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The Generalized Zhang's Operator and Kastler–Kalau–Walze Type Theorems

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Abstract— In this paper, we obtain two Lichnerowicz type formulas for the generalized Zhang's operator. And we give the proof of the Kastler–Kalau–Walze type theorem for the generalized Zhang's operator on 4-dimensional oriented compact manifolds with (respectively, without) boundary.

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1. INTRODUCTION

The noncommutative residue found in [6, 15] plays a prominent role in noncommutative geometry. For arbitrary closed compact n-dimensional manifolds, the noncommutative residue was introduced by Wodzicki in [15] using the theory of zeta functions of elliptic pseudo-differential operators. In [2], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analog. Furthermore, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein–Hilbert action in [3]. Let s be the scalar curvature and Wres denote the noncommutative residue. Then Kastler–Kalau–Walze type theorems give an operator-theoretic explanation of the gravitational action and say that for a 4–dimensional closed spin manifold, there exists a constant c_0 such that

$$\mathrm{Wres}(D^{-2}) = c_0 \int_M s d\mathrm{vol}_M. \tag{1}$$

In [8], Kastler gave a brute-force proof of this theorem. In [7], Kalau and Walze proved this theorem in the normal coordinates system simultaneously. And then, Ackermann proved that the Wodzicki residue $\mathrm{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of D^2 in [1].

On the other hand, Wang generalized the Connes' results to the case of manifolds with boundary in [11, 12], and proved the Kastler–Kalau–Walze type theorem for the Dirac operator and the signature operator on lower-dimensional manifolds with boundary [13]. In [13, 14], Wang computed $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-1}]$ and $\widetilde{\mathrm{Wres}}[\pi^+D^{-2}\circ\pi^+D^{-2}]$, where the two operators are symmetric, in these cases, the boundary term vanished. But for $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-3}]$, the authors got a nonvanishing boundary term [10], and gave a theoretical explanation for gravitational action on boundary. In others words, Wang provides a kind of method to study the Kastler–Kalau–Walze type theorem for manifolds with boundary.

In [18], Wu and Wang proved Kastler–Kalau–Walze type theorems for the operators $\sqrt{-1}\widehat{c}(V)(d+\delta)$ and $-\sqrt{-1}(d+\delta)\widehat{c}(V)$ on 3,4-dimensional oriented compact manifolds with (respectively, without) boundary. In Definition 2.1 and Proposition 2.2 of [17], Zhang defined a natural operator $D_V = \widehat{c}(V)(d+\delta) - \frac{1}{2}\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)$, where $|V| = 1$. This operator plays a very important role in unifying the Gauss–Bonnet–Chern theorem and the Hirzebruch-signature theorem. *The motivation of this paper* is to prove the Kastler–Kalau–Walze type theorem associated with the operators $\sqrt{-1}\Big(\widehat{c}(V)(d+\delta) + t\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$ and $-\sqrt{-1}\Big((d+\delta)\widehat{c}(V) + \bar{t}\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$, which we call the generalized Zhang's operator for 4 - dimensional manifolds with (respectively, without) boundary. Because we need to take $\widetilde{D}_V^{-1}$, it follows that $\widetilde{D}_V$ must be an elliptic operator. $\widetilde{D}_V = \sqrt{-1}\Big(\widehat{c}(V)(d+\delta) + t\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$, so $\widehat{c}(V)$ is an elliptic operator, and V cannot have zero points. Therefore, we assume that V is a nowhere zero vector field.

In this paper, we obtain two Lichnerowicz type formulas for the generalized Zhang's operator, and for a nowhere zero vector field, we prove the following main theorems.

Theorem 1.1. *Let M be a 4-dimensional oriented compact manifold with the boundary ∂M and let the metric g^{TM} be as in Section 3, the operators $\widetilde{D}_V=\sqrt{-1}\Big(\widehat{c}(V)(d+\delta)+t\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$ and let $\widetilde{D}^*_V=-\sqrt{-1}\Big((d+\delta)\widehat{c}(V)+\bar{t}\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$ be on $\widetilde{M}$ ($\widetilde{M}$ is a collar neighborhood of M); then*

$$\begin{aligned}\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}_V^{-1}\circ\pi^+(\widetilde{D}^*_V)^{-1}]=32\pi^2\int_M|V|^{-2}\Big(&-\frac{4}{3}s+8\left|\mathrm{grad}|V|^2\right|^2\frac{1}{|V|^4}-8\sum_j g(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\frac{1}{|V|^2}\\ &-8(t^2+\bar{t}^2+t+\bar{t})\sum_i\left(e_i(|V|^2)\right)^2\frac{1}{|V|^4}+16(t^2+t\bar{t}+\bar{t}^2)\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\frac{1}{|V|^2}\\ &+16\bar{t}g^{TM}(\Delta^{TM}V,V)\frac{1}{|V|^2}\Big)d\mathrm{Vol_M}+\int_{\partial M}\Big(\frac{9(i-2)}{4}h'(0)-\frac{6i+5}{2}h'(0)|V|^2\\ &+\frac{1}{4}\partial_{x_n}(|V|^2)\Big)\pi\Omega_3d\mathrm{Vol_{\partial M}},\end{aligned}\tag{2}$$

where s is the scalar curvature, h is defined by (62), *and Ω_3 is the canonical volume of S^2.*

Theorem 1.2. *Let M be a 4-dimensional oriented compact manifold with the boundary ∂M and the metric g^{TM} as in Section 3, and let the operator $\widetilde{D}_V=\sqrt{-1}\Big(\widehat{c}(V)(d+\delta)+t\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$ be on $\widetilde{M}$ ($\widetilde{M}$ is a collar neighborhood of M); then*

$$\begin{aligned}\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}_V^{-1}\circ\pi^+\widetilde{D}_V^{-1}]=32\pi^2\int_M|V|^{-2}\Big(&-\frac{4}{3}s+8\left|\mathrm{grad}|V|^2\right|^2\frac{1}{|V|^4}-8\sum_j g(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\frac{1}{|V|^2}\\ &+12\sum_r\left(e_r(|V|^2)\right)^2\frac{1}{|V|^4}+16t^2\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\frac{1}{|V|^2}\\ &+8(2t-1)g^{TM}(\Delta^{TM}V,V)\frac{1}{|V|^2}\Big)d\mathrm{Vol_M}\\ &+\int_{\partial M}\Big(\frac{9(i-2)}{4}h'(0)-\frac{6i+5}{2}h'(0)|V|^2+\frac{1}{4}\partial_{x_n}(|V|^2)\Big)\pi\Omega_3d\mathrm{Vol_{\partial M}},\end{aligned}\tag{3}$$

where s is the scalar curvature, h is defined by (62), *and Ω_3 is the canonical volume of S^2.*

We note that two operators in Theorem 1.1 are symmetric, but we still get the nonvanishing boundary term.

The paper is organized in the following way. In Section 2, by using the definition of the generalized Zhang's operator, we compute the Lichnerowicz formulas for the generalized Zhang's operator. In Section 3, we prove the Kastler–Kalau–Walze type theorem for 4-dimensional manifolds with (respectively, without) boundary associated with the generalized Zhang's operator.

2. THE GENERALIZED ZHANG'S OPERATOR AND THEIR LICHNEROWICZ FORMULAS

Firstly, we introduce some notation about the operators $\sqrt{-1}\Big(\widehat{c}(V)(d+\delta)+t\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$ and $-\sqrt{-1}\Big((d+\delta)\widehat{c}(V)+\bar{t}\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$. Let M be an n-dimensional ($n\geqslant 3$) oriented compact Riemannian manifold with a Riemannian metric g^{TM}.

Let ∇^L be the Levi-Civita connection about g^{TM}. In the fixed orthonormal frame $\{e_1,...,e_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(e_1,...,e_n)=(e_1,...,e_n)(\omega_{s,t}).\tag{4}$$

Let $\epsilon(e_j^*)$, $\iota(e_j^*)$ be the exterior and interior multiplications, respectively, where $e_j^*=g^{TM}(e_j,\cdot)$. And let $c(e_j)$ be the Clifford action. Write

$$\widehat{c}(e_j)=\epsilon(e_j^*)+\iota(e_j^*);\quad c(e_j)=\epsilon(e_j^*)-\iota(e_j^*),\tag{5}$$

which satisfies

$$\begin{aligned}
&\widehat{c}(e_i)\widehat{c}(e_j)+\widehat{c}(e_j)\widehat{c}(e_i)=2g^{TM}(e_i,e_j);\\
&c(e_i)c(e_j)+c(e_j)c(e_i)=-2g^{TM}(e_i,e_j);\\
&c(e_i)\widehat{c}(e_j)+\widehat{c}(e_j)c(e_i)=0.
\end{aligned} \tag{6}$$

By [16], we have

$$\widetilde{D}=d+\delta=\sum_{i=1}^{n}c(e_i)\Big[e_i+\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)[\widehat{c}(e_s)\widehat{c}(e_t)-c(e_s)c(e_t)]\Big]. \tag{7}$$

Let $e_1,e_2,...,e_n$ be the orthonormal basis of TM, the operators $\widetilde{D}_V$ and $\widetilde{D}_V^*$ acting on $\wedge^*T^*M\otimes\mathbb{C}$ are defined by

$$\begin{aligned}
\widetilde{D}_V&=\sqrt{-1}\Big(\widehat{c}(V)(d+\delta)+t\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)\\
&=\sqrt{-1}\widehat{c}(V)\sum_{i=1}^{n}c(e_i)\Big[e_i+\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)[\widehat{c}(e_s)\widehat{c}(e_t)-c(e_s)c(e_t)]\Big]+\sqrt{-1}t\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V);
\end{aligned} \tag{8}$$

$$\begin{aligned}
\widetilde{D}_V^*&=-\sqrt{-1}\Big((d+\delta)\widehat{c}(V)+\bar{t}\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)\\
&=-\sqrt{-1}\sum_{i=1}^{n}c(e_i)\Big[e_i+\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)[\widehat{c}(e_s)\widehat{c}(e_t)-c(e_s)c(e_t)]\Big]\widehat{c}(V)-\sqrt{-1}\bar{t}\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V),
\end{aligned} \tag{9}$$

where t is a complex number and V is a nowhere zero vector field on M.

Set $A=\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)$, so we can get the following theorem about the Lichnerowicz formulas.

Theorem 2.1. *The following equalities hold*:

$$\begin{aligned}
\widetilde{D}_V^*\widetilde{D}_V=&\Big\{-\Big[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j}-\nabla_{\nabla^L_{\partial_i}\partial_j})\Big]-\frac{1}{8}\sum_{ijkl}R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)+\frac{1}{4}s+\sum_{i=1}^{n}\Big[\frac{1}{2}c(e_i)c(\mathrm{grad}|V|^2)\\
&-c(e_i)\widehat{c}(V)tA-\bar{t}A\widehat{c}(V)c(e_i)\Big]^2\frac{1}{|V|^4}-\frac{1}{2}\sum_{j=1}^{n}\Big[c(e_j)c(\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)+2\nabla^{\wedge^*T^*M}_{e_j}(\bar{t}A)\widehat{c}(V)c(e_j)\Big]\frac{1}{|V|^2}\\
&+t\bar{t}A^2\frac{1}{|V|^2}\Big\}|V|^2;\\
\widetilde{D}_V^2=&\Big\{-\Big[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j}-\nabla_{\nabla^L_{\partial_i}\partial_j})\Big]-\frac{1}{8}\sum_{ijkl}R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)+\frac{1}{4}s+\sum_{i=1}^{n}\Big[\frac{1}{2}c(e_i)c(\mathrm{grad}|V|^2)\\
&+\frac{1}{2}\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_i)+\widehat{c}(V)c(e_i)tA+\widehat{c}(V)tAc(e_i)\Big]^2\frac{1}{|V|^4}-\frac{1}{2}\sum_{j=1}^{n}\Big[c(e_j)c(\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\\
&-\nabla^{\wedge^*T^*M}_{e_j}\big(\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)\big)c(e_j)-2\widehat{c}(V)\nabla^{\wedge^*T^*M}_{e_j}(tA)c(e_j)\Big]\frac{1}{|V|^2}-t^2A^2\frac{1}{|V|^2}\Big\}|V|^2,
\end{aligned} \tag{10}$$

where s is the scalar curvature.

Proof. Let M be an n-dimensional smooth compact oriented Riemannian manifold without boundary and N be a vector bundle on M. If P is a differential operator of Laplace type, then it has locally the form

$$P=-(g^{ij}\partial_i\partial_j+A^i\partial_i+B), \tag{11}$$

where ∂_i is a natural local frame on TM and $(g^{ij})_{1\leqslant i,j\leqslant n}$ is the inverse matrix associated to the metric matrix $(g_{ij})_{1\leqslant i,j\leqslant n}$ on M, and A^i and B are smooth sections of $\mathrm{End}(N)$ on M (endomorphism). If a Laplace type operator P satisfies (11), then there is a unique connection ∇ on N and a unique endomorphism E such that

$$P=-\Big[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j}-\nabla_{\nabla^L_{\partial_i}\partial_j})+E\Big], \tag{12}$$

where ∇^L is the Levi-Civita connection on M. Moreover (with local frames of T^*M and N), $\nabla_{\partial_i}=\partial_i+\omega_i$ and E are related to g^{ij}, A^i, and B through

$$\omega_i=\frac{1}{2}g_{ij}\big(A^i+g^{kl}\Gamma^j_{kl}\mathtt{id}\big), \tag{13}$$

$$E=B-g^{ij}\big(\partial_i(\omega_j)+\omega_i\omega_j-\omega_k\Gamma^k_{ij}\big), \tag{14}$$

where Γ^j_{kl} is the Christoffel coefficient of ∇^L.

