\hat{\boldsymbol{\chi}})\hat{\mathbf{G}} + 2\delta\hat{\boldsymbol{\chi}}^{(S)} + 2\varsigma\hat{\boldsymbol{\chi}}^{(A)}, \quad (8)$$

where μ and λ are the first and second classical viscosity coefficients, respectively. This can be easily seen after writing the symmetric component of the stress tensor in Cartesian coordinates. For an incompressible liquid, it has the form of

$$t_{ij}^{(S)} = -p\delta_{ij} + 2\mu\gamma_{ij}^{(S)} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

and completely coincides with the classical stress tensor. The antisymmetric component of the deformation rate tensor, $\gamma_{ij}^{(A)} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right) - \varepsilon_{ijk}\omega_k$, is related only to the rotational motions, thereby justifying the term "vorticity" or "microrotation" viscosity applied for coefficient κ. Symbols α, δ and ς denote angular-viscosity coefficients.

Substitution of Eqs. (7) and (8) into system (1)–(3) yields the motion equations for a micropolar liquid:

$$\begin{gathered}\nabla \cdot \mathbf{v} = 0,\\ \rho\dot{\mathbf{v}} = \rho\mathbf{F} - \nabla p + (\mu + \kappa)\Delta\mathbf{v} + 2\kappa\nabla\times\boldsymbol{\omega},\\ \rho\hat{\mathbf{J}}\cdot\dot{\boldsymbol{\omega}} = \rho\mathbf{L} + (\alpha + \delta - \varsigma)\nabla\nabla\cdot\boldsymbol{\omega}\\ +\,(\delta + \varsigma)\Delta\boldsymbol{\omega} + 2\kappa\nabla\times\mathbf{v} - 4\kappa\boldsymbol{\omega}.\end{gathered} \quad (9)$$

These equations will be referred to as the "Nowacki set of equations."

It is obvious that, at $\kappa = 0$, the first and second equations of set (9) are reduced to the equations of classical hydrodynamics, while viscosity of a micropolar liquid μ remains equal to the dynamic viscosity of a nonpolar liquid with no stipulations.

It can be easily seen that, at $\kappa = \frac{\kappa_v}{2}$, $\mu = \mu_v + \frac{\kappa_v}{2}$, $\delta = \frac{\gamma + \beta}{2}$ and $\varsigma = \frac{\gamma - \beta}{2}$, expressions (7) and (8) are transformed into Eqs. (4) and (5), respectively, while set of equations (9) coincides with system (6). That is, the state equations and motion equations in the Eringen and Nowacki forms are identical with an accuracy of the denotations of viscosity coefficients.

In our opinion, Nowacki's formulation of the problem is advantageous in the fact that it explicitly comprises the classical dynamic-viscosity coefficient, thereby enabling one not only to perform the formal passage to the classical limiting case in the obtained solutions, but also to compare the obtained solutions with the classical ones in a more convenient form. Therefore, the subsequent consideration will concern equations represented in Nowacki's form. A less compact notation of the moment equation, in particular, the use of sum $\delta + \varsigma$ instead of coefficient γ, which appears in Eringen's formulation, does not seem to be inconvenient, because it additionally reminds us that terms of both symmetric and antisymmetric components of the curvature–twist-rate tensor are involved in the solution.

Some arbitrariness in the notation of the dynamic microrotation and angular-viscosity coefficients seems to play no decisive role, because the definition and evaluation of these parameters encounter, probably, the greatest difficulties when developing the theory of micropolar liquids. The design of experiments on the determination of these characteristics is a separate complex problem. One of the few researchers to have experimentally studied the properties of micropolar liquids was the Belarusian scientist Migun. In his dissertation, he has developed and realized a method for the experimental determination of micropolar liquid parameters that combine the aforementioned viscosities. His technique was presented in monograph [15]. The characteristics of several liquids with micropolar properties have been reported in another monograph by the same authors [22]. The values of the viscosity coefficients for some micropolar liquids may be found in [23, 24].

In view of the difficulties relevant to the determination of the viscosity coefficients for micropolar liquids, all the analytical solutions that have been found within the framework of this theory appear to be especially valuable. They enable one to explicitly study the dependence of solution results on the coefficients or their combinations, to realize the measure of the influence of each of them, and to advance in elaborating procedures for experimental investigation of micropolar media.

Let us note the determination of the total time derivative, which is denoted by the dots over the symbols in the left-hand sides of the motion and moment equations. In classical hydrodynamics, it implies a material or substantial derivative, which has the following form: $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$. In this representation, it was used in Eringen's works and subsequent studies, a review of which may be found in [10, 16, 17]. A substantial derivative describes the evolution of a polar field of vectors of linear velocities. At the same time, in classical hydrodynamics, the evolution of the axial vector of vorticity field $\mathbf{\Omega} \equiv \mathrm{curl}\mathbf{v}$ is described by the Helmholtz equation. For a viscous incompressible liquid, it has the form of $\frac{\partial \mathbf{\Omega}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{\Omega} - (\mathbf{\Omega} \cdot \nabla)\mathbf{v} = \frac{\mu}{\rho}\Delta\mathbf{\Omega}$. In addition to the terms of the substantial derivative, the left-hand side of the equation comprises the term $-(\mathbf{\Omega} \cdot \nabla)\mathbf{v}$. In [25], it has been rigorously shown that the evolution of axial vector field $\boldsymbol{\omega}$ in a micropolar liquid must be described by an equation having a pattern the same as the Helmholtz equation has; namely:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} = \text{“viscous terms”}.$$

Thus, the left-hand sides of the moment equations in sets (6) and (9) must be written as $\dot{\boldsymbol{\omega}} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}$, where the dot over the symbol has the same meaning. The set of equations for a micropolar liquid taking into account the aforementioned correction has been considered as a modified set of equations for a micropolar liquid [25].

Note that, in planar and axially symmetric cases, the modified set of equations for a micropolar liquid coincides with the initial one. Differences arise when the flow is essentially three-dimensional. The Burgers vortex is the most common three-dimensional problem, which has an exact solution. Problems concerning the diffusion of vortex lines and vortex lattices also belong to this class of problems. Their solutions obtained for a micropolar liquid using the modified set of equations are presented in a recently published book by Brutyan [26].

A series of studies devoted to nonstationary micropolar flows of different geometries have been presented in [27–30] and other communications by the same authors. The authors used the Laplace transform; however, the passage to the region of original variables was performed numerically. In [31], the nonstationary equations for a planar flow have been solved in a completely analytical form; however, the dependence of the unknown functions on coordinates has a particular pattern.

Since, in the case of micropolar media, analytical solutions can be obtained predominantly for linear problems, slow flows will be considered below under the Stokes approximation. Equations written with allowance for inertial terms under the Oseen approximation are used much more seldom, although they also imply analytical solutions. The Stokes and Oseen flows have been considered in the general vector form in [32]; the characteristics of the flows have been obtained for two cases, i.e., at nonzero external forces and zero moments and vice versa.

BOUNDARY CONDITIONS AT A SOLID WALL

Since, in contrast to the set of Navier–Stokes equations, the set of constitutive equations for a micropolar liquid contains one more unknown vector function and one additional second-order vector equation, additional boundary conditions are required for its solution. The number of these conditions depends on the dimensionality of a problem. Two conditions are imposed on vector $\boldsymbol{\omega}$ for each independent coordinate. For problems of an external flow, they are conditions preset at a surface streamlined and at infinity. For flows inside closed areas, such as cylinders, they are a condition imposed on a surface and a condition of finiteness of the solution inside the flow region. In other configurations, there are commonly two different boundaries on which corresponding conditions are imposed.

It should be noted that, even in classical thermodynamics, the history of which is almost three centuries long, there is no consensus about the presence or absence of slip at solid surfaces and a distinct criterion for the selection of this or that condition at a boundary. From the point of view of formal mathematics, one may provide a boundary with either the Dirichlet condition, i.e., set the value of a desired function at a boundary, or the Neuman condition, i.e., specify the value of the derivative of the desired function. A mixed condition may also be used in the form of a linear combination of a function and its derivative with some coefficient. The first condition is based on the idea of sticking a liquid to a solid surface and has the form of $\mathbf{v}|_\mathrm{b} = \mathbf{U}_\mathrm{b}$, where the subscript b denotes “boundary” and $\mathbf{U}_\mathrm{b}$ is the velocity of the boundary, which may, in particular, be quiescent. This condition is most often used due to the authority of Stokes, who has stated that slip is impossible. At the same time, Navier [33] has suggested to impose the slip condition for the velocity component tangential to a surface, while retaining the normal component equal to zero. According to Navier’s idea, the tangential component of the flow velocity at a solid surface is linearly related to the tangential stress at it. From the mathematical point of view, this condition is attributed to the mixed type, where the coefficient has the dimensionality of length and is referred to as the slip length. At the zero slip length, the no-slip condition is realized, while, at the infinite one, the stress tensor component, which, for a Newtonian liquid, is expressed via derivatives of velocity, vanishes. Note that, even when the opinion of no-slip at solid surface–liquid prevailed, some exceptions to this rule took place, among which liquid polymers were noted. They represent media that are adequately described by the micropolar model. For a micropolar liquid, the stress tensor is determined by relation (7), and the Navier slip condition is somewhat more complex; an example of its use will be presented below. Brunn [34, 35] discussed in detail the necessity to use the slip condition for polar liquids proceeding from mechanical considerations and taking into account experimental data.

Plane-parallel stretching is a boundary condition of practical interest. It is realized when producing polymer materials and, as a rule, implies the applicability of the boundary layer approximation. Among numerous works devoted to such problems, an analytical solution obtained within the framework of magnetic hydrodynamics of micropolar boundary layers has been reported in [36].

The situation with boundary conditions for the angular velocity of a micropolar liquid is still less clear. The condition of the zero angular velocity at a quiescent boundary is most often used, because it yields a simpler solution of the boundary value problem. It was formulated by analogy with the no-slip condition for the linear velocity; however, it has not received a reliable physicochemical justification. Moreover, Kirwan [37] has shown that it is not universally applicable.

In some works, it has been proposed to impose condition $\boldsymbol{\omega}|_\mathrm{b} = \frac{1}{2}\mathrm{curl}\mathbf{v}|_\mathrm{b}$ on solid boundaries. From the mechanical point of view, namely this condition rather than condition $\boldsymbol{\omega}|_\mathrm{b} = 0$ ensures the stop of the microrotation, because it expresses the equality of the angular velocity of a liquid to its solid-state rotation rate. Note that the solid-state rotation rate may be nonzero in a purely translational flow of a Newtonian liquid. For example, the solution of the aforementioned equation for the evolution of vorticity for the Poiseuille planar flow of a Newtonian liquid yields a linear distribution of vorticity along a cross section of a channel. Therewith, the maximum absolute values of vorticity equal to $\frac{1}{2}\mathrm{curl}\mathbf{v}|_\mathrm{b}$ are reached at walls. One can directly ascertain that the use of this boundary condition in the absence of external couples of forces leads to the disappearance of the effects of micropolarity by the example of the solution presented below for the Poiseuille flow: taking $\boldsymbol{\omega}|_\mathrm{b} = \frac{1}{2}\mathrm{curl}\mathbf{v}|_\mathrm{b}$, we, at once, arrive at the coincidence of the obtained solution with the classical profile.