Let $g^{ij}=g(dx_i,dx_j)$, $\xi=\sum_j\xi_jdx_j$ and $\nabla^L_{\partial_i}\partial_j=\sum_k\Gamma^k_{ij}\partial_k$; we write

$$\begin{aligned}&\sigma_i=-\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)c(e_s)c(e_t);\quad a_i=\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)\widehat{c}(e_s)\widehat{c}(e_t);\\&\xi^j=g^{ij}\xi_i;\quad \Gamma^k=g^{ij}\Gamma^k_{ij};\quad \sigma^j=g^{ij}\sigma_i;\quad a^j=g^{ij}a_i.\end{aligned} \tag{15}$$

Then the operators $\widetilde{D}_V$ and $\widetilde{D}^*_V$ can be written as

$$\widetilde{D}_V=\sqrt{-1}\widehat{c}(V)\sum_{i=1}^n c(e_i)[e_i+a_i+\sigma_i]+\sqrt{-1}tA; \tag{16}$$

$$\widetilde{D}^*_V=-\sqrt{-1}\sum_{i=1}^n c(e_i)[e_i+a_i+\sigma_i]\widehat{c}(V)-\sqrt{-1}\bar{t}A. \tag{17}$$

By [1, 16], we have

$$(d+\delta)^2=-\triangle_0-\frac{1}{8}\sum_{ijkl}R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)+\frac{1}{4}K, \tag{18}$$

$$-\triangle_0=\Delta=-g^{ij}(\nabla^L_i\nabla^L_j-\Gamma^k_{ij}\nabla^L_k). \tag{19}$$

By (16) and (17), we get

$$\begin{aligned}\widetilde{D}^*_V\widetilde{D}_V=&-\sqrt{-1}(d+\delta)\widehat{c}(V)\sqrt{-1}\widehat{c}(V)(d+\delta)-\sqrt{-1}(d+\delta)\widehat{c}(V)\sqrt{-1}tA\\&-\sqrt{-1}\bar{t}A\sqrt{-1}\widehat{c}(V)(d+\delta)-\sqrt{-1}\bar{t}A\sqrt{-1}tA\\=&(d+\delta)|V|^2(d+\delta)+(d+\delta)\widehat{c}(V)tA+\bar{t}A\widehat{c}(V)(d+\delta)+t\bar{t}A^2\\=&\bigg[(d+\delta)^2-(d+\delta)c(\mathrm{grad}|V|^2)\frac{1}{|V|^2}+\big((d+\delta)\widehat{c}(V)tA+\bar{t}A\widehat{c}(V)(d+\delta)+t\bar{t}A^2\big)\frac{1}{|V|^2}\bigg]|V|^2\\=&\bigg\{-\sum_{ij}g^{ij}\Big[\partial_i\partial_j+2\sigma_i\partial_j+2a_i\partial_j-\Gamma^k_{ij}\partial_k+(\partial_i\sigma_j)+(\partial_ia_j)+\sigma_i\sigma_j+\sigma_ia_j+a_i\sigma_j+a_ia_j\\&-\Gamma^k_{ij}\sigma_k-\Gamma^k_{ij}a_k\Big]-\sum_{ij}g^{ij}\Big[c(\partial_i)c(\mathrm{grad}|V|^2)\partial_j+c(\partial_i)\partial_j(c(\mathrm{grad}|V|^2))+c(\partial_i)\sigma_ic(\mathrm{grad}|V|^2)\\&+c(\partial_i)a_ic(\mathrm{grad}|V|^2)\Big]\frac{1}{|V|^2}+\sum_{ij}g^{ij}\Big[c(\partial_i)\widehat{c}(V)tA+\bar{t}A\widehat{c}(V)c(\partial_i)\Big]\partial_j\frac{1}{|V|^2}+\sum_{ij}g^{ij}\Big[\bar{t}A\widehat{c}(V)c(\partial_i)\sigma_i\\&+\bar{t}A\widehat{c}(V)c(\partial_i)a_i+c(\partial_i)\widehat{c}(V)\partial_i(tA)+c(\partial_i)\widehat{c}(V)\sigma_itA+c(\partial_i)\widehat{c}(V)a_itA\Big]\frac{1}{|V|^2}+t\bar{t}A^2\frac{1}{|V|^2}\\&-\frac{1}{8}\sum_{ijkl}R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)+\frac{1}{4}s\bigg\}|V|^2\\:=&F^*|V|^2,\end{aligned} \tag{20}$$

where

$$\mathrm{grad}|V|^2=\sum_{j=1}^{n}\langle \mathrm{grad}|V|^2,e_j\rangle e_j=\sum_{j=1}^{n}e_j(|V|^2)e_j. \tag{21}$$

So, F^* is a generalized Laplacian.

Similarly, by $d+\delta=\sum_{q=1}^{n}c(e_q)\nabla_{e_q}^{\wedge^* T^*M}$, we get

$$\begin{aligned}
\widetilde{D}_V^2=&(d+\delta)|V|^2(d+\delta)-\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla_{e_q}^L V)\widehat{c}(V)(d+\delta)-\widehat{c}(V)(d+\delta)tA-tA\widehat{c}(V)(d+\delta)-t^2A^2\\
=&\Big\{-\sum_{ij}g^{ij}\Big[\partial_i\partial_j+2\sigma_i\partial_j+2a_i\partial_j-\Gamma_{ij}^k\partial_k+(\partial_i\sigma_j)+(\partial_i a_j)+\sigma_i\sigma_j+\sigma_i a_j+a_i\sigma_j+a_i a_j-\Gamma_{ij}^k\sigma_k\\
&-\Gamma_{ij}^k a_k\Big]-\sum_{ij}g^{ij}\Big[c(\partial_i)c(\mathrm{grad}|V|^2)\partial_j+c(\partial_i)\partial_j(c(\mathrm{grad}|V|^2))+c(\partial_i)\sigma_i c(\mathrm{grad}|V|^2)+c(\partial_i)a_i\\
&c(\mathrm{grad}|V|^2)\Big]\frac{1}{|V|^2}-\sum_{ij}g^{ij}\Big[\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla_{e_q}^L V)\widehat{c}(V)c(\partial_i)\partial_j+\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla_{e_q}^L V)\widehat{c}(V)c(\partial_i)\sigma_i\\
&+\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla_{e_q}^L V)\widehat{c}(V)c(\partial_i)a_i\Big]\frac{1}{|V|^2}-\sum_{ij}g^{ij}\Big[\widehat{c}(V)c(\partial_i)tA+\widehat{c}(V)tAc(\partial_i)\Big]\partial_j\frac{1}{|V|^2}\\
&-\sum_{ij}g^{ij}\Big[\widehat{c}(V)tAc(\partial_i)\sigma_i+\widehat{c}(V)tAc(\partial_i)a_i+\widehat{c}(V)c(\partial_i)\partial_i(tA)+\widehat{c}(V)c(\partial_i)\sigma_i tA+\widehat{c}(V)c(\partial_i)a_i tA\Big]\frac{1}{|V|^2}\\
&-t^2A^2\frac{1}{|V|^2}-\frac{1}{8}\sum_{ijkl}R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)+\frac{1}{4}s\Big\}|V|^2\\
:=&F|V|^2.
\end{aligned} \tag{22}$$

By (15)–(22), we obtain

$$(\omega_i)_{F^*}=\sigma_i+a_i+\frac{1}{2}c(\partial_i)c(\mathrm{grad}|V|^2)\frac{1}{|V|^2}-c(\partial_i)\widehat{c}(V)tA\frac{1}{|V|^2}-\bar{t}A\widehat{c}(V)c(\partial_i)\frac{1}{|V|^2}, \tag{23}$$

$$\begin{aligned}
E_{F^*}=&\frac{1}{8}\sum_{ijkl}R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)-\frac{1}{4}s-t\bar{t}A^2\frac{1}{|V|^2}+\sum_{ij}\Big[c(\partial_i)\partial^i c(\mathrm{grad}|V|^2)\\
&+c(\partial_i)\sigma^i c(\mathrm{grad}|V|^2)+c(\partial_i)a^i c(\mathrm{grad}|V|^2)-\big[\bar{t}A\widehat{c}(V)c(\partial_i)\sigma_i+\widehat{c}(V)tAc(\partial_i)a_i\\
&+c(\partial_i)\widehat{c}(V)\partial_i(tA)+c(\partial_i)\widehat{c}(V)\sigma_i tA+c(\partial_i)\widehat{c}(V)a_i tA\big]-\partial^j\big[\frac{1}{2}c(\partial_j)c(\mathrm{grad}|V|^2)\\
&-c(\partial_j)\widehat{c}(V)tA-\bar{t}A\widehat{c}(V)c(\partial_j)\big]-\big[\frac{1}{2}c(\partial_i)c(\mathrm{grad}|V|^2)-c(\partial_i)\widehat{c}(V)tA-\bar{t}A\widehat{c}(V)c(\partial_i)\big]\\
&(\sigma^i+a^i)-(\sigma^j+a^j)\big[\frac{1}{2}c(\partial_j)c(\mathrm{grad}|V|^2)-c(\partial_j)\widehat{c}(V)tA-\bar{t}A\widehat{c}(V)c(\partial_j)\big]\\
&+\Gamma^k\big[\frac{1}{2}c(\partial_k)c(\mathrm{grad}|V|^2)-c(\partial_k)\widehat{c}(V)tA-\bar{t}A\widehat{c}(V)c(\partial_k)\big]\Big]\frac{1}{|V|^2}\\
&-\sum_{ij}g^{ij}\Big[\frac{1}{2}c(\partial_i)c(\mathrm{grad}|V|^2)-c(\partial_i)\widehat{c}(V)tA-\bar{t}A\widehat{c}(V)c(\partial_i)\Big]\Big[\frac{1}{2}c(\partial_j)c(\mathrm{grad}|V|^2)\\
&-c(\partial_j)\widehat{c}(V)tA-\bar{t}A\widehat{c}(V)c(\partial_j)\Big]\frac{1}{|V|^4}.
\end{aligned} \tag{24}$$

Since E is globally defined on M, it follows that, taking normal coordinates at x_0, we have $\sigma^i(x_0)=0$, $a^i(x_0)=0$, $\partial^j[c(\partial_j)](x_0)=0$, $\Gamma^k(x_0)=0$, $g^{ij}(x_0)=\delta_i^j$, and then

$$\begin{aligned}E_{F^*}(x_0)=&\frac{1}{8}\sum_{ijkl}R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)-\frac{1}{4}s-t\bar{t}A^2\frac{1}{|V|^2}\\&-\sum_{i=1}^{n}\Big[\frac{1}{2}c(e_i)c(\mathrm{grad}|V|^2)-c(e_i)\widehat{c}(V)tA-\bar{t}A\widehat{c}(V)c(e_i)\Big]^2\frac{1}{|V|^4}\\&+\frac{1}{2}\sum_{j=1}^{n}\Big[c(e_j)c(\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)+2\nabla^{\wedge^*T^*M}_{e_j}(\bar{t}A)\widehat{c}(V)c(e_j)\Big]\frac{1}{|V|^2}.\end{aligned}\tag{25}$$

Similarly, we have

$$\begin{aligned}E_F(x_0)=&\frac{1}{8}\sum_{ijkl}R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)-\frac{1}{4}s+t^2A^2\frac{1}{|V|^2}-\sum_{i=1}^{n}\Big[\frac{1}{2}c(e_i)c(\mathrm{grad}|V|^2)\\&+\frac{1}{2}\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_i)+\widehat{c}(V)c(e_i)tA+\widehat{c}(V)tAc(e_i)\Big]^2\frac{1}{|V|^4}\\&+\frac{1}{2}\sum_{j=1}^{n}\Big[c(e_j)c(\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)-\nabla^{\wedge^*T^*M}_{e_j}\big(\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)\big)c(e_j)\\&-2\widehat{c}(V)\nabla^{\wedge^*T^*M}_{e_j}(tA)c(e_j)\Big]\frac{1}{|V|^2}.\end{aligned}\tag{26}$$

Hence, by (12), (25), and (26), we get Theorem 2.1.

Corollary 2.2. *When $|V|=1$, the equalities in Theorem 2.1 become*

$$\begin{aligned}\widetilde{D}_V^*\widetilde{D}_V=&-\Big[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j}-\nabla_{\nabla^L_{\partial_i}\partial_j})\Big]-\frac{1}{8}\sum_{ijkl}R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)+\frac{1}{4}s\\&-\sum_{i=1}^{n}\Big[c(e_i)\widehat{c}(V)tA+\bar{t}A\widehat{c}(V)c(e_i)\Big]^2-\sum_{j=1}^{n}\Big[\nabla^{\wedge^*T^*M}_{e_j}(\bar{t}A)\widehat{c}(V)c(e_j)\Big]+t\bar{t}A^2;\\\widetilde{D}_V^2=&-\Big[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j}-\nabla_{\nabla^L_{\partial_i}\partial_j})\Big]-\frac{1}{8}\sum_{ijkl}R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)+\frac{1}{4}s\\&+\sum_{i=1}^{n}\Big[\frac{1}{2}\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_i)+\widehat{c}(V)c(e_i)tA+\widehat{c}(V)tAc(e_i)\Big]^2\\&+\frac{1}{2}\sum_{j=1}^{n}\Big[\nabla^{\wedge^*T^*M}_{e_j}\big(\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)\big)c(e_j)+2\widehat{c}(V)\nabla^{\wedge^*T^*M}_{e_j}(tA)c(e_j)\Big]-t^2A^2,\end{aligned}\tag{27}$$

where s is the scalar curvature.

From [1], we know that the noncommutative residue of a generalized Laplacian $\overline{\Delta}$ is expressed as

$$(n-2)\Phi_2(\overline{\Delta})=(4\pi)^{-\frac{n}{2}}\Gamma(\frac{n}{2})\mathrm{Wres}(\overline{\Delta}^{-\frac{n}{2}+1}),\tag{28}$$

where $\Phi_2(\overline{\Delta})$ denotes the integral over the diagonal part of the second coefficient of the heat kernel expansion of $\overline{\Delta}$. Now let $\overline{\Delta}=\widetilde{D}_V^*\widetilde{D}_V$, since $\widetilde{D}_V^*\widetilde{D}_V$ is a generalized Laplacian, and $\widetilde{D}_V^*\widetilde{D}_V=\overline{\Delta}-E_{\widetilde{D}_V^*\widetilde{D}_V}$, then, for $n=4$, we have (see [5])

$$\mathrm{Wres}(\widetilde{D}_V^*\widetilde{D}_V)^{-1}=\mathrm{Wres}(F^*|V|^2)^{-1}=\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M|V|^{-2}\mathrm{tr}(\frac{1}{6}s+E_{F^*})d\mathrm{Vol_M};\tag{29}$$

$$\mathrm{Wres}(\widetilde{D}_V^2)^{-1}=\mathrm{Wres}(F|V|^2)^{-1}=\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M|V|^{-2}\mathrm{tr}(\frac{1}{6}s+E_F)d\mathrm{Vol_M},\tag{30}$$

where Wres denotes the noncommutative residue.

Next, we need to compute $\mathrm{tr}(\frac{1}{6}s+E_{F^*})$ and $\mathrm{tr}(\frac{1}{6}s+E_F)$. Obviously, we get

(1)

$$\mathrm{tr}\Big(-\frac{1}{12}s\Big)=-\frac{1}{12}s\mathrm{tr}[\mathtt{id}]. \tag{31}$$

(2)

$$\sum_{ijkl}\mathrm{tr}[R_{ijkl}\widehat{c}(e_i)\widehat{c}(e_j)c(e_k)c(e_l)]=0. \tag{32}$$

(3)

$$\mathrm{tr}\sum_{i=1}^{n}[\widehat{c}(V)c(e_i)tA+\widehat{c}(V)tAc(e_i)]=0. \tag{33}$$

(4) By $|V|^2=\sum_{i=1}^n g^{TM}(e_i,V)^2$, we have $\sum_{i=1}^n\big(g^{TM}(e_i,\mathrm{grad}|V|^2)\big)^2=|\mathrm{grad}|V|^2|^2$, so

$$\begin{aligned}\mathrm{tr}\sum_{i=1}^{n}[c(e_i)c(\mathrm{grad}|V|^2)]^2=&\mathrm{tr}\sum_{i=1}^{n}[c(e_i)c(\mathrm{grad}|V|^2)c(e_i)c(\mathrm{grad}|V|^2)]\\ =&-2\sum_{i=1}^{n}g^{TM}(e_i,\mathrm{grad}|V|^2)\mathrm{tr}[c(e_i)c(\mathrm{grad}|V|^2)]\\ &-\mathrm{tr}\sum_{i=1}^{n}[c(e_i)c(e_i)c(\mathrm{grad}|V|^2)c(\mathrm{grad}|V|^2)]\\ =&-2\sum_{i=1}^{n}g^{TM}(e_i,\mathrm{grad}|V|^2)\mathrm{tr}[c(e_i)c(\mathrm{grad}|V|^2)]-n\left|\mathrm{grad}|V|^2\right|^2\mathrm{tr}[\mathtt{id}]\\ =&2\sum_{i=1}^{n}\big(g^{TM}(e_i,\mathrm{grad}|V|^2)\big)^2\mathrm{tr}[\mathtt{id}]-n\left|\mathrm{grad}|V|^2\right|^2\mathrm{tr}[\mathtt{id}]\\ =&(2-n)\left|\mathrm{grad}|V|^2\right|^2\mathrm{tr}[\mathtt{id}].\end{aligned} \tag{34}$$

(5) By $\widehat{c}(V)\widehat{c}(\nabla^L_{e_r}V)+\widehat{c}(\nabla^L_{e_r}V)\widehat{c}(V)=2g^{TM}(V,\nabla^L_{e_r}V)=e_r(|V|^2)$, we obtain

$$\begin{aligned}\mathrm{tr}\sum_{i,q=1}^{n}[c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_i)]^2=&\mathrm{tr}\sum_{i,q,r=1}^{n}[c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_i)c(e_r)\widehat{c}(\nabla^L_{e_r}V)\widehat{c}(V)c(e_i)]\\ =&\mathrm{tr}\sum_{i,q,r=1}^{n}[c(e_q)c(e_i)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)\widehat{c}(\nabla^L_{e_r}V)\widehat{c}(V)c(e_r)c(e_i)]\\ =&2\sum_{r=1}^{n}g^{TM}(V,\nabla^L_{e_r}V)\Big(\mathrm{tr}\sum_{i,q=1}^{n}[c(e_q)c(e_i)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_r)c(e_i)]\Big)\\ &-\mathrm{tr}\sum_{i,q,r=1}^{n}[c(e_q)c(e_i)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(\nabla^L_{e_r}V)\widehat{c}(V)\widehat{c}(V)c(e_r)c(e_i)].\end{aligned} \tag{35}$$

(5-a)

$$\begin{aligned}&\mathrm{tr}\sum_{i,q=1}^{n}[c(e_q)c(e_i)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_r)c(e_i)]\\ &\qquad=2\sum_{i,q=1}^{n}g^{TM}(V,\nabla^L_{e_q}V)\mathrm{tr}[c(e_q)c(e_i)c(e_r)c(e_i)]-\mathrm{tr}\sum_{i,q=1}^{n}[c(e_q)c(e_i)\widehat{c}(V)\widehat{c}(\nabla^L_{e_q}V)c(e_r)c(e_i)],\end{aligned}$$

so

$$\begin{aligned}\mathrm{tr}\sum_{i,q=1}^{n}[c(e_q)c(e_i)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_r)c(e_i)] &= \sum_{i,q=1}^{n} g^{TM}(V,\nabla^L_{e_q}V)\mathrm{tr}[c(e_q)c(e_i)c(e_r)c(e_i)]\\ &=2\sum_{i,q=1}^{n} g^{TM}(V,\nabla^L_{e_q}V)\delta_{qi}\delta_{ri}\mathrm{tr}[\mathtt{id}]-\sum_{i,q=1}^{n} g^{TM}(V,\nabla^L_{e_q}V)\delta_{qr}\mathrm{tr}[\mathtt{id}]\\ &=2g^{TM}(V,\nabla^L_{e_r}V)\mathrm{tr}[\mathtt{id}]-ng^{TM}(V,\nabla^L_{e_r}V)\mathrm{tr}[\mathtt{id}]\\ &=(2-n)g^{TM}(V,\nabla^L_{e_r}V)\mathrm{tr}[\mathtt{id}]=\frac{2-n}{2}e_r(|V|^2)\mathrm{tr}[\mathtt{id}]. \end{aligned} \tag{36}$$

(5-b)

$$\mathrm{tr}\sum_{i,q,r=1}^{n}[c(e_q)c(e_i)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(\nabla^L_{e_r}V)\widehat{c}(V)\widehat{c}(V)c(e_r)c(e_i)]=(2-n)\sum_{r=1}^{n}|\nabla^L_{e_r}V|^2|V|^2\mathrm{tr}[\mathtt{id}]. \tag{37}$$

Therefore,

$$\mathrm{tr}\sum_{i,q=1}^{n}[c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_i)]^2=\frac{2-n}{2}\sum_{r=1}^{n}\big(e_r(|V|^2)\big)^2\mathrm{tr}[\mathtt{id}]-(2-n)\sum_{r=1}^{n}|\nabla^L_{e_r}V|^2|V|^2\mathrm{tr}[\mathtt{id}]. \tag{38}$$

(6)

$$\begin{aligned}\mathrm{tr}\sum_{i,q=1}^{n}[c(e_i)c(\mathrm{grad}|V|^2)c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_i)] &=-n\mathrm{tr}\sum_{q=1}^{n}[c(\mathrm{grad}|V|^2)c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)]\\ &=2n\sum_{q=1}^{n}g^{TM}(e_q,\mathrm{grad}|V|^2)\mathrm{tr}[\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)]\\ &\quad+n\mathrm{tr}\sum_{q=1}^{n}[c(e_q)c(\mathrm{grad}|V|^2)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)], \end{aligned} \tag{39}$$

so

$$\begin{aligned}\mathrm{tr}\sum_{q=1}^{n}[c(\mathrm{grad}|V|^2)c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)] &=-\sum_{q=1}^{n}g^{TM}(e_q,\mathrm{grad}|V|^2)\mathrm{tr}[\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)]\\ &=-\frac{1}{2}\sum_{q=1}^{n}\big(e_q(|V|^2)\big)^2\mathrm{tr}[\mathtt{id}]. \end{aligned} \tag{40}$$

Then

$$\mathrm{tr}\sum_{i,q=1}^{n}[c(e_i)c(\mathrm{grad}|V|^2)c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_i)]=\frac{n}{2}\sum_{q=1}^{n}\big(e_q(|V|^2)\big)^2\mathrm{tr}[\mathtt{id}]. \tag{41}$$

(7) By $\nabla^{\wedge^*T^*M}_{e_j}(c(\mathrm{grad}|V|^2))=c(\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)$, we get

$$\begin{aligned}\mathrm{tr}\sum_{j=1}^{n}[c(e_j)\nabla^{\wedge^*T^*M}_{e_j}(c(\mathrm{grad}|V|^2))] &=\mathrm{tr}\sum_{j=1}^{n}[c(e_j)c(\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)]\\ &=-2\sum_{j=1}^{n}g^{TM}(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\mathrm{tr}[\mathtt{id}]-\mathrm{tr}\sum_{j=1}^{n}[c(\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)c(e_j)], \end{aligned} \tag{42}$$

so

$$\mathrm{tr}\sum_{j=1}^{n}[c(e_j)\nabla^{\wedge^*T^*M}_{e_j}(c(\mathrm{grad}|V|^2))]=-\sum_{j=1}^{n}g^{TM}(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\mathrm{tr}[\mathtt{id}]. \tag{43}$$

(8) By $\nabla_{e_j}^{\wedge^* T^*M}(\alpha\beta)=(\nabla_{e_j}^{\wedge^* T^*M}\alpha)\beta+\alpha(\nabla_{e_j}^{\wedge^* T^*M}\beta)$, we get

$$\begin{aligned}\mathrm{tr}\sum_{j,q=1}^{n}&[\nabla_{e_j}^{\wedge^* T^*M}(c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V))c(e_j)]\\ &=\mathrm{tr}\sum_{j,q=1}^{n}[\nabla_{e_j}^{\wedge^* T^*M}(c(e_q))\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_j)]+\mathrm{tr}\sum_{j,q=1}^{n}[c(e_q)\nabla_{e_j}^{\wedge^* T^*M}(\widehat{c}(\nabla^L_{e_q}V))\widehat{c}(V)c(e_j)]\\ &\quad+\mathrm{tr}\sum_{j,q=1}^{n}[c(e_q)\widehat{c}(\nabla^L_{e_q}V)\nabla_{e_j}^{\wedge^* T^*M}(\widehat{c}(V))c(e_j)].\end{aligned}$$

(8-a) By $(\nabla^L_{e_j}e_q)(x_0)=0$, we get

$$\mathrm{tr}\sum_{j,q=1}^{n}[\nabla_{e_j}^{\wedge^* T^*M}(c(e_q))\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_j)](x_0)=\mathrm{tr}\sum_{j,q=1}^{n}[c(\nabla^L_{e_j}e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V)c(e_j)](x_0)=0. \tag{44}$$

(8-b)

$$\begin{aligned}\mathrm{tr}\sum_{j,q=1}^{n}[c(e_q)\nabla_{e_j}^{\wedge^* T^*M}(\widehat{c}(\nabla^L_{e_q}V))\widehat{c}(V)c(e_j)]=&\mathrm{tr}\sum_{j,q=1}^{n}[c(e_q)\widehat{c}(\nabla^L_{e_j}\nabla^L_{e_q}V)\widehat{c}(V)c(e_j)]\\ =&2\sum_{j,q=1}^{n}g^{TM}(\nabla^L_{e_j}\nabla^L_{e_q}V,V)\mathrm{tr}[c(e_q)c(e_j)]\\ &-\mathrm{tr}\sum_{j,q=1}^{n}[c(e_q)\widehat{c}(V)\widehat{c}(\nabla^L_{e_j}\nabla^L_{e_q}V)c(e_j)],\end{aligned}\tag{45}$$

so

$$\begin{aligned}\mathrm{tr}\sum_{j,q=1}^{n}[c(e_q)\widehat{c}(\nabla^L_{e_j}\nabla^L_{e_q}V)\widehat{c}(V)c(e_j)]&=\sum_{j,q=1}^{n}g^{TM}(\nabla^L_{e_j}\nabla^L_{e_q}V,V)\mathrm{tr}[c(e_q)c(e_j)]\\ &=-\sum_{q=1}^{n}g^{TM}(\nabla^L_{e_q}\nabla^L_{e_q}V,V)\mathrm{tr}[\mathtt{id}]=g^{TM}(\Delta^{TM}V,V)\mathrm{tr}[\mathtt{id}].\end{aligned}\tag{46}$$