The detailed analysis of all possible mechanisms for the appearance of vorticity may be found in Smith’s book [38], in which it has been shown that namely the interaction of a flow with solid boundaries is responsible for the generation of vorticity. Thus a correct formulation of boundary conditions may play the key role when considering micropolar flows.

In [37] it was porposed a condition of rather general form $\boldsymbol{\omega}|_b = n\,\mathrm{curl}\mathbf{v}|_b$, where coefficient n may acquire values strictly less than $\frac{1}{2}+\frac{\mu}{2\kappa}$, [39]. At $n = 1/2$, we obtain the aforementioned case of the possible degeneration of the micropolar flow into the classical one. At $n < 1/2$, and, in particular, at $n = 0$, the flow slows down relative to the classical one because of the braking action of a wall on its vorticity. At $n > 1/2$, an opposite situation is observed, as has been shown in [39], where different forms of the representation of the $n = n(\mu/\kappa)$ dependence and their possible applications were proposed. Examples of the use of this condition may be found in, e.g., [40–43]. However, the authors have not presented a reliable mechanical substantiation of the used conditions and have noted that there is a marked discrepancy between the results obtained.

In monograph [12], it has been proposed to use, as alternative boundary conditions for micropolar liquids, the values of forces and moments that are determined via the tensors of force and couple stresses. Or, passing to a more general mixed condition, this idea may be formulated as $\boldsymbol{\omega}|_b = m\left.\frac{\partial\boldsymbol{\omega}}{\partial n}\right|_b$, where the differentiation is performed in the direction normal to a wall, while coefficient m has the meaning of the slip length. Taking into account the pattern of the components of the couple-stress tensor, we arrive at a complete analogy with the Navier slip law. At present, this analogy is finding increasingly wide application for studying flows of micropolar media [28–30]. Moreover, the attention of researchers is focused on problems relevant to moving and, in particular, pulsing boundaries. It has been proposed to introduce a velocity jump for them [44]. A planar flow with preset slip velocity and normal stress was considered in [45].

Since the near-boundary interaction of a liquid with a solid body is governed by surface phenomena, the conditions at each boundary must depend on its properties, such as roughness, hydrophobicity or hydrophilicity, etc. For Newtonian liquids, these and other factors of the interaction with solid surfaces have been considered in detailed review [46] devoted to the experimental study of slip. The results of simulating a micropolar liquid flowing in microchannels within the framework of molecular dynamics have been presented in [47]. The calculations have shown essential dependences of the flow parameters on the channel width and materials of its walls. In addition, nonzero values were obtained for the angular velocity at boundaries, and an obvious slip effect was found, the scale of which depended on a wall material.

Hence, it is obvious that there is no universal boundary condition and that criteria are necessary for the selection of conditions for a specific medium and a boundary. Moreover, experimental studies of the interaction with solid surfaces at the micro- and nanolevels are extremely urgent for both Newtonian and non-Newtonian liquids.

ANALYTICAL SOLUTIONS AVAILABLE FOR SLOW FLOWS OF A MICROPOLAR LIQUID

In view of the peculiar properties of micropolar media, microscale slow flows with low Reynolds numbers are of greatest interest for engineering applications. Stationary problems of creeping or Stokes flows are the most widely known problems of classical hydrodynamics at low Reynolds numbers admitting analytical solutions. The Poiseuille flow between parallel planes and in a cylindrical tube, the flow between coaxial cylinders, the Couette flow, and the transverse flow around a cylinder and a sphere are among them. The generalizations of their solutions for micropolar liquids will be presented in this review. In most cases, these solutions may be obtained at nonzero external forces and moments. Nevertheless, we shall take $\mathbf{F} = 0$ and $\mathbf{L} = 0$, because, in this case, the representations of the solutions are much more compact, and their features are more evident as compared with the classical solutions, which were also obtained in the absence of external forces.

Planar Flow

Let us consider a planar flow in a Cartesian coordinate system oriented in a manner such that the xOz plane coincides with the flow plane. The channel width is denoted as $2R$ in order to make convenient the comparison of the solution with an analogous flow in a cylindrical channel with this diameter. Let the flow be bounded by planes $x = \pm R$, directed along the Oz axis and caused by constant pressure gradient $-dp/dz$. Then, $\mathbf{v} = \{0; 0; u(x)\}, \boldsymbol{\omega} = \{0; \omega(x); 0\}$, the continuity equation is identically fulfilled, and set of equations (9) acquires the following form:

$$(\mu + \kappa)u'' + 2\kappa\omega' = \frac{\partial p}{\partial z},$$
$$(\delta + \varsigma)\omega'' - 2\kappa u' - 4\kappa\omega = 0,$$

where the single and double prime symbols denote the first and second derivatives of the unknown functions over variable x, respectively.

Let us introduce dimensionless variables as follows:

$$\tilde{x} = \frac{x}{R}, \quad \tilde{u} = \frac{u\rho R}{\mu}, \quad \tilde{\omega} = \frac{\omega\rho R^2}{\mu}, \quad \frac{\partial\tilde{p}}{\partial\tilde{z}} = \frac{\partial p}{\partial z}\frac{\rho R^3}{\mu^2} \equiv \delta p. \tag{10}$$

Note that the dimensionless velocity formally coincides with the definition of the Reynolds number for a given flow. Thus, paying attention to the maximum value of the flow velocity, we can monitor the accuracy of the used Stokes approximation. In dimensionless variables, the set of equations is as follows:

$$\frac{\mu+\kappa}{\kappa}\tilde{u}''+2\tilde{\omega}'=\frac{\mu}{\kappa}\delta p,$$
$$\frac{\delta+\varsigma}{\mu R^2}\tilde{\omega}''-2\frac{\kappa}{\mu}\tilde{u}'-4\frac{\kappa}{\mu}\tilde{\omega}=0. \quad (11)$$

Note that viscosity coefficients are still written in dimensional form; however, ratios between them appear everywhere. Expression $\frac{\delta+\varepsilon}{\mu}$ has the dimensionality of squared length and may be used as a characteristic microscale of the problem. The presence of the ratio between this value and squared characteristic microscale R^2 of the problem in the dimensionless equations represents a fundamental difference between the theory of micropolar liquids and classical hydrodynamics.

The solution will be most compact and obvious, if we introduce denotations for combinations of the constants. We shall follow the denotations proposed by Lukaszewicz [10]; namely:

$$N^2=\frac{\kappa}{\mu+\kappa};\quad \tau^2=\frac{1}{4}\frac{\delta+\varsigma}{\mu};\quad L^2=\frac{\tau^2}{R^2}. \quad (12)$$

So-called micropolarity number N is related to vorticity viscosity κ. It characterizes the measure of the relation between the translational and rotational motions of a continuum. Sometimes, it is interpreted as the measure of engagement between particles forming a medium and a measure of the efficiency of rotational momentum transmission. The maximum N value corresponds to a strong micropolarity, while limiting value $N = 0$ enables us to separate the motion and moment equations and pass to the classical limiting case, i.e., the motion equation for a Newtonian liquid. Characteristic length τ is, sometimes, associated with "particles" forming a medium. The higher τ, the stronger the non-Newtonian effects. Nevertheless, this parameter is not a real size of any physical object and may, formally, take any value allowed by constraints on the values of the viscosity coefficients. These constraints follow from the second law of thermodynamics, and, as in the classical case, they have the following form [9]:

$$\mu\ge 0,\quad 3\lambda+2\mu\ge 0;\quad \kappa\ge 0;$$
$$\delta\ge 0,\quad \delta+\varsigma\ge 0,\quad 3\alpha+2\delta\ge 0,$$
$$-(\delta+\varsigma)\le\delta-\varsigma\le\delta+\varsigma.$$

Thus, dimensionless parameters N and L, which are determined by relations (12), take values lying in the ranges $N\in[0;1)$ and $L\in[0;\infty)$. So, the dimensionless formulation of the problem for the Poiseuille flow is as follows:

$$\frac{1}{N^2}u''+2\omega'=\left(\frac{1}{N^2}-1\right)\delta p,$$
$$L^2\omega''-\frac{1}{2}\frac{N^2}{1-N^2}u'-\frac{N^2}{1-N^2}\omega=0. \quad (13)$$

The tilde symbol is omitted in Eqs. (13) and below. Note that the coefficient at the higher-order derivative in the second equation of set (13) may take arbitrarily small values. This fact will be necessary to take into account when solving the system by approximate or numerical methods. However, the existence of an explicit analytical solution eliminates this necessity, and, therefore, it is especially valuable. It has the following form:

$$u(x)=C_1+\frac{\delta p}{2}x^2+C_2x$$
$$-LN\left(C_3\cosh(xN/L)+C_4\sinh(xN/L)\right), \quad (14)$$

$$\omega(x)=-\frac{\delta p}{2}x-\frac{C_2}{2}$$
$$+\frac{1}{2}\left(C_3\sinh(xN/L)+C_4\cosh(xN/L)\right), \quad (15)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. Let us write the partial solution only for the conditions of complete no-slip and no-spin at both walls, i.e., $u(\pm1)=0$, $\omega(\pm1)=0$, to make more convenient the comparison of the solution with the classical Poiseuille profile, which is also obtained under the no-slip condition. Note that, in addition to the pressure gradient, the solution depends on only two more dimensionless parameters N and L. In order to emphasize this peculiarity, parameters N and L are included in the list of the arguments of the following functions:

$$u(x,N,L)=-\frac{\delta p}{2}(1-x^2)$$
$$+\delta pLN\frac{\cosh(N/L)-\cosh(xN/L)}{\sinh(N/L)}, \quad (16)$$

$$\omega(x,N,L)=-\frac{\delta p}{2}\left(x-\frac{\sinh(xN/L)}{\sinh(N/L)}\right). \quad (17)$$

It can be easily seen that the first term in the expression for the linear velocity represents the Poiseuille parabola, while the correction to it has the opposite sign; i.e., the flow velocity of a micropolar liquid is lower than the velocity of a nonpolar liquid at the same pressure gradient. This fact is quite obvious, because additional energy consumptions are required to overcome new type of viscous friction. The angular flow velocity also consists of a term equal to $\frac{1}{2}\mathrm{curl}\,\mathbf{v}$ and a correction term with the opposite sign.