(8-c)

$$\begin{aligned}\mathrm{tr}\sum_{j,q=1}^{n}[c(e_q)\widehat{c}(\nabla^L_{e_q}V)\nabla_{e_j}^{\wedge^* T^*M}(\widehat{c}(V))c(e_j)]=&\mathrm{tr}\sum_{j,q=1}^{n}[c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(\nabla^L_{e_j}V)c(e_j)]\\ =&2\sum_{j,q=1}^{n}g^{TM}(\nabla^L_{e_q}V,\nabla^L_{e_j}V)\mathrm{tr}[c(e_q)c(e_j)]\\ &-\sum_{j,q=1}^{n}\mathrm{tr}[c(e_q)\widehat{c}(\nabla^L_{e_j}V)\widehat{c}(\nabla^L_{e_q}V)c(e_j)],\end{aligned}\tag{47}$$

so

$$\mathrm{tr}\sum_{j,q=1}^{n}[c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(\nabla^L_{e_j}V)c(e_j)]=\sum_{j,q=1}^{n}g^{TM}(\nabla^L_{e_q}V,\nabla^L_{e_j}V)\mathrm{tr}[c(e_q)c(e_j)]=-\sum_{q=1}^{n}|\nabla^L_{e_q}V|^2\mathrm{tr}[\mathtt{id}]. \tag{48}$$

Hence, by (44), (46), and (48), we have

$$\mathrm{tr}\sum_{j,q=1}^{n}[\nabla_{e_j}^{\wedge^* T^*M}(c(e_q)\widehat{c}(\nabla^L_{e_q}V)\widehat{c}(V))c(e_j)]=g^{TM}(\Delta^{TM}V,V)\mathrm{tr}[\mathtt{id}]-\sum_{q=1}^{n}|\nabla^L_{e_q}V|^2\mathrm{tr}[\mathtt{id}]. \tag{49}$$

(9) Similar to (8), we have

$$\begin{aligned}&\mathrm{tr}\sum_{i,j=1}^{n}[\widehat{c}(V)\nabla_{e_j}^{\wedge^*T^*M}(c(e_i)\widehat{c}(\nabla_{e_i}^{TM}V))c(e_j)]\\&=\mathrm{tr}\sum_{i,j=1}^{n}[\widehat{c}(V)\nabla_{e_j}^{\wedge^*T^*M}(c(e_i))\widehat{c}(\nabla_{e_i}^{TM}V)c(e_j)]+\mathrm{tr}\sum_{i,j=1}^{n}[\widehat{c}(V)c(e_i)\nabla_{e_j}^{\wedge^*T^*M}(\widehat{c}(\nabla_{e_i}^{TM}V))c(e_j)].\end{aligned}$$

(9-a) Similar to (8-a), we get

$$\mathrm{tr}\sum_{i,j=1}^{n}[\widehat{c}(V)\nabla_{e_j}^{\wedge^*T^*M}(c(e_i))\widehat{c}(\nabla_{e_i}^{TM}V)c(e_j)](x_0)=\mathrm{tr}\sum_{i,j=1}^{n}[\widehat{c}(V)c(\nabla_{e_j}^{L}e_i)\widehat{c}(\nabla_{e_i}^{TM}V)c(e_j)](x_0)=0. \tag{50}$$

(9-b)

$$\begin{aligned}\mathrm{tr}\sum_{i,j=1}^{n}[\widehat{c}(V)c(e_i)\widehat{c}(\nabla_{e_j}^{L}\nabla_{e_i}^{TM}V)c(e_j)]&=-\,\mathrm{tr}\sum_{i,j=1}^{n}[c(e_i)\widehat{c}(\nabla_{e_j}^{L}\nabla_{e_i}^{TM}V)\widehat{c}(V)c(e_j)]\\&=-\,2\sum_{i,j=1}^{n}g^{TM}(\nabla_{e_j}^{L}\nabla_{e_i}^{TM}V,V)\mathrm{tr}[c(e_i)c(e_j)]\\&\quad+\mathrm{tr}\sum_{i,j=1}^{n}[c(e_i)\widehat{c}(V)\widehat{c}(\nabla_{e_j}^{L}\nabla_{e_i}^{TM}V)c(e_j)],\end{aligned} \tag{51}$$

so

$$\begin{aligned}\mathrm{tr}\sum_{i,j=1}^{n}[\widehat{c}(V)c(e_i)\widehat{c}(\nabla_{e_j}^{L}\nabla_{e_i}^{TM}V)c(e_j)]&=-\sum_{i,j=1}^{n}g^{TM}(\nabla_{e_j}^{L}\nabla_{e_i}^{TM}V,V)\mathrm{tr}[c(e_i)c(e_j)]\\&=\sum_{i=1}^{n}g^{TM}(\nabla_{e_i}^{L}\nabla_{e_i}^{TM}V,V)\mathrm{tr}[\mathtt{id}]\\&=-\,g^{TM}(\Delta^{TM}V,V)\mathrm{tr}[\mathtt{id}].\end{aligned} \tag{52}$$

Hence, by (50) and (52), we obtain

$$\mathrm{tr}\sum_{i,j=1}^{n}[\widehat{c}(V)\nabla_{e_j}^{\wedge^*T^*M}(c(e_i)\widehat{c}(\nabla_{e_i}^{TM}V))c(e_j)]=-g^{TM}(\Delta^{TM}V,V)\mathrm{tr}[\mathtt{id}]. \tag{53}$$

(10)

$$\begin{aligned}\mathrm{tr}[A^2]=&\mathrm{tr}\sum_{i,r=1}^{n}[c(e_i)\widehat{c}(\nabla_{e_i}^{TM}V)c(e_r)\widehat{c}(\nabla_{e_r}^{TM}V)]\\&=-\,\mathrm{tr}\sum_{i,r=1}^{n}[c(e_i)\widehat{c}(\nabla_{e_i}^{TM}V)\widehat{c}(\nabla_{e_r}^{TM}V)c(e_r)]\\&=-\,2\sum_{i,r=1}^{n}g^{TM}(\nabla_{e_i}^{TM}V,\nabla_{e_r}^{TM}V)\mathrm{tr}[c(e_i)c(e_r)]+\mathrm{tr}\sum_{i,r=1}^{n}[c(e_i)\widehat{c}(\nabla_{e_r}^{TM}V)\widehat{c}(\nabla_{e_i}^{TM}V)c(e_r)],\end{aligned} \tag{54}$$

so

$$\mathrm{tr}[A^2]=-\sum_{i,r=1}^{n}g^{TM}(\nabla_{e_i}^{TM}V,\nabla_{e_r}^{TM}V)\mathrm{tr}[c(e_i)c(e_r)]=\sum_{i=1}^{n}|\nabla_{e_i}^{TM}V|^2\mathrm{tr}[\mathtt{id}]. \tag{55}$$

(11) By $\widehat{c}(V)\widehat{c}(\nabla^{TM}_{e_i}V)+\widehat{c}(\nabla^{TM}_{e_i}V)\widehat{c}(V)=2g^{TM}(V,\nabla^{TM}_{e_i}V)=e_i(|V|^2)$, then

$$\begin{aligned}\mathrm{tr}\sum_{i=1}^n[\widehat{c}(V)\widehat{c}(\nabla^{TM}_{e_i}V)]^2 =&\mathrm{tr}\sum_{i=1}^n[\widehat{c}(V)\widehat{c}(\nabla^{TM}_{e_i}V)\widehat{c}(V)\widehat{c}(\nabla^{TM}_{e_i}V)]\\ =&2\sum_{i=1}^n g^{TM}(V,\nabla^{TM}_{e_i}V)\mathrm{tr}[\widehat{c}(V)\widehat{c}(\nabla^{TM}_{e_i}V)]-\mathrm{tr}\sum_{i=1}^n[\widehat{c}(\nabla^{L}_{e_i}V)\widehat{c}(V)\widehat{c}(V)\widehat{c}(\nabla^{TM}_{e_i}V)]\\ =&\frac{1}{2}\sum_{i=1}^n\big(e_i(|V|^2)\big)^2\mathrm{tr}[\mathtt{id}]-\sum_{i=1}^n\left|\nabla^{TM}_{e_i}V\right|^2|V|^2\mathrm{tr}[\mathtt{id}]. \end{aligned}\tag{56}$$

(12)

$$\begin{aligned}&\mathrm{tr}\sum_{i=1}^n[c(e_i)c(\mathrm{grad}|V|^2)\widehat{c}(V)\widehat{c}(\nabla^{TM}_{e_i}V)]\\ &\qquad=2\sum_{i=1}^n g^{TM}(V,\nabla^{TM}_{e_i}V)\mathrm{tr}[c(e_i)c(\mathrm{grad}|V|^2)]-\mathrm{tr}\sum_{i=1}^n[c(e_i)c(\mathrm{grad}|V|^2)\widehat{c}(\nabla^{TM}_{e_i}V)\widehat{c}(V)],\end{aligned}$$

so

$$\begin{aligned}\mathrm{tr}\sum_{i=1}^n[c(e_i)c(\mathrm{grad}|V|^2)\widehat{c}(V)\widehat{c}(\nabla^{TM}_{e_i}V)]=&\sum_{i=1}^n g^{TM}(V,\nabla^{TM}_{e_i}V)\mathrm{tr}[c(e_i)c(\mathrm{grad}|V|^2)]\\ =&-\sum_{i=1}^n g^{TM}(V,\nabla^{TM}_{e_i}V)g^{TM}(e_i,\mathrm{grad}|V|^2)\\ =&-\frac{1}{2}\sum_{i=1}^n\big(e_q(|V|^2)\big)^2\mathrm{tr}[\mathtt{id}].\end{aligned}\tag{57}$$

Therefore, we get

$$\begin{aligned}\mathrm{tr}(\frac{1}{6}s+E_{F^*})=&\Big(-\frac{1}{12}s+\frac{n-2}{4}\left|\mathrm{grad}|V|^2\right|^2\frac{1}{|V|^4}-\frac{1}{2}\sum_j g^{TM}(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\frac{1}{|V|^2}\\ &-\frac{t^2+\bar{t}^2+t+\bar{t}}{2}\sum_i\big(e_i(|V|^2)\big)^2\frac{1}{|V|^4}+(t^2+t\bar{t}+\bar{t}^2)\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\frac{1}{|V|^2}\\ &+\bar{t}g^{TM}(\Delta^{TM}V,V)\frac{1}{|V|^2}\Big)\mathrm{tr}[\mathtt{id}];\end{aligned}\tag{58}$$

$$\begin{aligned}\mathrm{tr}(\frac{1}{6}s+E_{F})=&\Big(-\frac{1}{12}s+\frac{n-2}{4}\left|\mathrm{grad}|V|^2\right|^2\frac{1}{|V|^4}-\frac{1}{2}\sum_j g(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\frac{1}{|V|^2}\\ &+\frac{n-1}{4}\sum_r\big(e_r(|V|^2)\big)^2\frac{1}{|V|^4}-\frac{n-4}{4}\sum_r\left|\nabla^{L}_{e_r}V\right|^2\frac{1}{|V|^2}+t^2\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\frac{1}{|V|^2}\\ &+(t-\frac{1}{2})g^{TM}(\Delta^{TM}V,V)\frac{1}{|V|^2}\Big)\mathrm{tr}[\mathtt{id}].\end{aligned}\tag{59}$$

Then by (29) and (30), we have the following theorem and corollary.

Theorem 2.3. *If M is a 4-dimensional compact oriented manifold without boundary, then we get the* *following equalities*:

$$\begin{aligned}\mathrm{Wres}(\widetilde{D}_V^*\widetilde{D}_V)^{-1} =&32\pi^2\int_M|V|^{-2}\Big(-\frac{4}{3}s+8\left|\mathrm{grad}|V|^2\right|^2\frac{1}{|V|^4}-8\sum_j g(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\frac{1}{|V|^2}\\&-8(t^2+\bar{t}^2+t+\bar{t})\sum_i\big(e_i(|V|^2)\big)^2\frac{1}{|V|^4}+16(t^2+t\bar{t}+\bar{t}^2)\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\frac{1}{|V|^2}\\&+16\bar{t}g^{TM}(\Delta^{TM}V,V)\frac{1}{|V|^2}\Big)d\mathrm{Vol_M};\\ \mathrm{Wres}(\widetilde{D}_V^2)^{-1} =&32\pi^2\int_M|V|^{-2}\Big(-\frac{4}{3}s+8\left|\mathrm{grad}|V|^2\right|^2\frac{1}{|V|^4}-8\sum_j g(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\frac{1}{|V|^2}\\&+12\sum_r\big(e_r(|V|^2)\big)^2\frac{1}{|V|^4}+16t^2\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\frac{1}{|V|^2}\\&+8(2t-1)g^{TM}(\Delta^{TM}V,V)\frac{1}{|V|^2}\Big)d\mathrm{Vol_M}.\end{aligned} \tag{60}$$

Corollary 2.4. *If M is an n-dimensional compact oriented manifold without boundary, and n is even, then when $|V|=1$, we get the following equalities*:

$$\begin{aligned}\mathrm{Wres}(\widetilde{D}_V^*\widetilde{D}_V)^{-\frac{n-2}{2}} =&\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M 2^n\Big(-\frac{1}{12}s+(t^2+t\bar{t}+\bar{t}^2)\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\\&+\bar{t}g^{TM}(\Delta^{TM}V,V)\Big)d\mathrm{Vol_M};\\ \mathrm{Wres}(\widetilde{D}_V^2)^{-\frac{n-2}{2}} =&\frac{(n-2)(4\pi)^{\frac{n}{2}}}{(\frac{n}{2}-1)!}\int_M 2^n\Big(-\frac{1}{12}s-\frac{n-4}{4}\sum_r\left|\nabla^{L}_{e_r}V\right|^2+t^2\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\\&+(t-\frac{1}{2})g^{TM}(\Delta^{TM}V,V)\Big)d\mathrm{Vol_M}.\end{aligned} \tag{61}$$

3. THE KASTLER–KALAU–WALZE TYPE THEOREM FOR 4-DIMENSIONAL MANIFOLDS WITH BOUNDARY

In this section, we prove the Kastler–Kalau–Walze type theorem for 4-dimensional oriented compact manifolds with boundary. We firstly recall some basic facts and formulas about Boutet de Monvel's calculus and the definition of the noncommutative residue for manifolds with boundary which will be used in the following. For more details, see Section 2 in [13].

Let M be a 4-dimensional compact oriented manifold with the boundary ∂M. We assume that the metric g^{TM} on M has the following form near the boundary:

$$g^{TM}=\frac{1}{h(x_n)}g^{\partial M}+dx_n^2, \tag{62}$$

where $g^{\partial M}$ is the metric on ∂M and $h(x_n)\in C^\infty([0,1)):=\{\widehat{h}|_{[0,1)}|\widehat{h}\in C^\infty((-\varepsilon,1))\}$ for some $\varepsilon>0$ and $h(x_n)$ satisfies $h(x_n)>0$, $h(0)=1$, where x_n denotes the normal directional coordinate. Let $U\subset M$ be a collar neighborhood of ∂M which is diffeomorphic with $\partial M\times[0,1)$. By the definition of $h(x_n)\in C^\infty([0,1))$ and $h(x_n)>0$, there exists an $\widehat{h}\in C^\infty((-\varepsilon,1))$ such that $\widehat{h}|_{[0,1)}=h$ and $\widehat{h}>0$ for some sufficiently small $\varepsilon>0$. Then there exists a metric g' on $\widetilde{M}=M\bigcup_{\partial M}\partial M\times(-\varepsilon,0]$ which has the following form on $U\bigcup_{\partial M}\partial M\times(-\varepsilon,0]$:

$$g'=\frac{1}{\widehat{h}(x_n)}g^{\partial M}+dx_n^2, \tag{63}$$

such that $g'|_M=g$. We fix a metric g' on the $\widetilde{M}$ such that $g'|_M=g$.

Let the Fourier transformation F' be

$$F':L^2(\mathbf{R}_t)\to L^2(\mathbf{R}_v);\ F'(u)(v)=\int_{\mathbb{R}}e^{-ivt}u(t)dt,$$

and let

$$r^{+}: \quad C^{\infty}(\mathbf{R}) \to C^{\infty}(\widetilde{\mathbf{R}^{+}}); \quad f \to f|\widetilde{\mathbf{R}^{+}}; \quad \widetilde{\mathbf{R}^{+}} = \{x \geqslant 0; \quad x \in \mathbf{R}\}.$$

We define $H^{+} = F'(\Phi(\widetilde{\mathbf{R}^{+}}))$; $H_0^{-} = F'(\Phi(\widetilde{\mathbf{R}^{-}}))$ which satisfies $H^{+} \perp H_0^{-}$, where $\Phi(\widetilde{\mathbf{R}^{+}}) = r^{+}\Phi(\mathbf{R})$, $\Phi(\widetilde{\mathbf{R}^{-}}) = r^{-}\Phi(\mathbf{R})$ and $\Phi(\mathbf{R})$ denotes the Schwartz space. We have the following property: $h \in H^{+}$ (respectively, H_0^{-}) if and only if $h \in C^{\infty}(\mathbf{R})$ which has an analytic extension to the lower (respectively, upper) complex half-plane $\{\mathrm{Im}\xi < 0\}$ (respectively, $\{\mathrm{Im}\xi > 0\}$) such that, for all nonnegative integers l,

$$\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l}{d\xi^l}\left(\frac{c_k}{\xi^k}\right),$$

as $|\xi| \to +\infty, \mathrm{Im}\xi \leqslant 0$ (respectively, $\mathrm{Im}\xi \geqslant 0$) and where $c_k \in \mathbb{C}$ are some constants.

Let H' be the space of all polynomials and $H^{-} = H_0^{-} \bigoplus H'$; $H = H^{+} \bigoplus H^{-}$. Denote by π^{+} (respectively, π^{-}) the projection on H^{+} (respectively, H^{-}). Let $\widetilde{H} = \{$rational functions having no poles on the real axis$\}$. Then, on $\widetilde{H}$,

$$\pi^{+}h(\xi_0) = \frac{1}{2\pi i} \lim_{u \to 0^{-}} \int_{\Gamma^{+}} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \tag{64}$$

where Γ^{+} is a Jordan closed curve included $\mathrm{Im}(\xi) > 0$ surrounding all the singularities of h in the upper half-plane and $\xi_0 \in \mathbf{R}$. In our computations, we only compute $\pi^{+}h$ for h in $\widetilde{H}$. Similarly, define π' on $\widetilde{H}$,

$$\pi' h = \frac{1}{2\pi} \int_{\Gamma^{+}} h(\xi) d\xi. \tag{65}$$

So $\pi'(H^{-}) = 0$. For $h \in H \bigcap L^1(\mathbf{R})$, $\pi' h = \frac{1}{2\pi}\int_{\mathbf{R}} h(v)dv$ and, for $h \in H^{+} \bigcap L^1(\mathbf{R})$, $\pi' h = 0$.

An operator of order $m \in \mathbf{Z}$ and type d is a matrix

$$\widetilde{A} = \begin{pmatrix} \pi^{+}P + G & K \\ T & \widetilde{S} \end{pmatrix}: \begin{matrix} C^{\infty}(M, E_1) \\ \bigoplus \\ C^{\infty}(\partial M, F_1) \end{matrix} \longrightarrow \begin{matrix} C^{\infty}(M, E_2) \\ \bigoplus \\ C^{\infty}(\partial M, F_2) \end{matrix},$$

where M is a manifold with boundary ∂M and E_1, E_2 (respectively, F_1, F_2) are vector bundles over M (respectively, ∂M). Here, $P: C_0^{\infty}(\Omega, \overline{E_1}) \to C^{\infty}(\Omega, \overline{E_2})$ is a classical pseudo-differential operator of order m on Ω, where Ω is a collar neighborhood of M and $\overline{E_i}|M = E_i$ $(i = 1, 2)$. P has an extension: $\mathcal{E}'(\Omega, \overline{E_1}) \to \mathcal{D}'(\Omega, \overline{E_2})$, where $\mathcal{E}'(\Omega, \overline{E_1})$ $(\mathcal{D}'(\Omega, \overline{E_2}))$ is the dual space of $C^{\infty}(\Omega, \overline{E_1})$ $(C_0^{\infty}(\Omega, \overline{E_2}))$. Let $e^{+}: C^{\infty}(M, E_1) \to \mathcal{E}'(\Omega, \overline{E_1})$ denote extension by zero from M to Ω and $r^{+}: \mathcal{D}'(\Omega, \overline{E_2}) \to \mathcal{D}'(\Omega, E_2)$ denote the restriction from Ω to X, then define

$$\pi^{+}P = r^{+}Pe^{+}: C^{\infty}(M, E_1) \to \mathcal{D}'(\Omega, E_2).$$

In addition, P is supposed to have the transmission property; this means that, for all j, k, α, the homogeneous component p_j of order j in the asymptotic expansion of the symbol p of P in local coordinates near the boundary satisfies:

$$\partial_{x_n}^{k}\partial_{\xi'}^{\alpha} p_j(x', 0, 0, +1) = (-1)^{j-|\alpha|}\partial_{x_n}^{k}\partial_{\xi'}^{\alpha} p_j(x', 0, 0, -1),$$

then $\pi^{+}P: C^{\infty}(M, E_1) \to C^{\infty}(M, E_2)$ by [11]. Let G, T be, respectively, the singular Green operator and the trace operator of order m and type d. Let K be a potential operator and S be a classical pseudo-differential operator of order m along the boundary. Denote by $B^{m,d}$ the collection of all operators of order m and type d, and let $\mathcal{B}$ be the union over all m and d.

Recall that $B^{m,d}$ is a Fréchet space. The composition of the above operator matrices yields a continuous mapping: $B^{m,d} \times B^{m',d'} \to B^{m+m', \max\{m'+d, d'\}}$. Write

$$\widetilde{A} = \begin{pmatrix} \pi^{+}P + G & K \\ T & \widetilde{S} \end{pmatrix} \in B^{m,d}, \quad \widetilde{A}' = \begin{pmatrix} \pi^{+}P' + G' & K' \\ T' & \widetilde{S}' \end{pmatrix} \in B^{m',d'}.$$

The composition $\widetilde{A}\widetilde{A}'$ is obtained by multiplication of the matrices (for more details, see [9]). For example, $\pi^{+}P \circ G'$ and $G \circ G'$ are singular Green operators of type d' and

$$\pi^{+}P \circ \pi^{+}P' = \pi^{+}(PP') + L(P, P').$$

Here PP' is the usual composition of pseudo-differential operators and $L(P,P')$, called thr leftover term, is a singular Green operator of type $m'+d$. For our case, P, P' are classical pseudo-differential operators, in other words, $\pi^+P\in\mathcal{B}^\infty$ and $\pi^+P'\in\mathcal{B}^\infty$.

Let M be an n-dimensional compact oriented manifold with the boundary ∂M. Denote by $\mathcal{B}$ the Boutet de Monvel's algebra. We recall the main theorem and the definition in [4, 13].

Theorem 3.1. (Fedosov–Golse–Leichtnam–Schrohe)[4] *Let X and ∂X be connected,* $\dim M=n\geqslant 3$, *and let $\widetilde{S}$ (respectively, $\widetilde{S}'$) be the unit sphere about ξ (respectively, ξ') and $\sigma(\xi)$ (respectively, $\sigma(\xi')$) be the corresponding canonical $n-1$ (respectively, $(n-2)$) volume form. Set $\widetilde{A}=\begin{pmatrix}\pi^+P+G & K\\ T & \widetilde{S}\end{pmatrix}\in\mathcal{B}$, and denote by p, b, and s the local symbols of P,G and $\widetilde{S}$, respectively. Define*:

$$\widetilde{\mathrm{Wres}}(\widetilde{A})=\int_X\int_{\widetilde{\mathbf{S}}}\mathrm{tr}_E\left[p_{-n}(x,\xi)\right]\sigma(\xi)dx$$
$$+2\pi\int_{\partial X}\int_{\widetilde{\mathbf{S}}'}\left\{\mathrm{tr}_E\left[(\mathrm{tr}b_{-n})(x',\xi')\right]+\mathrm{tr}_F\left[s_{1-n}(x',\xi')\right]\right\}\sigma(\xi')dx', \tag{66}$$

where $\widetilde{\mathrm{Wres}}$ *denotes the noncommutative residue of an operator in the Boutet de Monvel's algebra.*
Then (a) $\widetilde{\mathrm{Wres}}([\widetilde{A},B])=0$, *for any $\widetilde{A},B\in\mathcal{B}$;* (b) *This is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.*

Definition 3.2. [13] *The lower-dimensional volumes of spin manifolds with boundary are defined by*

$$\mathrm{Vol}_n^{(p_1,p_2)}M:=\widetilde{\mathrm{Wres}}[\pi^+D^{-p_1}\circ\pi^+D^{-p_2}]. \tag{67}$$

By [13], we get

$$\widetilde{\mathrm{Wres}}[\pi^+D^{-p_1}\circ\pi^+D^{-p_2}]=\int_M\int_{|\xi'|=1}\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\sigma_{-n}(D^{-p_1-p_2})]\sigma(\xi)dx+\int_{\partial M}\Psi, \tag{68}$$

and

$$\Psi=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\partial_{x_n}^j\partial_{\xi'}^\alpha\partial_{\xi_n}^k\sigma_r^+(D^{-p_1})(x',0,\xi',\xi_n)$$
$$\times\partial_{x'}^\alpha\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l(D^{-p_2})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx', \tag{69}$$

where the sum is taken over $r+l-k-|\alpha|-j-1=-n,\ \ r\leqslant -p_1, l\leqslant -p_2$.