Many authors have dealt with detailed studying the properties of planar flows of micropolar liquids under different boundary conditions. Some results have been presented in aforementioned monographs [10, 15]. Exact solutions have been obtained for a planar flow between infinite rotating discs simulating an orthogonal rheometer [48, 49]. Some specific types of stationary planar flows were analyzed in the plane of a hodograph [50] using the Legendre transform of the stream function. The Poiseuille planar flow was compared in [51] with a flow squeezed between two approaching planes for three real micropolar liquids. Characteristic interplanar distances have been determined beginning from which the micropolarity effects become essential, and it has been shown that these distances for approaching planes are an order of magnitude smaller than corresponding values for the Poiseuille flow. Work [52], in which a nonstationary Couette planar flow has been considered under the conditions of linear velocity slip at both boundaries, deserves attention. A problem, which was similarly formulated but under the boundary conditions of complete no-slip, has been solved in [53]. Planar and essentially two-dimensional flows were considered in [54], where the analytical solution was obtained using the Lie group method. However, the boundary conditions imposed by the authors must be additionally discussed.

Flow along the Axis of Coaxial Cylinders

Let a flow between coaxial cylinders be directed along their symmetry axis, which coincides with the Oz axis, and is caused by constant pressure gradient $-dp/dz$. We denote the radii of the external and internal cylinders as R and A, respectively. Then, in cylindrical coordinates (r, θ, z), $\mathbf{v} = \{0; 0; u(r)\}$, $\boldsymbol{\omega} = \{0; \omega(r); 0\}$, and set of Eqs. (9) takes the following form:

$$(\mu+\kappa)u'' + \frac{\mu+\kappa}{r}u' + 2\kappa\omega' + \frac{2\kappa}{r}\omega = \frac{\partial p}{\partial z},$$
$$(\delta+\varsigma)\omega'' + \frac{\delta+\varsigma}{r}\omega' - 2\kappa u' - \left(4\kappa + \frac{\delta+\varsigma}{r^2}\right)\omega = 0,$$

where prime symbols denote differentiation over coordinate r.

Let us write this set in dimensionless variables (10) at $\tilde{r} = r/R$:

$$\frac{\mu+\kappa}{\kappa}\tilde{u}'' + \frac{\mu+\kappa}{\kappa}\frac{\tilde{u}'}{\tilde{r}} + 2\tilde{\omega}' + 2\frac{\tilde{\omega}}{\tilde{r}} = \frac{\mu}{\kappa}\delta p,$$
$$\frac{\delta+\varsigma}{\mu R^2}\tilde{\omega}'' + \frac{\delta+\varsigma}{\mu R^2}\frac{\tilde{\omega}'}{\tilde{r}} - 2\frac{\kappa}{\mu}\tilde{u}' - \left(4\frac{\kappa}{\mu} + \frac{\delta+\varsigma}{\mu R^2}\frac{1}{\tilde{r}^2}\right)\tilde{\omega} = 0.$$

In denotations (12), the dimensionless formulation of the problem is as follows:

$$\frac{1}{N^2}u'' + \frac{1}{N^2}\frac{u'}{r} + 2\omega' + 2\frac{\omega}{r} = \left(\frac{1}{N^2} - 1\right)\delta p,$$
$$L^2\omega'' + L^2\frac{\omega'}{r} - \frac{1}{2}\frac{N^2}{1-N^2}u' - \left(\frac{N^2}{1-N^2} + \frac{L^2}{r^2}\right)\omega = 0. \quad (18)$$

Here and below, the tilde symbol is omitted and the remarks concerning the coefficient at the higher-order derivative in the planar case remain valid for the cylindrical geometry.

The general solution of set (18) contains four arbitrary constants C_i and comprises modified Bessel $\mathrm{I}_0(rN/L)$, $\mathrm{I}_1(rN/L)$ and Macdonald $\mathrm{K}_0(rN/L)$, $\mathrm{K}_1(rN/L)$ functions of the zero and first orders, respectively:

$$u(r) = C_1 + \frac{\delta p}{4}r^2 + C_2\ln r - C_3\,\mathrm{I}_0(rN/L) - C_4\,\mathrm{K}_0(rN/L), \quad (19)$$

$$\omega(r) = -\frac{\delta p}{4}r - \frac{C_2}{2r} + \frac{1}{2NL}\left(C_3\,\mathrm{I}_1(rN/L) - C_4\,\mathrm{K}_1(rN/L)\right). \quad (20)$$

The classical profile of the Poiseuille flow between coaxial cylinders has the following form:

$$u(r) = -\frac{\delta p}{4}\left(1 - r^2\right) + \frac{\delta p}{4}(1-\tilde{a}^2)\frac{\ln r}{\ln\tilde{a}},$$

where $\tilde{a} = A/R$. Comparing Eq. (19) with this expression and Eq. (20) with a corresponding coordinate of the vorticity vector of the classical flow of this geometry

$$\frac{1}{2}(\mathrm{curl}\,\mathbf{v})_\theta = -\frac{1}{2}u'(r) = -\frac{\delta p}{4}r - \frac{\delta p}{4}\frac{1-\tilde{a}^2}{\ln\tilde{a}}\frac{1}{2r},$$

it may, again, be noted that the structure of solutions (19) and (20) is represented in the form of the sum of classical terms with correction terms, which depend on characteristics N and L of a micropolar medium.

In the particular case $A \to 0$, we obtain the flow inside one cylinder. The condition of a finiteness solution of the problem enables us to exclude infinitely high function values at $r \to 0$ and represent solutions (19) and (20) in the following form containing only two constants:

$$u(r) = C_1 + \frac{\delta p}{4}r^2 - C_3\,\mathrm{I}_0(rN/L),$$

$$\omega(r) = -\frac{\delta p}{4}r + \frac{C_3}{2NL}\mathrm{I}_1(rN/L).$$

Using no-silp and no-spin conditions $u(1) = 0$ and $\omega(1) = 0$, we obtain a solution of the Poiseuille type and represent it in a form convenient for the analysis of its dependence on dimensionless parameters N and L:

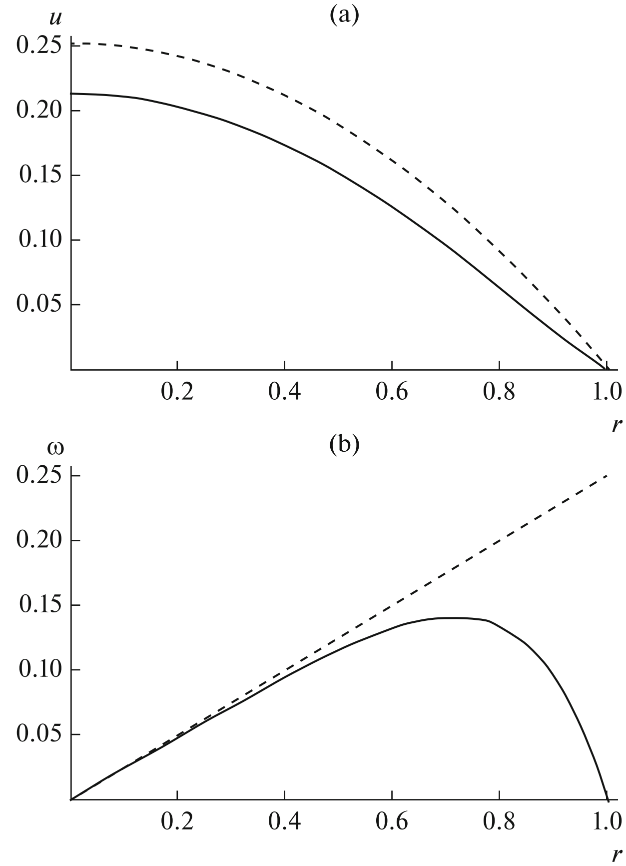

Fig. 1. Profiles of (a) linear and (b) angular velocities for Poiseuille flows of micropolar (solid lines) and Newtonian (dashed lines) liquids.

$$u(r,N,L) = -\frac{\delta p}{4}\left((1-r^2) - 2NL\frac{\mathrm{I}_0(N/L) - \mathrm{I}_0(rN/L)}{\mathrm{I}_1(N/L)}\right),$$

$$\omega(r,N,L) = -\frac{\delta p}{4}\left(r - \frac{\mathrm{I}_1(rN/L)}{\mathrm{I}_1(N/L)}\right).$$

Let us present an example of a brief parametric study of the obtained solution as an illustration. Primarily, we are comparing the obtained profiles of the linear and angular velocities with corresponding profiles for the classical Poiseuille flow. Figure 1 shows velocity curves for $\delta p = -1$ and values $N^2 = 0.5$ and $L = 0.1$ corresponding to a rather well-developed micropolarity. The profile of the linear velocity of a micropolar flow (Fig. 1a) is located below the classical Poiseuille parabola; i.e., the micropolar liquid flows more slowly than a Newtonian one does, because the flow energy is partly consumed to overcome additional viscous forces. As N and L grow, this difference will increase, because the micropolar properties of the liquid will become more pronounced. In Fig. 1b, the angular velocity of a micropolar liquid is compared with the angular velocity of the solid-state rotation of a Newtonian liquid. The strongest difference between the profiles is observed near the channel walls and is explained by the boundary conditions selected for the angular velocity. As has been mentioned above, under the condition $\boldsymbol{\omega}|_b = \frac{1}{2}\mathrm{curl}\mathbf{v}|_b$, the profile of the angular velocity coincides with the straight line corresponding to the classical flow of a Newtonian liquid. This fact once more evidently shows the extent of the influence and the importance of the proper selection of boundary conditions for problem formulation.

The extent of the influence of parameters N and L on the solution can be seen in Figs. 2 and 3. Figure 2 shows the surfaces, which may be considered to be the evolutions of the profiles of (a) linear and (b) angular velocities with an increase in parameter N at fixed value $L = 0.1$. Figure 3 illustrates the evolution of the same profiles with a rise in parameter L at fixed value $N^2 = 0.5$. Remember that parameter N may vary from zero to unity, while parameter L may vary from zero to infinity. At the same time, they enter the solution in a manner such that the limiting case resulting from their simultaneous tending to zero is uncertain.

Figure 2a indicates that, at $N = 0$, the velocity profile coincides with the classical parabola and, as parameter N increases, the profile goes down. At $N = 1$, it reaches the limiting position, which is determined by the value of parameter L. Another situation is presented in Fig. 2b; namely, the profile of the angular velocity increases from zero value $N = 0$ to some highest position at $N = 1$. This effect is explained by two factors, i.e., the procedure of determining parameter N and the boundary conditions preset for $\boldsymbol{\omega}$. According to the definition of parameter N, condition $N = 0$ is equivalent to zero microrotation viscosity κ. The latter, in turn, plays the role of the measure of "engagement" between medium microparticles and characterizes the rate of torque transmission. At zero engagement, we have the zero rotation at any r values because of the condition imposed on the walls. As κ and, hence, N grow, increasingly efficient rotational interaction of medium microparticles better transmits the torques and aligns the profile, leading it to zero at a wall $r = 1$.

The tendency of parameter L toward zero may be interpreted either as the free rotation of microparticles at infinitely low angular viscosities or as a vanishingly small size of microparticles at a fixed channel radius. In both cases, we obtain the classical limits at a nonzero fixed value of parameter N. As parameter L grows, both the linear and angular velocities decrease, and, already at $L \sim 1$, they reach their asymptotic values, which correspond to $L \to \infty$. The angular velocity tends to zero as might be expected at infinite values of angular viscosities. The linear velocity reaches the classical parabolic profile scaled by coefficient $1 - N^2$.

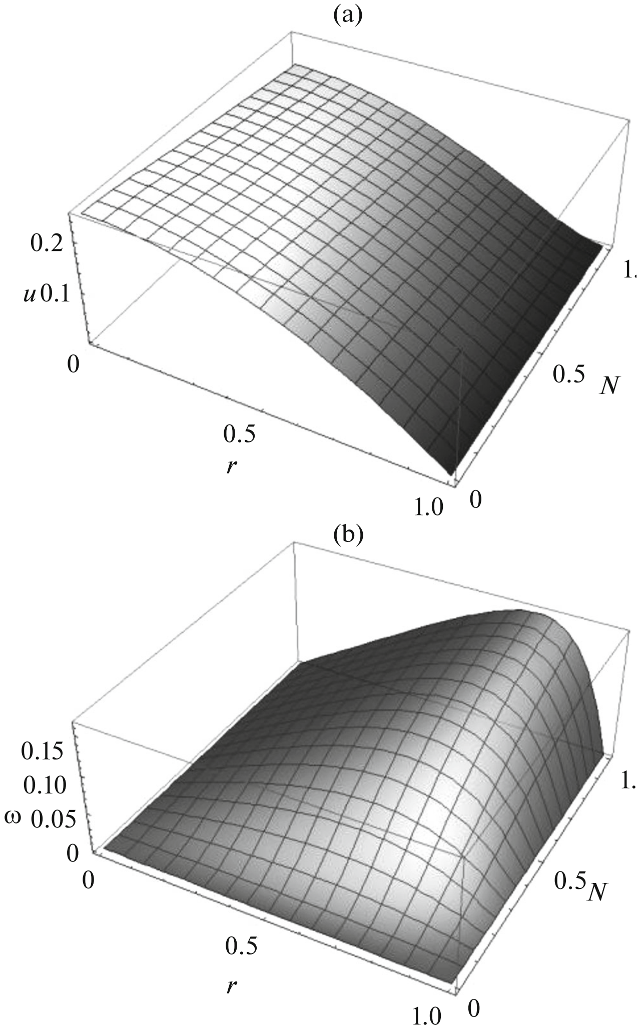

Fig. 2. The (a) $u(r,N,L_{\text{fix}})$ and (b) $\omega(r,N,L_{\text{fix}})$ dependences at $L_{\text{fix}} = 0.1$.

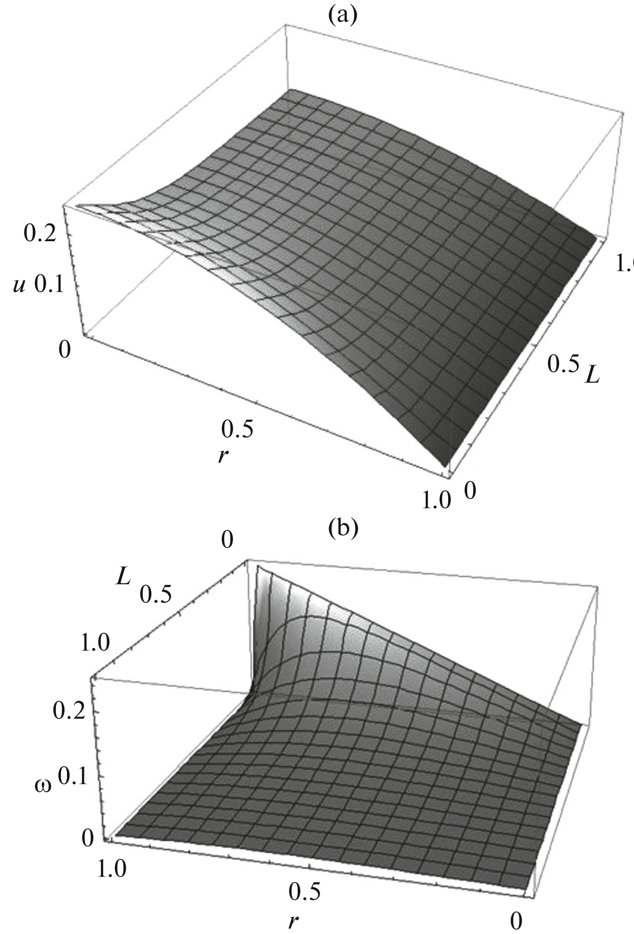

Fig. 3. The (a) $u(r,N_{\text{fix}},L)$ and (b) $\omega(r,N_{\text{fix}},L)$ dependences at $N_{\text{fix}}^2 = 0.5$.

Formally this means that a micropolar flow may be completely stopped at $L \to \infty, N \to 1$. In practice, the aforementioned values of the parameters are, of course, unattainable. Nevertheless, this important property may be of practical significance.

Flows in capillaries are drawing increasing attention of researchers in view of the great importance of specifically this geometry for engineering and medical applications. Corresponding references will be given below when considering simulation of hemodynamics. Moreover, the cylindrical shape of a tube is, as a rule, most suitable for describing flows in channels of arbitrary or unknown shapes.

The authors of [21] quantitatively analyzed flows of several micropolar liquids in cylindrical channels and obtained the characteristic radii at which the flow characteristics substantially differ from the classical ones.

Couette Concentric Flow between Coaxial Cylinders

Let us consider a stationary flow in a ring-shaped gap between coaxial cylinders caused by their rotation relative to each other at constant angular velocity $\boldsymbol{\Psi}$ and the zero pressure gradient. As before, the radii of the external and internal cylinders are denoted as R and A, respectively. In this case, the motion occurs along concentric circles perpendicular to the axis of the cylinders, which coincides with the Oz axis of a cylindrical coordinate system (r,θ,z). The vectors of the linear and angular velocities of the liquid have coordinates $\mathbf{v} = \{0; u(r); 0\}, \boldsymbol{\omega} = \{0; 0; \omega(r)\}$, while set of equations (9) is transformed as follows:

$$(\mu + \kappa)\left(u'' + \frac{u'}{r} - \frac{u}{r^2}\right) - 2\kappa\omega' = 0,$$

$$(\delta + \varsigma)\left(\omega'' + \frac{\omega'}{r}\right) + 2\kappa\left(u' + \frac{u}{r}\right) - 4\kappa\omega = 0.$$

In dimensionless variables (10) and at $\tilde{r} = r/R$, it has the following form:

$$\frac{\mu+\kappa}{\kappa}\left(\tilde{u}''+\frac{\tilde{u}'}{\tilde{r}}-\frac{\tilde{u}}{\tilde{r}^2}\right)-2\tilde{\omega}'=0,$$

$$\frac{\delta+\varsigma}{\mu R^2}\left(\tilde{\omega}''+\frac{\tilde{\omega}'}{\tilde{r}}\right)+2\frac{\kappa}{\mu}\left(\tilde{u}'+\frac{\tilde{u}}{\tilde{r}}\right)-4\frac{\kappa}{\mu}\tilde{\omega}=0.$$

Rewriting the set in denotations (12) and omitting the tilde symbol, we arrive at the following formulation of the problem:

$$\begin{gathered}\frac{1}{N^2}\left(u''+\frac{u'}{r}-\frac{u}{r^2}\right)-2\omega'=0,\\ L^2\left(\omega''+\frac{\omega'}{r}\right)+\frac{1}{2}\frac{N^2}{1-N^2}\left(u'+\frac{u}{r}\right)-\frac{N^2}{1-N^2}\omega=0.\end{gathered}\tag{21}$$

The general solution of set (21) has a structure the same as the solution of set (18):

$$u(r)=C_1r+C_2/r-C_3\,\mathrm{I}_1(rN/L)-C_4\,\mathrm{K}_1(rN/L),\tag{22}$$

$$\omega(r)=C_1+\frac{1}{2NL}\left(C_3\,\mathrm{I}_0(rN/L)-C_4\,\mathrm{K}_0(rN/L)\right).\tag{23}$$

Analogously to all cases considered here, solutions (22) and (23) are presented as the sum of classical terms and correction terms, which take into account the micropolarity. One may become convinced of this by looking at the example of the classical problem relevant to the rotation of a spindle in a coaxial bearing. Its solution under boundary conditions $u(\tilde{a})=\tilde{a}\Psi$, $u(1)=0$, where $\tilde{a}=A/R$, in the accepted notations has the form of

$$u(r)=\frac{\tilde{a}^2\Psi}{1-\tilde{a}^2}\left(\frac{1}{r}-r\right),$$

while the z coordinate of the vorticity vector of this flow is equal to

$$\frac{1}{2}(\mathrm{curl}\,\mathbf{v})_z=\frac{1}{2}\left(u'+\frac{u}{r}\right)=C_1=-\frac{\tilde{a}^2\Psi}{1-\tilde{a}^2}.$$

Thus, the linear velocity is described by the first two terms of Eq. (22), while the angular velocity is described by the first term of Eq. (23). The last two terms in Eqs. (22) and (23) are the correction terms comprising micropolar medium characteristics N and L.

In the case of an eccentric position of a spindle in a bearing, the solution is much more cumbersome, but, however, possible. In view of the high potential of using micropolar lubricants, the studies in this field have been continued from the very beginning of the development of the theory of micropolar liquids. The main difficulty, which hinders the development and application of micropolar lubricants, is the deficiency of data on the characteristic viscosities of these media.

In addition, works devoted to flows over or inside a cylinder subjected to longitudinal, transverse, or torsional vibrations [55–58] deserve attention. Such problems may be encountered upon operation of a cylindrical bearing, preparation of polymer solutions, and drilling oil wells in the shelf under the conditions of ocean waves.

Infinite Flow past a Cylinder

The problem of a transverse flow past an infinite cylinder has no solution within the framework of the micropolar theory, as well as in terms of classical hydrodynamics [59, 60]. Namely, the so-called "Stokes paradox" appears here, which consists in the fact that, when an infinite cylindrical body with an arbitrary cross sectional area is flowed around, the conditions of liquid no-slip on its surface and limited velocity at infinity cannot be met simultaneously. A nontrivial solution for this geometry may be found within the framework of cell models in addition to spherical cells, because the external size of a cell is finite and fixed.

Infinite Flow past a Sphere

Let us consider an analog of the classical problem of a stationary infinite slow uniform flow around a solid sphere.