For any fixed point $x_0\in\partial M$, we choose the normal coordinates U of x_0 in ∂M (not in M) and compute $\Phi(x_0)$ in the coordinates $\widetilde{U}=U\times[0,1)\subset M$ and the metric $\frac{1}{h(x_n)}g^{\partial M}+dx_n^2$. The dual metric of g^{TM} on $\widetilde{U}$ is $h(x_n)g^{\partial M}+dx_n^2$. Write $g_{ij}^{TM}=g^{TM}(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})$; $g^{ij}_{TM}=g^{TM}(dx_i,dx_j)$, then

$$[g_{ij}^{TM}]=\begin{bmatrix}\frac{1}{h(x_n)}[g_{ij}^{\partial M}] & 0\\ 0 & 1\end{bmatrix};\quad [g^{ij}_{TM}]=\begin{bmatrix}h(x_n)[g^{ij}_{\partial M}] & 0\\ 0 & 1\end{bmatrix}, \tag{70}$$

and

$$\partial_{x_s}g_{ij}^{\partial M}(x_0)=0,\quad 1\leqslant i,j\leqslant n-1;\quad g_{ij}^{TM}(x_0)=\delta_{ij}. \tag{71}$$

From [13], we can get the following three lemmas.

Lemma 3.3. [13]. *With the metric g^{TM} on M near the boundary*

$$\partial_{x_j}(|\xi|^2_{g^{TM}})(x_0)=\begin{cases}0, & for\ j<n,\\ h'(0)|\xi'|^2_{g^{\partial M}}, & for\ j=n,\end{cases} \tag{72}$$

$$\partial_{x_j}(c(\xi))(x_0)=\begin{cases}0, & for\ j<n,\\ \partial_{x_n}[c(\xi')](x_0), & for\ j=n,\end{cases} \tag{73}$$

where $\xi=\xi'+\xi_ndx_n$.

Lemma 3.4. [13]. *With the metric g^{TM} on M near the boundary*

$$\omega_{s,t}(e_i)(x_0)=\begin{cases}\omega_{n,i}(e_i)(x_0)=\frac{1}{2}h'(0), & for\ s=n,t=i,i<n,\\ \omega_{i,n}(e_i)(x_0)=-\frac{1}{2}h'(0), & for\ s=i,t=n,i<n,\\ \omega_{s,t}(e_i)(x_0)=0, & other\ cases,\end{cases} \tag{74}$$

where $(\omega_{s,t})$ denotes the connection matrix of Levi-Civita connection ∇^L.

Lemma 3.5. [13]. *When $i<n$, we have*

$$\Gamma^k_{st}(x_0)=\begin{cases}\Gamma^n_{ii}(x_0)=\frac{1}{2}h'(0), & for\ s=t=i,k=n,\\ \Gamma^i_{ni}(x_0)=-\frac{1}{2}h'(0), & for\ s=n,t=i,k=i,\\ \Gamma^i_{in}(x_0)=-\frac{1}{2}h'(0), & for\ s=i,t=n,k=i,\end{cases} \tag{75}$$

in other cases, $\Gamma^i_{st}(x_0)=0$.

By (68) and (69), we firstly compute

$$\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}_V^{-1}\circ\pi^+(\widetilde{D}_V^*)^{-1}]=\int_M\int_{|\xi'|=1}\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\sigma_{-4}((\widetilde{D}_V^*\widetilde{D}_V)^{-1})]\sigma(\xi)dx+\int_{\partial M}\Psi, \tag{76}$$

where

$$\begin{aligned}\Psi=&\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{\xi_n}\sigma^+_r(\widetilde{D}_V^{-1})(x',0,\xi',\xi_n)\\&\times\partial^\alpha_{x'}\partial^{j+1}_{\xi_n}\partial^k_{x_n}\sigma_l((\widetilde{D}_V^*)^{-1})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx',\end{aligned} \tag{77}$$

and the sum is taken over $r+l-k-j-|\alpha|=-3,\ \ r\leqslant-1,l\leqslant-1$.

Since $[\sigma_{-n}(D^{-p_1-p_2})]|_M$ has the same expression as $\sigma_{-n}(D^{-p_1-p_2})$ in the case of manifolds without boundary, so locally we can compute the interior of $\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}_V^{-1}\circ\pi^+(\widetilde{D}_V^*)^{-1}]$ by Theorem 2.3; then we have

$$\begin{aligned}&\int_M\int_{|\xi'|=1}\mathrm{tr}_{\wedge^*T^*M\otimes\mathbb{C}}[\sigma_{-4}((\widetilde{D}_V^*\widetilde{D}_V)^{-1})]\sigma(\xi)dx\\&=32\pi^2\int_M|V|^{-2}\Big(-\frac{4}{3}s+8\left|\mathrm{grad}|V|^2\right|^2\frac{1}{|V|^4}-8\sum_j g(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\frac{1}{|V|^2}\\&-8(t^2+\bar{t}^2+t+\bar{t})\sum_i\left(e_i(|V|^2)\right)^2\frac{1}{|V|^4}+16(t^2+t\bar{t}+\bar{t}^2)\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\frac{1}{|V|^2}\\&+16\bar{t}g^{TM}(\Delta^{TM}V,V)\frac{1}{|V|^2}\Big)d\mathrm{Vol_M}.\end{aligned}$$

Also, we have the following lemmas.

Lemma 3.6. *The following identities hold*:

$$\begin{aligned}\sigma_1(\widetilde{D}_V)=&\sigma_1(\widetilde{D}_V^*)=-\widehat{c}(V)c(\xi);\\ \sigma_0(\widetilde{D}_V)=&\frac{i\widehat{c}(V)}{4}\Big(\sum_{i,s,t}\omega_{s,t}(e_i)c(e_i)\widehat{c}(e_s)\widehat{c}(e_t)-\sum_{i,s,t}\omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t)\Big)+itA;\\ \sigma_0(\widetilde{D}_V^*)=&\frac{i\widehat{c}(V)}{4}\Big(\sum_{i,s,t}\omega_{s,t}(e_i)c(e_i)\widehat{c}(e_s)\widehat{c}(e_t)-\sum_{i,s,t}\omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t)\Big)\\&-i\sum_{q=1}^n c(e_q)\widehat{c}(\nabla^L_{e_q}V)-i\bar{t}A.\end{aligned} \tag{78}$$

(79)

Write

$$D_x^\alpha \quad = (-i)^{|\alpha|}\partial_x^\alpha; \quad \sigma(\widetilde{D}_V) = p_1 + p_0; \quad \sigma(\widetilde{D}_V^{-1}) = \textstyle\sum_{j=1}^{\infty} q_{-j}. \tag{80}$$

By the composition formula of pseudo-differential operators, we have

$$\begin{aligned}1 = \sigma(\widetilde{D}_V \circ \widetilde{D}_V^{-1}) &= \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha[\sigma(\widetilde{D}_V)]D_x^\alpha[\sigma(\widetilde{D}_V^{-1})]\\ &= (p_1+p_0)(q_{-1}+q_{-2}+q_{-3}+\cdots)\\ &\quad + \sum_j(\partial_{\xi_j}p_1+\partial_{\xi_j}p_0)(D_{x_j}q_{-1}+D_{x_j}q_{-2}+D_{x_j}q_{-3}+\cdots)\\ &= p_1q_{-1} + \left(p_1q_{-2}+p_0q_{-1}+\sum_j\partial_{\xi_j}p_1D_{x_j}q_{-1}\right)+\cdots,\end{aligned} \tag{81}$$

so

$$q_{-1} = p_1^{-1};\ q_{-2} = -p_1^{-1}\left[p_0p_1^{-1}+\sum_j\partial_{\xi_j}p_1D_{x_j}(p_1^{-1})\right]. \tag{82}$$

Lemma 3.7. *The following identities hold*:

$$\begin{aligned}\sigma_{-1}(\widetilde{D}_V^{-1}) &= \sigma_{-1}((\widetilde{D}_V^*)^{-1}) = -\frac{\widehat{c}(V)c(\xi)}{|\xi|^2};\\ \sigma_{-2}(\widetilde{D}_V^{-1}) &= \frac{c(\xi)\sigma_0(\widetilde{D}_V)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)\Big];\\ \sigma_{-2}((\widetilde{D}_V^*)^{-1}) &= \frac{c(\xi)\sigma_0(\widetilde{D}_V^*)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)\Big].\end{aligned} \tag{83}$$

Now we need to compute $\int_{\partial M}\Psi$. When $n=4$, we have $\mathrm{tr}_{\wedge^*T^*M\otimes\mathbb{C}}[\mathtt{id}] = 2^n = 16$, and the sum is taken over $r+l-k-j-|\alpha| = -3,\ \ r\leqslant -1,\ \ l\leqslant -1$; then we have the following five cases.

Case (a-I). $r=-1,\ l=-1,\ k=j=0,\ |\alpha|=1.$

By (77), we get

$$\Psi_1 = -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi'}^\alpha\pi_{\xi_n}^+\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial_{x'}^\alpha\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{84}$$

By Lemma 3.3, for $i<n$,

$$\begin{aligned}\partial_{x_i}\left(-\frac{\widehat{c}(V)c(\xi)}{|\xi|^2}\right)(x_0) &= -\frac{\partial_{x_i}(\widehat{c}(V))c(\xi)}{|\xi|^2}(x_0) - \widehat{c}(V)\partial_{x_i}\left(\frac{c(\xi)}{|\xi|^2}\right)(x_0)\\ &= -\sum_{l=1}^n\partial_{x_i}(V_l)\widehat{c}(e_l)\frac{c(\xi)}{|\xi|^2}(x_0),\end{aligned} \tag{85}$$

where $\widehat{c}(V) = \sum_{l=1}^n V_l\widehat{c}(e_l), V_l = g^{TM}(V,e_l)$. Then

$$\partial_{x'}^\alpha\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})(x_0) = \frac{(\xi_n^2-1)\sum_{l=1}^n\partial_{x_i}(V_l)\widehat{c}(e_l)}{(1+\xi_n^2)^2}c(dx_n) + \frac{2\xi_n\sum_{l=1}^n\partial_{x_i}(V_l)\widehat{c}(e_l)}{(1+\xi_n^2)^2}c(\xi'). \tag{86}$$

By $c(\xi)=\sum_{j=1}^n \xi_j c(dx_j), |\xi|^2=\sum_{ij} g^{ij}\xi_i\xi_j$, for $i<n$,

$$\begin{aligned}\partial_{\xi_i}\pi^+_{\xi_n}\left(-\frac{\widehat{c}(V)c(\xi)}{|\xi|^2}\right)(x_0) =&\pi^+_{\xi_n}\partial_{\xi_i}\left(-\frac{\widehat{c}(V)\sum_{j=1}^n\xi_j c(dx_j)}{|\xi|^2}\right)(x_0)\\ =&\pi^+_{\xi_n}\left(\frac{-\widehat{c}(V)c(dx_i)}{|\xi|^2}+\frac{2\sum_{j=1}^n\xi_i\xi_j\widehat{c}(V)c(dx_j)}{|\xi|^4}\right)(x_0)\\ =&\frac{i}{2(\xi_n-i)}\widehat{c}(V)c(dx_i)-\frac{i\xi_n+2}{2(\xi_n-i)^2}\sum_{j=1}^{n-1}\xi_i\xi_j\widehat{c}(V)c(dx_j)-\frac{i}{2(\xi_n-i)^2}\xi_i\widehat{c}(V)c(dx_n).\end{aligned} \tag{87}$$

Then

$$\begin{aligned}&\sum_{|\alpha|=1}\operatorname{tr}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)|_{|\xi'|=1} \qquad (88)\\ &=\frac{i(\xi_n^2-1)}{2(\xi_n-i)^3(\xi_n+i)^2}\sum_{l=1}^n\sum_{i=1}^{n-1}\operatorname{tr}[\partial_{x_i}(V_l)\widehat{c}(V)c(dx_i)\widehat{c}(e_l)c(dx_n)]\\ &+\frac{i\xi_n}{(\xi_n-i)^3(\xi_n+i)^2}\sum_{l=1}^n\sum_{i=1}^{n-1}\operatorname{tr}[\partial_{x_i}(V_l)\widehat{c}(V)c(dx_i)\widehat{c}(e_l)c(\xi')]\\ &-\frac{i\xi_n^3+2\xi_n^2-i\xi_n+2}{2(\xi_n-i)^4(\xi_n+i)^2}\sum_{l=1}^n\sum_{i,j=1}^{n-1}\operatorname{tr}[\xi_i\xi_j\partial_{x_i}(V_l)\widehat{c}(V)c(dx_j)\widehat{c}(e_l)c(dx_n)]\\ &-\frac{i\xi_n^2+2\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}\sum_{l=1}^n\sum_{i,j=1}^{n-1}\operatorname{tr}[\xi_i\xi_j\partial_{x_i}(V_l)\widehat{c}(V)c(dx_j)\widehat{c}(e_l)c(\xi')]\\ &+\frac{i(1-\xi_n^2)}{2(\xi_n-i)^4(\xi_n+i)^2}\sum_{l=1}^n\sum_{i=1}^{n-1}\operatorname{tr}[\xi_i\partial_{x_i}(V_l)\widehat{c}(V)c(dx_n)\widehat{c}(e_l)c(dx_n)]\\ &-\frac{i\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}\sum_{l=1}^n\sum_{i=1}^{n-1}\operatorname{tr}[\xi_i\partial_{x_i}(V_l)\widehat{c}(V)c(dx_n)\widehat{c}(e_l)c(\xi')].\end{aligned}$$