Assume that a uniform flow of a micropolar liquid has constant velocity U at infinity, while the radius of the sphere flowed around is R_s. The $\theta=0$ axis of a spherical coordinate system (r,θ,φ) is directed along vector $\mathbf{U}$ $(0\le\theta\le\pi,\ 0\le\varphi<2\pi)$. The unknown functions are $\mathbf{v}=\{u(r,\theta);v(r,\theta);0\}$, $\boldsymbol{\omega}=\{0;0;\omega(r,\theta)\}$, and $p(r,\theta)$. At infinity, the finiteness condition is imposed on the angular-velocity vector, while the nonzero coordinates of the linear-velocity vector must be equal to $u(r,\theta)=U\cos\theta$, $v(r,\theta)=-U\sin\theta$, respectively. The boundary conditions at the sphere surface will be discussed after obtaining the general solution of the set of Eqs. (9) by the Lamb method. After denotation $\boldsymbol{\Omega}=\mathrm{curl}\,\mathbf{v}$ is introduced, system (9) is transformed as follows:

$$\begin{gathered}\nabla\cdot\mathbf{v}=0,\\ -(\mu+\kappa)\nabla\times\boldsymbol{\Omega}+2\kappa\nabla\times\boldsymbol{\omega}=\nabla p,\\ -(\delta+\varsigma)\nabla\times\nabla\times\boldsymbol{\omega}+2\kappa\boldsymbol{\Omega}-4\kappa\boldsymbol{\omega}=0.\end{gathered}\tag{24}$$

In dimensionless variables

$$\tilde{r}=\frac{r}{R_s},\quad \tilde{u}=\frac{u}{U},\quad \tilde{v}=\frac{v}{U},\quad \tilde{\omega}=\omega\frac{R_s}{U},$$

$$\tilde{p}=p\frac{R_s}{U\mu},\quad \delta p=p_r'\frac{R_s^2}{U\mu}$$

and with the use of relations (12), in which R is replaced by R_s, the second equation of set (24) is written as

$$-\frac{1}{N^2}\nabla\times\boldsymbol{\Omega}+2\nabla\times\boldsymbol{\omega}=\frac{\mu}{\kappa}\nabla p,$$

where the tilde symbol is omitted.

Using the curl operation to both sides of this equation, we obtain

$$\nabla \times \nabla \times (\mathbf{\Omega} - 2N^2\boldsymbol{\omega}) = 0. \quad (25)$$

Passing to the coordinate notation of Eq. (25) and searching for the solution by the method of separation of variables, we, for the third coordinate of vector $\mathbf{\Omega} - 2N^2\boldsymbol{\omega}$, will have $\Omega - 2N^2\omega = \left(Ar + \frac{B}{r^2}\right)\sin\theta$. The solution for $\omega(r,\theta)$ will be sought in the form of $\omega(r,\theta) = f_\omega(r)\sin\theta$; then, $\Omega = \left(2N^2 f_\omega(r) + Ar + \frac{B}{r^2}\right)\sin\theta$. The substitution of this expression into the dimensionless-coordinate form of the third equation of system (24), which, in turn, acquires the pattern of

$$L^2\left(f_\omega'' + 2\frac{f_\omega'}{r} - 2\frac{f_\omega}{r^2}\right) + \frac{1}{2}\frac{\kappa}{\mu}\frac{\Omega}{\sin\theta} - \frac{\kappa}{\mu}f_\omega = 0,$$

yields the inhomogeneous Bessel equation. Its general solution contains the following modified Bessel $\mathrm{I}_{3/2}(rN/L)$ and Macdonald $\mathrm{K}_{3/2}(rN/L)$ functions with fractional subscripts:

$$f_\omega(r) = \frac{Ar}{2(1-N^2)} + \frac{1}{2(1-N^2)}\frac{B}{r^2} + C\frac{\mathrm{I}_{3/2}(rN/L)}{\sqrt{r}} + D\frac{\mathrm{K}_{3/2}(rN/L)}{\sqrt{r}}.$$

Taking into account the finiteness condition for $f_\omega(r)$ at $r \to \infty$, we obtain

$$\omega(r,\theta) = \left(\frac{1}{2(1-N^2)}\frac{B}{r^2} + D\frac{\mathrm{K}_{3/2}(rN/L)}{\sqrt{r}}\right)\sin\theta. \quad (26)$$

With allowance for solution (26), the coordinate of the velocity curl may be represented as follows:

$$\Omega = f_\Omega(r)\sin\theta = \left(\frac{1}{1-N^2}\frac{B}{r^2} + 2N^2 D\frac{\mathrm{K}_{3/2}(rN/L)}{\sqrt{r}}\right)\sin\theta.$$

Taking into account the conditions at infinity, the components of velocity will be sought in the form of $u(r,\theta) = f_u(r)\cos\theta$, $v(r,\theta) = f_v(r)\sin\theta$. Then, the continuity equation and expression for the velocity curl in the dimensionless-coordinate form will compose a set of equations for finding the $f_u(r)$ and $f_v(r)$ functions:

$$f_u' + 2\frac{f_u + f_v}{r} = 0,$$

$$f_v' + \frac{f_u + f_v}{r} = \frac{1}{1-N^2}\frac{B}{r^2} + 2N^2 D\frac{\mathrm{K}_{3/2}(rN/L)}{\sqrt{r}}.$$

Its solution is as follows:

$$f_u(r) = \frac{1}{1-N^2}\frac{B}{r} - \frac{E}{r^3} + F - 4L^2 D\frac{\mathrm{K}_{3/2}(rN/L)}{\sqrt{r^3}},$$

$$f_v(r) = \frac{1}{N^2-1}\frac{B}{2r} - \frac{E}{2r^3} - F - 2L^2 D\left(\frac{\mathrm{K}_{3/2}(rN/L)}{\sqrt{r^3}} + \frac{N}{L}\frac{\mathrm{K}_{1/2}(rN/L)}{\sqrt{r}}\right).$$

The expression for pressure is found from the θ-projection of the second equation of set (24):

$$p(r,\theta) = \frac{1}{1-N^2}\frac{B}{r^2}\cos\theta.$$

The redefinition of the constants for a more compact notation of the solution with allowance for the condition at infinity yields the following final solution in the dimensional form:

$$u(r,\theta) = U\left(C_1\frac{R_s^3}{r^3} + C_2\frac{R_s}{r} + 1 + C_4\frac{\mathrm{K}_{3/2}(rN/(R_sL))}{(r/R_s)^{3/2}}\right)\cos\theta,$$

$$v(r,\theta) = U\left[\frac{C_1}{2}\frac{R_s^3}{r^3} - \frac{C_2}{2}\frac{R_s}{r} - 1 + \frac{C_4}{2}\left(\frac{\mathrm{K}_{3/2}\left(\frac{rN}{R_sL}\right)}{(r/R_s)^{3/2}} + \frac{N}{L}\frac{\mathrm{K}_{1/2}\left(\frac{rN}{R_sL}\right)}{\sqrt{r/R_s}}\right)\right]\sin\theta,$$

$$\omega(r,\theta) = \frac{U}{R_s}\left(\frac{C_2}{2}\frac{R_s^2}{r^2} - \frac{C_4}{4L^2}\frac{\mathrm{K}_{3/2}(rN/(R_sL))}{\sqrt{r/R_s}}\right)\sin\theta,$$

$$p(r,\theta) = \mu U R_s\frac{C_2}{r^2}\cos\theta.$$

In the problem under consideration, the calculation of the drag force and torque that an oncoming stream applies to a sphere that it flows over is of practical interest.

The projection of the drag force onto the direction of the oncoming stream and the torque are calculated by integrating corresponding force and couple stresses over the sphere surface:

$$F_\mathrm{d} = \int_0^\pi \left(t_{rr}\big|_{r=R_s}\cos\theta - t_{r\theta}\big|_{r=R_s}\sin\theta\right)2\pi R_s^2\sin\theta\,d\theta, \quad (27)$$

$$\mathbf{M}_\mathrm{d} = \int_0^{2\pi}\int_0^\pi \mathbf{e}_\varphi m_{r\varphi}\big|_{r=R_s} R_s^2\sin\theta\,d\theta\,d\varphi,$$

where $t_{rr} = -p + 2\mu \frac{\partial u}{\partial r}$, $t_{r\theta} = (\mu + \kappa)\frac{\partial v}{\partial r} + (\mu - \kappa)\frac{1}{r}\left(\frac{\partial u}{\partial \theta} - v\right) - 2\kappa\omega$, and $m_{r\varphi} = (\delta + \varsigma)\frac{\partial \omega}{\partial r} + (\varsigma - \delta)\frac{\omega}{r}$. Since the components of the couple-stress tensor and, in particular, $m_{r\varphi}$ are independent of φ, while vector $\mathbf{e}_\varphi$ is collinear to the equatorial plane of the spherical coordinate system, its integration over φ yields zero. That is, the integral expressing $\mathbf{M}_d$ is equal to zero. This result is independent of the values of constants C_i and, hence, remains correct under any boundary conditions at the sphere surface, as might be expected from the considerations of symmetry. Another situation takes place for the drag force.

The classical Stokes equation was obtained under the condition of no-slip at the sphere surface. Its analogs under different conditions at the sphere surface, i.e., the drag forces applied to a sphere with radius R_s moving at velocity U in a micropolar liquid are calculated by Eq. (27). For this purpose, we initially find a particular solution of the flow problem under corresponding conditions and calculate the required components of the stress tensor on the sphere surface. The result of the calculation under conditions $u(R_s, \theta) = v(R_s, \theta) = \omega(R_s, \theta) = 0$ may be represented as

$$F_0 = 6\pi U R_s \mu \left(1 + \frac{\frac{\kappa}{\mu}}{1 + 2R_s\sqrt{\frac{\kappa(\mu+\kappa)}{\mu(\delta+\varsigma)}}}\right).$$

This formula in a somewhat different form was obtained in [61]. However, the authors did not take into account the aforementioned difference in the definitions of the dynamic viscosity and, as a result, arrived at a not quite true estimate of the correction factor when comparing the obtained expression with the Stokes formula for a Newtonian liquid. The extension of the formula for the drag force to the case of a nonstationary flow around was derived in [62] and analyzed for four different time dependences of oncoming-flow velocity, namely, for damped and undamped oscillations and uniform and jumpwise accelerations.

Let us write the Navier slip condition introducing slip length l: $\mu v(R_s, \theta) = l\, t_{r\theta}\big|_{r=R_s}$ and leaving the remaining two other boundary conditions unchanged, namely, $u(R_s, \theta) = \omega(R_s, \theta) = 0$. The calculation of the projection of the drag force yields

$$F_l = 6\pi U R_s \mu \left(1 + \frac{\frac{\kappa}{\mu} - \frac{l}{R_s}\left(1 + 2R_s\sqrt{\frac{\kappa(\mu+\kappa)}{\mu(\delta+\varsigma)}} - \frac{\kappa}{\mu}\right)}{1 + 2R_s\sqrt{\frac{\kappa(\mu+\kappa)}{\mu(\delta+\varsigma)}} + 3\frac{l}{R_s}\left(1 + 2R_s\sqrt{\frac{\kappa(\mu+\kappa)}{\mu(\delta+\varsigma)}} + \frac{\kappa}{3\mu}\right)}\right).$$

An analogous expression was derived in [63].