By the relation of the Clifford action and $\operatorname{tr}(AB)=\operatorname{tr}(BA)$, we have the following equalities:

$$\sum_{l=1}^n\sum_{i=1}^{n-1}\operatorname{tr}[\partial_{x_i}(V_l)\widehat{c}(V)c(dx_i)\widehat{c}(e_l)c(dx_n)]=0; \tag{89}$$

$$\begin{aligned}\sum_{l=1}^n\sum_{i,j=1}^{n-1}\operatorname{tr}[\xi_i\xi_j\partial_{x_i}(V_l)\widehat{c}(V)c(dx_j)\widehat{c}(e_l)c(dx_n)]=0;\\ \sum_{l=1}^n\sum_{i=1}^{n-1}\operatorname{tr}[\xi_i\partial_{x_i}(V_l)\widehat{c}(V)c(dx_n)\widehat{c}(e_l)c(\xi')]=0.\end{aligned} \tag{90}$$

We note that $i<n,\ \int_{|\xi'|=1}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\}\sigma(\xi')=0$, so we omit some items that have no contribution for computing Ψ_1. So

$$\Psi_1=0. \tag{91}$$

Case (a-II). $r=-1,\ l=-1,\ k=|\alpha|=0,\ j=1.$

By (77), we get

$$\Psi_2=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{92}$$

By Lemma 3.7, we have

$$\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}^*_V)^{-1})(x_0)=\widehat{c}(V)\left(\frac{6\xi_nc(dx_n)+2c(\xi')}{|\xi|^4}-\frac{8\xi_n^2c(\xi)}{|\xi|^6}\right), \tag{93}$$

and

$$\partial_{x_n}\sigma_{-1}(\widetilde{D}_V^{-1})(x_0)=-\frac{\partial_{x_n}(\widehat{c}(V))c(\xi)}{|\xi|^2}(x_0)-\frac{\widehat{c}(V)\partial_{x_n}[c(\xi')](x_0)}{|\xi|^2}+\frac{\widehat{c}(V)c(\xi)|\xi'|^2h'(0)}{|\xi|^4}(x_0). \tag{94}$$

By integrating the formula, we obtain

$$\begin{aligned}\pi^+_{\xi_n}\left[-\frac{\partial_{x_n}(\widehat{c}(V))c(\xi)}{|\xi|^2}\right](x_0)|_{|\xi'|=1}&=-\partial_{x_n}(\widehat{c}(V))\pi^+_{\xi_n}\left[\frac{(i\xi_n+2)c(\xi')+ic(dx_n)}{1+\xi_n^2}\right]\\&=i\partial_{x_n}(\widehat{c}(V))\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)}.\end{aligned} \tag{95}$$

Similarly, we have

$$\pi^+_{\xi_n}\left[\frac{\widehat{c}(V)\partial_{x_n}[c(\xi')]}{|\xi|^2}\right](x_0)|_{|\xi'|=1}=\frac{\widehat{c}(V)\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n-i)}, \tag{96}$$

and

$$\pi^+_{\xi_n}\left[\frac{\widehat{c}(V)c(\xi)|\xi'|^2h'(0)}{|\xi|^4}\right](x_0)|_{|\xi'|=1}=-ih'(0)\widehat{c}(V)\left[\frac{(i\xi_n+2)c(\xi')+ic(dx_n)}{4(\xi_n-i)^2}\right]. \tag{97}$$

Then

$$\begin{aligned}&\pi^+_{\xi_n}\partial_{x_n}\sigma_{-1}(\widetilde{D}_V^{-1})(x_0)|_{|\xi'|=1}\\&\quad=\frac{i\partial_{x_n}(\widehat{c}(V))(c(\xi')+ic(dx_n))-\widehat{c}(V)\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n-i)}-ih'(0)\widehat{c}(V)\left[\frac{(i\xi_n+2)c(\xi')+ic(dx_n)}{4(\xi_n-i)^2}\right].\end{aligned} \tag{98}$$

By the relation of the Clifford action and $\mathrm{tr}(AB)=\mathrm{tr}(BA)$ and by $\partial_{x_n}(V_l)V_l=\frac{1}{2}\partial_{x_n}(|V|^2)$, we have the following equalities:

$$\begin{aligned}\mathrm{tr}[\partial_{x_n}(\widehat{c}(V))c(\xi')\widehat{c}(V)c(dx_n)]&=0;\quad \mathrm{tr}[\partial_{x_n}(\widehat{c}(V))c(\xi')\widehat{c}(V)c(\xi')](x_0)|_{|\xi'|=1}=8\partial_{x_n}(|V|^2);\\ \mathrm{tr}[\partial_{x_n}(\widehat{c}(V))c(dx_n)\widehat{c}(V)c(dx_n)]&=8\partial_{x_n}(|V|^2);\quad \mathrm{tr}[\partial_{x_n}(\widehat{c}(V))c(dx_n)\widehat{c}(V)c(\xi')]=0;\\ \mathrm{tr}[\widehat{c}(V)\partial_{x_n}[c(\xi')]\widehat{c}(V)c(\xi')](x_0)|_{|\xi'|=1}&=8h'(0)|V|^2;\quad \mathrm{tr}[\widehat{c}(V)c(\xi')\widehat{c}(V)c(\xi')](x_0)|_{|\xi'|=1}=16|V|^2;\\ \mathrm{tr}[\widehat{c}(V)c(dx_n)\widehat{c}(V)c(\xi')]&=0;\quad \mathrm{tr}[\widehat{c}(V)\partial_{x_n}[c(\xi')]\widehat{c}(V)c(dx_n)]=0.\end{aligned} \tag{99}$$

(100)

By (93), (98), and (99), we have

$$\begin{aligned}&\mathrm{tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}^*_V)^{-1})](x_0)|_{|\xi'|=1}\\&\quad=\frac{(24\xi_n^2+8\xi_n-16i-8)h'(0)}{(\xi_n-i)^4(\xi_n+i)^2}|V|^2-\frac{(40\xi_n^3-64i\xi_n^2-24\xi_n)h'(0)}{(\xi_n-i)^5(\xi_n+i)^3}|V|^2\\&\quad\quad+\frac{8\xi_n^3-24i\xi_n^2-24\xi_n+8i}{(\xi_n-i)^4(\xi_n+i)^3}\partial_{x_n}(|V|^2).\end{aligned} \tag{101}$$

Therefore, we get

$$\begin{aligned}
\Psi_2 =& -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx' \qquad (102)\\
=& -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{(24\xi_n^2+8\xi_n-16i-8)h'(0)}{(\xi_n-i)^4(\xi_n+i)^2}|V|^2d\xi_n\sigma(\xi')dx'\\
&+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{(40\xi_n^3-64i\xi_n^2-24\xi_n)h'(0)}{(\xi_n-i)^5(\xi_n+i)^3}|V|^2d\xi_n\sigma(\xi')dx'\\
&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{8\xi_n^3-24i\xi_n^2-24\xi_n+8i}{(\xi_n-i)^4(\xi_n+i)^3}\partial_{x_n}(|V|^2)d\xi_n\sigma(\xi')dx'\\
=& -4h'(0)\Omega_3\frac{2\pi i}{3!}\left[\frac{3\xi_n^2+\xi_n-2i-1}{(\xi_n+i)^2}\right]^{(3)}\Big|_{\xi_n=i}|V|^2dx'\\
&+4h'(0)\Omega_3\frac{2\pi i}{4!}\left[\frac{5\xi_n^3-8i\xi_n^2-3\xi_n}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}|V|^2dx'\\
&-4\Omega_3\frac{2\pi i}{3!}\left[\frac{\xi_n^3-3i\xi_n^2-3\xi_n+i}{(\xi_n+i)^2}\right]^{(3)}\Big|_{\xi_n=i}\partial_{x_n}(|V|^2)dx'\\
=&\left(\frac{(1-6i)}{2}h'(0)|V|^2+\partial_{x_n}(|V|^2)\right)\pi\Omega_3dx', \qquad (103)
\end{aligned}$$

where Ω_3 is the canonical volume of S^2.

Case (a-III). $r=-1,\ l=-1,\ j=|\alpha|=0,\ k=1.$

By (77), we get

$$\Psi_3=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \qquad (104)$$

By Lemma 3.7, we have

$$\begin{aligned}
&\partial_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})(x_0)|_{|\xi'|=1} \qquad (105)\\
&=-\frac{\partial_{x_n}(\widehat{c}(V))c(dx_n)}{|\xi|^2}+\frac{2\partial_{x_n}(\widehat{c}(V))(\xi_nc(\xi')+\xi_n^2c(dx_n))}{|\xi|^4}+\frac{2\widehat{c}(V)\xi_n\partial_{x_n}[c(\xi')](x_0)}{|\xi|^4}\\
&+h'(0)\left(\frac{\widehat{c}(V)c(dx_n)}{|\xi|^4}-4\xi_n\frac{\widehat{c}(V)(c(\xi')+\xi_nc(dx_n))}{|\xi|^6}\right),
\end{aligned}$$

and

$$\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})(x_0)|_{|\xi'|=1}=-\frac{i\widehat{c}(V)c(\xi')-\widehat{c}(V)c(dx_n)}{2(\xi_n-i)^2}. \qquad (106)$$

By (99), (105), and (106), we have

$$\begin{aligned}
\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)|_{|\xi'|=1}=&\frac{8h'(0)}{(\xi_n-i)^4(\xi_n+i)^2}|V|^2+\frac{4(3i\xi_n-4\xi_n^2-i\xi_n^3)h'(0)}{(\xi_n-i)^5(\xi_n+i)^3}|V|^2 \qquad (107)\\
&-\frac{4(i\xi_n+1)}{(\xi_n-i)^4(\xi_n+i)^2}\partial_{x_n}(|V|^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}\Psi_3 =& -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\ =& -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{4h'(0)}{(\xi_n-i)^4(\xi_n+i)^2}|V|^2d\xi_n\sigma(\xi')dx'\\ &-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{2(3i\xi_n-4\xi_n^2-i\xi_n^3)h'(0)}{(\xi_n-i)^5(\xi_n+i)^3}|V|^2d\xi_n\sigma(\xi')dx'\\ &+\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{2(i\xi_n+1)}{(\xi_n-i)^4(\xi_n+i)^2}\partial_{x_n}(|V|^2)d\xi_n\sigma(\xi')dx'\\ =& -4h'(0)\Omega_3\frac{2\pi i}{3!}\left[\frac{1}{(\xi_n+i)^2}\right]^{(3)}|_{\xi_n=i}|V|^2dx'\\ &-2h'(0)\Omega_3\frac{2\pi i}{4!}\left[\frac{3i\xi_n-4\xi_n^2-i\xi_n^3}{(\xi_n+i)^3}\right]^{(4)}|_{\xi_n=i}|V|^2dx'\\ &+2\Omega_3\frac{2\pi i}{3!}\left[\frac{i\xi_n+1}{(\xi_n+i)^2}\right]^{(3)}|_{\xi_n=i}\partial_{x_n}(|V|^2)dx'\\ =&\left(-\frac{3}{2}h'(0)|V|^2-\frac{3}{4}\partial_{x_n}(|V|^2)\right)\pi\Omega_3dx'. \end{aligned}\tag{108}$$

Case (a-IV). $r=-2,\ l=-1,\ k=j=|\alpha|=0.$

By (77), we get

$$\Psi_4=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'.\tag{109}$$

By Lemma 3.7, we have

$$\sigma_{-2}(\widetilde{D}_V^{-1})(x_0)=\frac{c(\xi)\sigma_0(\widetilde{D}_V)(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\big[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)|\xi'|^2_{\partial M}\big],\tag{110}$$

where

$$\sigma_0(\widetilde{D}_V)(x_0)=\frac{i\widehat{c}(V)}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)\widehat{c}(e_s)\widehat{c}(e_t)-\frac{i\widehat{c}(V)}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t)+itA.\tag{111}$$

We denote

$$\begin{aligned}&B_0^1(x_0)=\frac{i\widehat{c}(V)}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)\widehat{c}(e_s)\widehat{c}(e_t)=i\widehat{c}(V)b_0^1(x_0);\\ &B_0^2(x_0)=-\frac{i\widehat{c}(V)}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t)=i\widehat{c}(V)b_0^2(x_0);\\ &B_0^3(x_0)=itA(x_0),\end{aligned}\tag{112}$$

where $b_0^2=c_0c(dx_n)$ and $c_0=-\frac{3}{4}h'(0)$. Then

$$\begin{aligned}\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}_V^{-1})(x_0)|_{|\xi'|=1}=&\pi^+_{\xi_n}\Big[\frac{c(\xi)B_0^1(x_0)c(\xi)}{(1+\xi_n^2)^2}\Big]+\pi^+_{\xi_n}\Big[\frac{c(\xi)B_0^2(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2}\\ &-h'(0)\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^3}\Big]+\pi^+_{\xi_n}\Big[\frac{c(\xi)B_0^3(x_0)c(\xi)}{(1+\xi_n^2)^2}\Big].\end{aligned}\tag{113}$$