A flow of a micropolar liquid past a sphere under boundary condition $\boldsymbol{\omega}\big|_b = n\,\mathrm{rot}\mathbf{v}\big|_b$ and zero linear velocity at the sphere surface was considered by Hoffmann et al. [41]. In the same work, the obtained result was analyzed in comparison with all previous studies and the possibility of continuous transition between them was shown. Nevertheless, the obtained expression was not quite suitable for practical application, because the values of coefficient n remained unknown.

In the case of complete slip for the angular velocity and no-slip for the linear one, i.e., at $m_{r\varphi}\big|_{r=R} = 0$ and $u(R_s, \theta) = v(R_s, \theta) = 0$, the corresponding expression for the projection of the drag force is as follows:

$$F_m = 6\pi U R_s \mu \left(1 + \frac{1}{\frac{(3\delta+\varsigma)(\mu+\kappa)}{4\kappa\delta} - 1 + \frac{R_s}{2}\frac{(3\delta+\varsigma)}{\delta}\sqrt{\frac{\mu(\mu+\kappa)}{\kappa(\delta+\varsigma)}} + R_s^2\frac{\mu}{\delta}}\right).$$

The simultaneous consideration of the conditions of finite slip for the linear and angular velocities encounters difficulties of only technical but not fundamental character. The resulting expressions are rather awkward and are not presented here.

Note that, at $l \to 0$, $F_l \to F_0$, while at $\kappa \to 0$ the classical value of the drag force is obtained under all considered conditions; namely:

$$\lim_{\kappa \to 0} F_l = 6\pi U R_s \mu \left(1 - \frac{l}{R_s + 3l}\right),$$

$$\lim_{\kappa\to 0} F_0 = \lim_{\substack{\kappa\to 0\\ l\to 0}} F_l = \lim_{\kappa\to 0} F_m = 6\pi U R_s \mu.$$

The case of a complete slip for the linear-velocity component corresponds to $l \to \infty$. Physically, this means that the tangential stress at the sphere surface is equal to zero, while the velocity component is finite. In this case, for drag force F_l at $\kappa \to 0$, we obtain a result known in classical hydrodynamics,

$$\lim_{\substack{\kappa\to 0\\ l\to\infty}} F_l = 6\pi U R_s \mu \frac{2}{3} = 4\pi U R_s \mu,$$

while, at $\kappa \neq 0$, we have

$$F_{l\to\infty} = 4\pi U R_s \mu \left(1 + \frac{\frac{2}{3}\frac{\kappa}{\mu}}{1 + 2R_s\sqrt{\frac{\kappa(\mu+\kappa)}{\mu(\delta+\varsigma)}} + \frac{\kappa}{3\mu}} \right),$$

i.e., as might be expected, the force is somewhat stronger than that in a Newtonian liquid under analogous conditions.

Finally, let us consider the limiting expressions for spheres with infinitesimal radius R_s, with these expressions being of importance for the subsequent consideration. Under the condition of complete no-slip, we obtain $F_0|_{R_s\to 0} \simeq 6\pi U R_s \mu \left(1 + \frac{\kappa}{\mu}\right)$; i.e., the projection of the drag force may be calculated using an analog of the classical Stokes formula, in which vorticity viscosity κ should be added to dynamic viscosity μ.

In the presence of finite nonzero slip, we have

$$F_l|_{R_s\to 0} \simeq 6\pi U R_s \mu \left(1 - \frac{1 - \frac{\kappa}{\mu}}{3 + \frac{\kappa}{\mu}} \right) = 4\pi U R_s \mu \frac{1 + \frac{\kappa}{\mu}}{1 + \frac{\kappa}{3\mu}}.$$

In the obtained expression, slip length is absent as well as at infinite slip, because the relation $R_s/l \to 0 \Leftrightarrow R_s \to 0 \wedge l \to \infty$ takes place. Nevertheless, it coincides with $F_{l\to\infty}$ only if we take $R_s = 0$ in $F_{l\to\infty}$. For a sphere of finite radius, the equation for the drag force at infinite slip contains this radius in the correction term.

Unfortunately, at $R_s \to 0$, the expression for the drag force with slip cannot be reduced to an analogous limit under the condition of complete no-slip. This is natural, because the R_s/l ratio requires an extension of the definition. Cases $R_s/l \to 0$ and $R_s/l \to \infty$ are reduced to the above-considered limits. At $R_s/l \to \text{const}$, in the denominator of the expression for the drag force, only one term may be omitted, thereby only slightly simplifying the equation for F_l.

Upon a slip for the angular velocity, the corresponding limit contains all viscosity coefficients of a micropolar liquid:

$$F_m|_{R_s\to 0} \simeq 6\pi U R_s \mu \left(1 + \frac{1}{\frac{(3\delta+\varsigma)(\mu+\kappa)}{4\kappa\delta} - 1} \right).$$

Note that, formally, there is a nonzero subset of viscosity coefficient values, at which the correction factor with classical term $6\pi U R_s \mu$ becomes negative. Nevertheless, this situation is not realized because of the thermodynamic constraints on the values of viscosities. To estimate the order of the correction term, we take $\varsigma = \delta$. Then, $F_m|_{R_s\to 0} \simeq 6\pi U R_s \mu \left(1 + \frac{\kappa}{\mu}\right)$, thereby coinciding with the expression obtained under the condition of complete no-slip.

In view of the great importance of problems concerning diverse motions of spherical or almost spherical particles in a micropolar liquid, there are many studies devoted to this scope. The solution of the problem concerning the stationary flow past a spheroid has been presented in [64], while the drag force applied to the spheroid has been calculated in [65]. In the case of a stationary flow past an arbitrary axially symmetric body, the formula for the drag was presented in frequently cited work [66]; later, it was reported in [67] in a more exact mathematical form.

Slow rotations of a solid sphere in a micropolar liquid were considered in, e.g., [68], while linear and torsional vibrations of a sphere were studied in [69]; then, the results were extended to the cases of vibrations and rotations of a spheroid in [70, 71] and the case of an arbitrary body of revolution in [72, 73]. In the aforementioned works, the classical conditions of no-slip at a solid surface were used; then, the problems implying the condition of slip for the linear velocity were solved, and, finally, the conditions of slip for both linear and angular velocities [30]. Rotation of a micropolar liquid sphere in a concentric Newtonian-liquid capsule was simulated in [74]; however, an aforementioned incorrect interpretation was given there for the viscosity coefficient of a micropolar liquid. In [75], the same authors considered torsional vibrations of a micropolar spherical droplet in a viscous spherical shell. The torque applied to the droplet was calculated. It is known that, in some specific cases, vibrations may acquire a resonant character associated with a decrease in the drag. Combinations of parameters, at which this effect is observed for a sphere subjected to translational vibrations in a micropolar liquid, were studied in [76].

The Hadamard-Rybczynski problem of a flow past a liquid droplet was developed for the case of a micropolar spherical droplet in [77].

FILTRATION OF A MICROPOLAR LIQUID IN A POROUS MEDIUM

Flows of Newtonian liquids in porous media are adequately described by the Darcy law, which establishes a linear relation between a constant filtration rate and a pressure gradient. The calculations performed by this law are in excellent agreement with experimental data for large regions of flows, in which boundary effects may be ignored. Analogs of the Darcy law for the linear and angular velocities of a micropolar medium were obtained for stationary and nonstationary flows in [78, 79]. The homogenization method was used under the assumption that the porous medium had a regular periodic structure, with the period of this structure playing the role of a small parameter. The permeability coefficients in the derived equations had a tensor nature and were calculated on the basis of the solution of a local problem for an individual cell.

In the case of liquid filtration in thin capillaries or membranes containing micro- and nanosized pores and upon a conjugate flow of a liquid in free and porous spaces, the Brinkman equation appears to be preferable. It describes the velocity of a filtration flow in regions of any scales. Far from the boundaries of a region, it yields a solution that coincides with the rate calculated by the Darcy law, while, at intermedium boundaries, it enables one to impose conditions corresponding to the physical character of a problem, because it comprises a differential term of the second order with respect to a coordinate.

The simulation of the flows of micropolar liquids in porous media is extremely urgent for problems of micro- and ultrafiltration, description of flows of blood and other physiological fluids, the problems of oil production that are associated with flows in microcracks, etc. Since the effects of liquid micropolarity manifest themselves namely in narrow channels, where the key role is played by the conditions at the boundaries, it is of special importance to use an adequate mathematical model of the filtration motion on these scales, e.g., in the form of an analog of the Brinkman equation.

In the original work by Brinkman [80], the motion equation was derived phenomenologically. The balance of forces applied to an element of a liquid was governed by the pressure gradient, on the one hand, and the drag resulting from the viscosity and the damping effect of a porous matrix on the other. The mathematical representation of the latter term follows from the notation of the Stokes force, if the porous matrix is interpreted as a conglomerate of spherical particles. Thus, the Brinkman equation for a Newtonian liquid has the following form:

$$\nabla p = \mu' \Delta \mathbf{v} - \frac{\mu}{k}\mathbf{v},$$

where k is the permeability of a porous medium and μ' is the effective viscosity of a permeating liquid, which is not necessarily equal to μ. In almost half a century, Ochoa-Tapia and Whitaker [81] derived the Brinkman equation as a correction to the Darcy law by averaging the Stokes equations over a small medium volume, which contained solid and liquid parts and simulated a porous body filled with a liquid. In the same work, it was shown that the pressure gradient should be averaged not over the entire representative volume, but rather over the part of it that is occupied by pores. Moreover, it was found that $\mu' = \mu/\varepsilon$, and this does not require other correction; ε is the porosity, which is equal to the medium volume fraction occupied by pores.

Repeating Brinkman's considerations on a pressure-gradient-induced slow stationary flow of a micropolar liquid in a porous medium, let us write the motion equation as follows:

$$\nabla p = (\mu' + \kappa')\Delta \mathbf{v} + 2\kappa' \nabla \times \boldsymbol{\omega} - \frac{\mu + \kappa}{k}\mathbf{v}, \qquad (28)$$

where μ' and κ' are the effective viscosities, while the Darcy term is written taking into account the drag forces for a solid sphere under the conditions of no-slip, no-spin on its surface. Note that, when the slip of microrotation is taken into account, the term of the Darcy type remains unchanged, while, at the slip for the linear velocity at a solid surface, this term should be represented in the form of $-\frac{\mu + \kappa}{k}\left(1 + \frac{\kappa}{3\mu}\right)^{-1}\mathbf{v}$.

In [82], a set of equations that determine the flow of a micropolar liquid in a porous medium has been rigorously derived by the standard method of averaging over a representative volume using the conditions of no-slip, no-spin at solid surfaces. The continuity equation written for the averaged velocity coincides with the unaveraged equation. The motion equation coincides with Eq. (28) with an accuracy of factor k under condition $\mu' = \mu/\varepsilon, \kappa' = \kappa/\varepsilon$. The integrals that describe this factor depend on the geometric characteristics of a porous medium, which were presented in [82] by a rather cumbersome function of several values difficult to determine. However, for applied problems, it is not only sufficient but, in effect, more correct to confine ourselves to one general coefficient k, if we take into account the contemporary deficiency of information on the values of viscosity coefficients for a micropolar liquid. In [83], one of the authors of [82] has shown that, for slow flows, the aforementioned factor may be written as $1/k$, which completely corresponds to Eq. (28), which may be presented in the final form as

$$\nabla p = \left(\frac{\mu}{\varepsilon} + \frac{\kappa}{\varepsilon}\right)\Delta \mathbf{v} + 2\frac{\kappa}{\varepsilon}\nabla \times \boldsymbol{\omega} - \frac{\mu + \kappa}{k}\mathbf{v}. \qquad (29)$$

The form of the averaged-moment equation essentially depends on the properties of vector field $\boldsymbol{\omega}$ and the geometry of a flow. In the general case of a slow stationary flow, the moment equation takes the following form [82]:

$$0 = (\alpha + \delta - \varsigma)\nabla\langle\nabla \cdot \boldsymbol{\omega}\rangle + (\delta + \varsigma)\Delta\boldsymbol{\omega} + 2\kappa\nabla \times \mathbf{v} - 4\kappa\boldsymbol{\omega},$$

where averaged angular-velocity divergence $\langle\nabla \cdot \boldsymbol{\omega}\rangle = \frac{1}{V}\int_V \nabla \cdot \boldsymbol{\omega} dV$ is used. If the geometry of a flow has a dimensionality lower than 3 or is symmetric, $\nabla \cdot \boldsymbol{\omega} = 0$, and this term vanishes. In the same work [82], for essentially three-dimensional flows, it has been proposed to impose condition $\nabla \cdot \boldsymbol{\omega} = 0$ and introduce additional unknown p^*, which is interpreted as the surface pressure necessary to damp the rotational motion when it approaches surfaces on which condition $\boldsymbol{\omega}|_b = 0$ has been imposed. Then, the averaged-moment equation acquires the following form:

$$\nabla p^* = (\delta + \varsigma)\Delta\boldsymbol{\omega} + 2\kappa\nabla \times \mathbf{v} - 4\kappa\boldsymbol{\omega} - f(\delta + \varsigma)\boldsymbol{\omega},$$

where coefficient f may be taken equal to $1/k$.

For practically important planar, cylindrical, and spherical geometries of a flow, condition $\nabla \cdot \boldsymbol{\omega} = 0$ is fulfilled automatically, and the averaged moment equation may be represented as

$$0 = (\delta + \varsigma)\Delta\boldsymbol{\omega} + 2\kappa\nabla \times \mathbf{v} - 4\kappa\boldsymbol{\omega}. \quad (30)$$

Note that it contains no additional terms of the Darcy type in contrast to the momentum equation. This result is also obtained in the Brinkman heuristic way with allowance for the zero moment applied to a sphere from the side of a micropolar liquid. In [84], the averaged equations for a filtration flow of a micropolar liquid have been obtained for porous media of any nature with a variable porosity, which takes any values from zero to unity. In a particular case of a constant porosity, they coincide with the equations presented above.

Thus, Eqs. (29) and (30), together with the continuity equation for the averaged velocity, represent a system that describes the filtration of a micropolar liquid through a porous medium with permeability k and fixed porosity ε. This formulation was used in [85] to describe a planar Poiseuille flow in a porous channel. The authors of [86] tried to use the Brinkman model for describing a micropolar liquid flow past a porous sphere. However the term of the Darcy type was written with coefficient μ rather than $\mu + \kappa$ without any explanations.

Many works devoted to flows associated with porous regions represent the effect of porosity via the blow–suction model in accordance with one or another law. This idea was proposed in [87] and has been further developed in the direction of complication of the types of motion and methods of the delivery–evacuation of a liquid through walls. An example of the use of the homotopy method for such problems has been presented in [88]. The nonstationarity of a flow has been taken into account for a planar geometry in [89]. Since the a priori specification of flow characteristics is, in authors' opinion, less preferable, the aforementioned model of the Brinkman type will be used below.

CELL MODELS

One of the most common and efficient methods for simulating flows in porous media is the cell method proposed by Happel and Brenner [90]. For Newtonian liquids, this method has been well developed and shown itself to be a reliable model for describing filtration flows, in particular, through membranes [91, 92]. According to this method, a porous medium is replaced by a regular system composed of identical cells of, as a rule, a spherical or cylindrical shape. Each cell consists of a core and a liquid shell. The core may be solid, porous, liquid, or consisting of a combination of these phases.

At present, cell models are being intensely developed as applied to micropolar flows. The cell method has been described in detail within the framework of a cylindrical geometry for a micropolar liquid in [93], where both parallel and transverse flows were considered. The core of the cell was assumed to be solid and impermeable, with conditions of slip for both linear and angular velocities being imposed on its surface. At the external boundary of the cell, the Happel condition was used, i.e., the equality of tangential stresses to zero, supplemented with the same condition for couple stresses. The Kozeny constant, which enters into the Kozeny–Carman equation for the permeability of a porous medium, was estimated as a resulting macrocharacteristic.

The development of a classical model for a spherical cell with a liquid core for the cases of a micropolar core surrounded with a nonpolar viscous shell and, vice versa, a viscous core in a micropolar shell was described in [94]. Four types of possible boundary conditions known for nonpolar liquids, namely, Happel [95], Kuwabara [96], Kvashin [97], and Cunningham [98] (Mehta–Morse [99]), were considered for the external surface of the cell. The zero angular velocity was used as the second condition on the external boundary of the cell. In all cases, analytical solutions and expressions for the drag force applied to the cell were obtained, and their dependences on the characteristic parameters of the problem were studied. The cell model for a viscous spheroid located in a micropolar spheroidal shell was developed in [100]. The effects of boundary conditions and parameters of a micropolar medium on the drag force applied to the viscous spheroid were studied.

As far as we know, the cell model with a combined solid–porous core streamlined by a micropolar fluid has been considered neither in the cylindrical nor in the spherical geometry so far. Here, we shall propose a formulation of such problem in a vector form, which is suitable for the consideration of both longitudinal and transverse flows in a cylindrical cell, as well as for a spherical cell after the transition to a corresponding coordinate system.

Assume that a cell consists of an impermeable core with radius a, a concentric porous shell with radius b, while the radius of an external liquid concentric layer of the cell is denoted as c. All variables in the region of the free flow, $b < r < c$, and in the porous region, $a < r < b$, will be denoted by subscripts 1 and 2, respectively. Variable r means the radial coordinate of a cylindrical or a spherical coordinate system. Owing to the system symmetry, we, for all three considered configurations of the flow, have $\nabla \cdot \boldsymbol{\omega} = 0$, and this relation is used when writing the moment equations for both the first and the second regions. Thus, a slow stationary flow of a micropolar liquid under the Stokes approximation in the absence of external forces and moments is described by the following set of equations:

$$\begin{gathered}\nabla \cdot \mathbf{v}_1 = 0,\\ \nabla p = (\mu + \kappa)\Delta \mathbf{v}_1 + 2\kappa \nabla \times \boldsymbol{\omega}_1,\\ 0 = (\delta + \varsigma)\Delta \boldsymbol{\omega}_1 + 2\kappa \nabla \times \mathbf{v}_1 - 4\kappa \boldsymbol{\omega}_1.\end{gathered} \tag{31}$$

For a filtration flow in the porous region, the set of motion equations has another form:

$$\begin{gathered}\nabla \cdot \mathbf{v}_2 = 0,\\ \nabla p = \frac{\mu + \kappa}{\varepsilon}\Delta \mathbf{v}_2 + 2\frac{\kappa}{\varepsilon}\nabla \times \boldsymbol{\omega}_2 - \frac{\mu + \kappa}{k}\mathbf{v}_2,\\ 0 = (\delta + \varsigma)\Delta \boldsymbol{\omega}_2 + 2\kappa \nabla \times \mathbf{v}_2 - 4\kappa \boldsymbol{\omega}_2.\end{gathered} \tag{32}$$

For a longitudinal flow in a cylindrical cell, the pressure gradient is commonly preset, while the Oz axis of a coordinate system is superposed with the cylinder symmetry axis. Then, the vectors of the linear and angular velocities have one nonzero component each, with the components being dependent only on the radial coordinate. The continuity equation is satisfied identically, while two other second-order equations in sets (31) and (32) require eight boundary conditions. Two of the conditions are imposed on internal boundary $r = a$, four of them are used at boundary $r = b$, and two other conditions are set at boundary $r = c$.

For a transverse flow through the cylindrical cell, the oncoming-flow velocity is preset, while the pressure is the unknown function. The vectors of the linear and angular velocities have two and one nonzero components, respectively. Although all four unknown functions depend on two coordinates, the dependence on the angular coordinate is determined from the conditions of symmetry, while functions of variable r are to be found. Each of systems (31) and (32) consists of one scalar first-order differential equation and three scalar second-order differential equations for velocity components and contains a first-order differential operator applied to pressure. Thus, in order to close the boun-dary problem, twelve boundary conditions are necessary in this case. Three conditions are imposed on each of internal boundary $r = a$ and external boundary $r = c$, while six conditions are imposed on boundary $r = b$.

The same number of conditions are required to be imposed on the boundaries of a spherical cell, if the symmetry axis of a spherical coordinate system is superposed with the direction of the velocity vector of the oncoming flow. Note that, in cell models, the oncoming flow is preset at the external boundary of a cell rather than at infinity.

When describing a filtration flow by system (32), the most correct conditions at internal boundary $r = a$ are the no-slip and no-spin conditions $\mathbf{v}|_{r=a} = 0$, $\boldsymbol{\omega}|_{r=a} = 0$, because the equations of set (32) were derived under the assumption of namely this type of the interaction between a flow and solid walls. Other types of boundary conditions at $r = a$ are also possible; however, they require substantial revision of the equations of set (32).

Researchers who dealt with studying Newtonian liquids have focused intense attention on the boundary $r = b$ between a porous medium and free flow. The continuity conditions of normal and tangential stresses, as well as of the velocity vector at this boundary, have been recognized to be most adequate. They can be easily satisfied owing to the use of the Brinkman equation rather than the Darcy equation, which implies the introduction of fitting parameters for conjugation of velocity fields in a porous medium and a free flow. Developing this idea for micropolar liquids, we obtain $\mathbf{v}_1|_{r=b} = \mathbf{v}_2|_{r=b}, \boldsymbol{\omega}_1|_{r=b} = \boldsymbol{\omega}_2|_{r=b}$. In addition to force stresses, it is necessary to consider the couple stresses. At a longitudinal flow in a cylindrical cell, normal stresses at the considered boundary are identically equal, while the tangential force and couple stresses are expressed via components t_{rz} and $m_{r\theta}$, respectively. Thus, for a longitudinal flow in a cylindrical cell, we obtain conditions $t_{rz1}|_{r=b} = t_{rz2}|_{r=b}$, $m_{r\theta 1}|_{r=b} = m_{r\theta 2}|_{r=b}$. For a spherical cell and a transverse flow in a cylindrical cell, the normal and tangential stresses are specified by components t_{rr} and $t_{r\theta}$, while the tangential couple stresses are preset by component m_{13}, i.e., by components $m_{r\varphi}$ and m_{rz} for the spherical and cylindrical geometries, respectively. In this case, the boundary conditions have the following form:

$$t_{rr1}|_{r=b} = t_{rr2}|_{r=b}, \quad t_{r\theta 1}|_{r=b} = t_{r\theta 2}|_{r=b},$$
$$m_{13}|_{r=b-0} = m_{13}|_{r=b+0}.$$

In conjugate flows of Newtonian liquids at a boundary between a porous medium and a free flow, a condition of a jump in tangential stresses is sometimes set [81]: the tangential stress at the interfacial boundary from the side of the porous region differs from the tangential stress from the side of the free flow by a value of $\eta v_{\tan}\mu/\sqrt{k}$, where $\eta \in [-1; 1]$ is a fitting parameter and $v_{\tan}$ is the flow-velocity component tangential to the surface under consideration. Note that the authors who have proposed this boundary condition have not explained its physicochemical sense. Moreover, they noted that it was introduced to simplify the calculations in the problems taking into account the heat exchange. This condition can be used for isothermal flows; however, it should be physically substantiated. It should also be noted that, when solving a rather simple boundary value problem in [101], it was shown that the limits of variations in parameter η are somewhat different: $-\infty < \eta < \sqrt{\mu'/\mu}$. A jump in the tangential stresses and tangential-couple stress in micropolar flows may also be ascribed to the boundary under consideration. Nevertheless, the derivation of specific expressions for the values of the aforementioned jumps represents an independent informal problem.

External boundary $r = c$ of the cell remains, so far, to be the least studied object. On the one hand, it is associated with the external flow and plays the role of an infinitely remote surface. On the other hand, it is necessary to take into account the interaction between the cells and impose corresponding conditions namely at $r = c$. In classical cell models, the continuity of normal velocity component v_r is traditionally used. For the longitudinal flow in a cylinder, this condition is fulfilled automatically, while, for the transverse flow past a cylinder and a sphere, it is represented as $v_{r1}\big|_{r=c} = U\cos\theta$, where U is the velocity of an oncoming flow.

As the second condition for Newtonian liquids at the external boundary, that of the four relations is commonly considered, in favor of which experimental data witness within the framework of a specific application. In the opposite case, all four conditions are considered to be equivalent. The Happel model [95] implies the absence of tangential stresses at the cell surface, $t_{r\theta 1}\big|_{r=c} = 0$. Kuwabara [96] has proposed to impose the condition of the absence of vorticity at this boundary, $\operatorname{curl}\mathbf{v}_1\big|_{r=c} = 0$. According to the Kvashin model [97], the profile of the flow velocity is symmetric; i.e., $\left.\frac{\partial v_{\tan 1}}{\partial r}\right|_{r=c} = 0$. At last, the Mehta–Morse condition [99] (which was initially formulated by Cunningham [98]) means the uniformity of a flow and is expressed as $v_{\tan 1}\big|_{r=c} = -U\sin\theta$. For the flow of a Newtonian liquid along the axis of a cylindrical cell, all four conditions coincide [92], while for the flow in a spherical cell and a transverse flow in a cylindrical cell, they are essentially different [91, 92]. In terms of the micropolar model for a longitudinal flow in a cylindrical cell, only three last conditions are equivalent. Owing to the presence of the angular velocity of microrotation in the expression for components of stress tensor, the Happel condition essentially differs from the others.

In order to close the boundary problem, it is necessary to impose one more condition on boundary $r = c$. It must, obviously, comprise information on the state of microrotations or couple properties of a medium. The absence of tangential couple stresses on the cell surface is an analog of the Happel condition. An example of its use for cylindrical cells of both orientations can be found in [93]. The absence of the solid-state rotation (the Kuwabara condition) should, obviously, be replaced by the complete absence of rotation, $\boldsymbol{\omega}_1\big|_{r=c} = 0$, although it is mathematically admissible to consider both conditions to be independent [94]. According to Kvashin, the concepts of the symmetry may be extended to the angular velocity as well. The condition of the uniformity of a flow of the Cunningham type coincides with $\boldsymbol{\omega}_1\big|_{r=c} = 0$. Thus, we obtain three types of new independent boundary conditions at the external boundary of the cell. They may be arbitrarily combined with the classical Happel, Kuwabara, Kvashin, and Cunningham conditions. The comparison of twelve solutions, which will be obtained, is the goal of a separate study. The consideration of the aforementioned conditions of the jump as an alternative to the continuity conditions at the $r = b$ boundary at least doubles the number of the resulting variants. The comparison of such number of solutions will possibly provide valuable information for understanding the specific features of the flows described by the cell models, peculiarities of boundary conditions, and the extent of their influence on the flow parameters.

CONCLUSIONS

In this review, problems relevant to simplest configurations of flows have been formulated within the framework of the theory of isothermal micropolar liquids. The overwhelming majority of the cited works, in which the aforementioned problems have been considered, yield their analytical solutions. The formulation proposed for the cell model of micropolar liquid filtration also admits analytical solutions for the cases of simple geometric shapes of the cells. They will be considered in our separate works.

In conclusion, let us briefly consider the main applications of the theory of micropolar liquids. They began to be intensely developed immediately after a corresponding mathematical apparatus had been elaborated. Micropolar models of blood are widely applied, because they enable one to consider both the translational and rotational motions of erythrocytes and other components present in blood plasma. The idea to simulate blood as a micropolar liquid was proposed in one of the earliest theoretical works by Eringen [79], because it was already known that rotation of erythrocytes causes their nonuniform distribution over the cross section of a capillary. Models that take into account the pulsing character of the bloodstream are widely used. A detailed review and bibliography of works, as well as comparison of a micropolar model of blood with a micromorphic one, i.e., one taking into account the possibility of erythrocyte deformation, have been presented in monograph by Migun and Prokhorenko [15]. Then, a drastic decrease in the concentration of blood particles near vessel walls was revealed, and models were elaborated for conjugate flows of micropolar and Newtonian liquids simulating a near-wall flow of pure plasma [102]. There are several models that take into account blood filtration through capillary walls, with porous regions being described by the Brinkman or Darcy equations [103]. At present, models of angiostenosis are being developed [104–106].

In addition to blood, other biological fluids—in particular, synovia in joints—are described in terms of the micropolar model [107]. More complex models take into account the porous structure [108, 109] and surface roughness [110] of joints. Contemporary models for the operation of joints take into account both their surface roughness and porous structure with possible penetration of synovia into it [111]. In some cases, mucus is also considered as a micropolar medium, in which the motion of elementary organisms or, e.g., propagation of spermatozoids through the cervical channel, is simulated [112].

The theory of micropolar liquids is rather efficiently used in the hydrodynamic theory of lubrication, because the gap between a spindle and a bearing is rather thin, while lubricants are either contaminated with metal chips or filled with special dopants. One of the parameters that determine the properties of a micropolar flow has the meaning of the microscale, thereby making it possible to simulate lubricants containing different additives [113]. Some works have been devoted to studying liquid films squeezed between two planes (see, e.g., [114–116]); the behavior of liquid lubricants in bearings of different geometries [117] and those with porous walls [118] has been studied. Microscale processes affecting the operation of a bearing essentially depend on the properties of lubricated surfaces, in particular, on their roughness. A model that takes into account the interaction of a micropolar lubricant with a rough surface has been proposed in [119]. The applicability of all aforementioned data on the theory of lubricants is limited by the conditions of no-slip, no-spin at solid surfaces. The new interpretation of dynamic boundary conditions with the use of the so-called boundary viscosity has been described in [120], while the development of the theory of lubricants has been presented in [121].

It has been shown that the micropolar theory is applicable for the simulation of suspensions [122, 123] with some stipulations relevant to dilute suspensions [124]. Moreover, the theory has found application in the description of the dynamics of drilling fluids. For example, the slip at well walls with clay shells formed on them was described introducing a thin layer of a micropolar liquid into the consideration [125].

The theory of micropolar flows is also applicable to describing the motion of granulated media [126–128]. The contemporary state of the theory of granulated media with the micropolar model appended to it can be found in [129].

Since the effects of the micropolarity are strongest near the boundaries of the flow region, the theory of boundary layers is widely used for micropolar liquids. This theory has been described for a planar case in [130]. The theory of a three-dimensional boundary layer for an arbitrary surface has been developed by Korzhov [131].

Many works have been devoted to the problems concerning the stability of micropolar liquids both isothermal compressible and incompressible and heated by different methods. Among the first works devoted to this problem, communications [132, 133] should be noted, while detailed reviews and bibliographies have been presented in works by Migun and Prokhorenko [15], Lukashevich [10], and Eringen [9].

Micropolar models of turbulence proposed for the first time by Peddiesson [134] and Ahmadi [135] are been gradually developed. Not only a linear, but also an inherent angular velocity is attributed to turbulent vortices. The contemporary state of the theory of turbulence, which has been called as postclassical, is described in review [136].

Liquid-penetrant test is one of scarcer experimental applications of the micropolar theory. For example, in [137], it was proposed to simulate liquids used for liquid-penetrant testing as micropolar ones, which has made it possible to explain the effect of viscosity at small capillary radii.

Advances in the use of numerical algorithms for solution of the problems under consideration, i.e., works devoted to the study of liquid crystals, which is the most developed field of the application of micropolar theory, have remained beyond the scope of this review, because they comprise electrodynamic aspects that require additional discussion and magnetohydrodynamic investigations.

The study of micropolar liquids flowing in porous media is a promising field, although less developed yet. The problem formulations proposed in this review within the frameworks of the cell model are destined to eliminate this drawback. While numerous analytical and numerical studies of the flows of micropolar liquids are available, experimental works are scarcer. The lack of data on the values of characteristic viscosities is, often, the main factor hindering the active use of the developed models in practice. The authors hope that this review will attract the attention of experimenters, thereby substantially stimulating the development of applications of models of micropolar liquids.

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