By computations, we have

$$\begin{aligned}\pi^+_{\xi_n}\Big[\frac{c(\xi)B_0^1(x_0)c(\xi)}{(1+\xi_n^2)^2}\Big] =&\pi^+_{\xi_n}\Big[\frac{c(\xi')B_0^1(x_0)c(\xi')}{(1+\xi_n^2)^2}\Big]+\pi^+_{\xi_n}\Big[\frac{\xi_nc(\xi')B_0^1(x_0)c(dx_n)}{(1+\xi_n^2)^2}\Big]\\ &+\pi^+_{\xi_n}\Big[\frac{\xi_nc(dx_n)B_0^1(x_0)c(\xi')}{(1+\xi_n^2)^2}\Big]+\pi^+_{\xi_n}\Big[\frac{\xi_n^2c(dx_n)B_0^1(x_0)c(dx_n)}{(1+\xi_n^2)^2}\Big]\\ =&-\frac{c(\xi')B_0^1(x_0)c(\xi')(2+i\xi_n)}{4(\xi_n-i)^2}-\frac{ic(\xi')B_0^1(x_0)c(dx_n)}{4(\xi_n-i)^2}\\ &-\frac{ic(dx_n)B_0^1(x_0)c(\xi')}{4(\xi_n-i)^2}-\frac{i\xi_nc(dx_n)B_0^1(x_0)c(dx_n)}{4(\xi_n-i)^2}.\end{aligned}\tag{114}$$

Since

$$c(dx_n)b_0^1(x_0)=-\frac{1}{4}h'(0)\sum_{i=1}^{n-1}c(e_i)\widehat{c}(e_i)c(e_n)\widehat{c}(e_n),\tag{115}$$

it follows that by the relation of the Clifford action and $\mathrm{tr}(AB)=\mathrm{tr}(BA)$, we have the following equalities:

$$\mathrm{tr}[c(e_i)\widehat{c}(e_i)c(e_n)\widehat{c}(e_n)]=0\ \ (i<n);\ \ \mathrm{tr}[b_0^1(x_0)c(dx_n)]=0;\quad \mathrm{tr}[\widehat{c}(\xi')\widehat{c}(dx_n)]=0.\tag{116}$$

Since

$$\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})(x_0)=-\widehat{c}(V)\left[\frac{(1-\xi_n^2)c(dx_n)}{(1+\xi_n^2)^2}-\frac{2\xi_nc(\xi')}{(1+\xi_n^2)^2}\right].\tag{117}$$

By (114) and (117), we have

$$\begin{aligned}&\mathrm{tr}[\pi^+_{\xi_n}\Big[\frac{c(\xi)B_0^1(x_0)c(\xi)}{(1+\xi_n^2)^2}\Big]\times\partial_{\xi_n}\sigma_{-1}((D_V{}^*)^{-1})](x_0)|_{|\xi'|=1}\\ &\qquad=-\frac{1}{2(1+\xi_n^2)^2}\mathrm{tr}[c(\xi')b_0^1(x_0)]-\frac{i}{2(1+\xi_n^2)^2}\mathrm{tr}[c(dx_n)b_0^1(x_0)]\\ &\qquad=-\frac{1}{2(1+\xi_n^2)^2}\mathrm{tr}[c(\xi')b_0^1(x_0)].\end{aligned}\tag{118}$$

We note that $i<n,\ \int_{|\xi'|=1}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\}\sigma(\xi')=0$, so $\mathrm{tr}[c(\xi')b_0^1(x_0)]$ has no contribution for computing Ψ_4.

By computations, we have

$$\pi^+_{\xi_n}\Big[\frac{c(\xi)B_0^2(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2}\Big]-h'(0)\pi^+_{\xi_n}\Big[\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^3}\Big]:=L_1-L_2,\tag{119}$$

where

$$\begin{aligned}L_1=&\frac{-1}{4(\xi_n-i)^2}\big[i(2+i\xi_n)c(\xi')\widehat{c}(V)b_0^2(x_0)c(\xi')-\xi_nc(dx_n)\widehat{c}(V)b_0^2(x_0)c(dx_n)\\ &+(2+i\xi_n)c(\xi')c(dx_n)\partial_{x_n}[c(\xi')]-c(dx_n)\widehat{c}(V)b_0^2(x_0)c(\xi')\\ &-c(\xi')\widehat{c}(V)b_0^2(x_0)c(dx_n)-i\partial_{x_n}[c(\xi')]\big],\end{aligned}\tag{120}$$

and

$$L_2=\frac{h'(0)}{2}\left[\frac{c(dx_n)}{4i(\xi_n-i)}+\frac{c(dx_n)-ic(\xi')}{8(\xi_n-i)^2}+\frac{3\xi_n-7i}{8(\xi_n-i)^3}\big[ic(\xi')-c(dx_n)\big]\right].\tag{121}$$

By the relation of the Clifford action and $\mathrm{tr}(AB)=\mathrm{tr}(BA)$, we have the following equalities:

$$\begin{aligned}\mathrm{tr}[c(\xi')\widehat{c}(V)c(\xi')]=0;&\quad \mathrm{tr}[c(dx_n)\widehat{c}(V)c(\xi')]=0;\\ \mathrm{tr}[c(dx_n)\widehat{c}(V)c(dx_n)]=0;&\quad \mathrm{tr}[A\widehat{c}(V)c(dx_n)]=0.\end{aligned}\tag{122}$$

By (117)–(122), we have

$$\mathrm{tr}[L_1\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)|_{|\xi'|=1}=\frac{3h'(0)\xi_n}{(\xi_n-i)^3(\xi_n+i)}|V|^2+\frac{3h'(0)(\xi_n^3-2i\xi_n^2-5\xi_n+2i)}{(\xi_n-i)^4(\xi_n+i)^2}|V|^2, \tag{123}$$

and

$$\begin{aligned}&\mathrm{tr}[L_2\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)|_{|\xi'|=1}\\&\quad=\frac{ih'(0)(\xi_n^2-3i\xi_n)}{4(\xi_n-i)^5(\xi_n+i)^2}\mathrm{tr}[c(\xi')\widehat{c}(V)c(\xi')]-\frac{ih'(0)(4\xi_n^3-12i\xi_n^2-9\xi_n+3i)}{8(\xi_n-i)^5(\xi_n+i)^2}\mathrm{tr}[c(dx_n)\widehat{c}(V)c(\xi')]\\&\qquad-\frac{ih'(0)(2\xi_n^4-6i\xi_n^3-9\xi_n^2+3i\xi_n+4)}{8(\xi_n-i)^5(\xi_n+i)^2}\mathrm{tr}[c(dx_n)\widehat{c}(V)c(dx_n)]=0.\end{aligned} \tag{124}$$

Moreover,

$$\begin{aligned}\pi^+_{\xi_n}\Big[\frac{c(\xi)B_0^3(x_0)c(\xi)}{(1+\xi_n^2)^2}\Big]=&it\Big\{\pi^+_{\xi_n}\Big[\frac{c(\xi')A(x_0)c(\xi')}{(1+\xi_n^2)^2}\Big]+\pi^+_{\xi_n}\Big[\frac{\xi_nc(\xi')A(x_0)c(dx_n)}{(1+\xi_n^2)^2}\Big]\\&+\pi^+_{\xi_n}\Big[\frac{\xi_nc(dx_n)A(x_0)c(\xi')}{(1+\xi_n^2)^2}\Big]+\pi^+_{\xi_n}\Big[\frac{\xi_n^2c(dx_n)A(x_0)c(dx_n)}{(1+\xi_n^2)^2}\Big]\Big\}\\=&-\frac{itc(\xi')A(x_0)c(\xi')(2+i\xi_n)}{4(\xi_n-i)^2}+\frac{tc(\xi')A(x_0)c(dx_n)}{4(\xi_n-i)^2}\\&+\frac{tc(dx_n)A(x_0)c(\xi')}{4(\xi_n-i)^2}+\frac{t\xi_nc(dx_n)A(x_0)c(dx_n)}{4(\xi_n-i)^2}.\end{aligned} \tag{125}$$

By (117), (122), and (125), we have

$$\begin{aligned}&\mathrm{tr}[\pi^+_{\xi_n}\Big[\frac{c(\xi)B_0^3(x_0)c(\xi)}{(1+\xi_n^2)^2}\Big]\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)|_{|\xi'|=1}\\&\quad=\frac{t(2\xi_n^3-3i\xi_n^2+2\xi_n-i)}{2(\xi_n-i)^4(\xi_n+i)^2}\mathrm{tr}[A\widehat{c}(V)c(dx_n)]+\frac{t(\xi_n^2-2i\xi_n-2)}{2(\xi_n-i)^4(\xi_n+i)^2}\mathrm{tr}[A\widehat{c}(V)c(\xi')]\\&\quad=\frac{t(\xi_n^2-2i\xi_n-2)}{2(\xi_n-i)^4(\xi_n+i)^2}\mathrm{tr}[A\widehat{c}(V)c(\xi')].\end{aligned} \tag{126}$$

Similarly, $\mathrm{tr}[A\widehat{c}(V)c(\xi')]$ has no contribution for computing Ψ_4. Therefore,

$$\begin{aligned}\Psi_4=&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_V^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\=&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{3h'(0)\xi_n}{(\xi_n-i)^3(\xi_n+i)}|V|^2d\xi_n\sigma(\xi')dx'\\&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{3h'(0)(\xi_n^3-2i\xi_n^2-5\xi_n+2i)}{(\xi_n-i)^4(\xi_n+i)^2}|V|^2d\xi_n\sigma(\xi')dx'\\=&-3ih'(0)\Omega_3\frac{2\pi i}{2!}\Big[\frac{\xi_n}{(\xi_n+i)}\Big]^{(2)}|_{\xi_n=i}|V|^2dx'\\&-3ih'(0)\Omega_3\frac{2\pi i}{3!}\Big[\frac{\xi_n^3-2i\xi_n^2-5\xi_n+2i}{(\xi_n+i)^2}\Big]^{(3)}|_{\xi_n=i}|V|^2dx'\\=&-\frac{3}{2}h'(0)|V|^2\pi\Omega_3dx'.\end{aligned} \tag{127}$$

Case (a-V). $r=-1,\ l=-2,\ k=j=|\alpha|=0.$

By (77), we get

$$\Psi_5=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\sigma_{-2}((\widetilde{D}_V^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{128}$$

By integrating formula and Lemma 3.7, we have

$$\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})(x_0)=\frac{i\widehat{c}(V)c(\xi')-\widehat{c}(V)c(dx_n)}{2(\xi_n-i)}. \tag{129}$$

Since

$$\sigma_{-2}((\widetilde{D}_V^*)^{-1})(x_0)=\frac{c(\xi)\sigma_0(\widetilde{D}_V^*)(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\big[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)|\xi'|^2_{\partial_M}\big], \tag{130}$$

where

$$\begin{aligned}\sigma_0(\widetilde{D}_V^*)(x_0)=&\frac{i\widehat{c}(V)}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)\widehat{c}(e_s)\widehat{c}(e_t)-\frac{i\widehat{c}(V)}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t)\\&-i\sum_{q=1}^n c(e_q)\widehat{c}(\nabla^L_{e_q}V)(x_0)-i\bar{t}A(x_0)\\=&i\widehat{c}(V)b_0^1(x_0)+i\widehat{c}(V)b_0^2(x_0)-i\sum_{q=1}^n c(e_q)\widehat{c}(\nabla^L_{e_q}V)(x_0)-i\bar{t}A(x_0).\end{aligned} \tag{131}$$

Then

$$\begin{aligned}&\partial_{\xi_n}\sigma_{-2}((\widetilde{D}_V^*)^{-1})(x_0)|_{|\xi'|=1}\\&=\partial_{\xi_n}\bigg\{\frac{c(\xi)[B_0^1(x_0)+B_0^2(x_0)-i\sum_{q=1}^n c(e_q)\widehat{c}(\nabla^L_{e_q}V)(x_0)-i\bar{t}A(x_0)]c(\xi)}{|\xi|^4}\\&\quad+\frac{c(\xi)}{|\xi|^6}c(dx_n)\big[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)\big]\bigg\}\\&=\partial_{\xi_n}\frac{c(\xi)B_0^1(x_0)c(\xi)}{|\xi|^4}+\partial_{\xi_n}\bigg\{\frac{c(\xi)B_0^2(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\big[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)\big]\bigg\}\\&\quad-\partial_{\xi_n}\frac{c(\xi)[i\sum_{q=1}^n c(e_q)\widehat{c}(\nabla^L_{e_q}V)](x_0)c(\xi)}{|\xi|^4}-\partial_{\xi_n}\frac{c(\xi)i\bar{t}A(x_0)c(\xi)}{|\xi|^4}\\&:=N_1+N_2-N_3-N_4.\end{aligned} \tag{132}$$

An easy calculation gives

$$\begin{aligned}N_1&=\partial_{\xi_n}\frac{c(\xi)B_0^1(x_0)c(\xi)}{|\xi|^4}\\&=i\frac{c(dx_n)\widehat{c}(V)b_0^1(x_0)c(\xi)}{|\xi|^4}+i\frac{c(\xi)\widehat{c}(V)b_0^1(x_0)c(dx_n)}{|\xi|^4}-i\frac{4\xi_nc(\xi)\widehat{c}(V)b_0^1(x_0)c(\xi)}{|\xi|^6},\end{aligned} \tag{133}$$

$$\begin{aligned}N_2=&\frac{1}{(1+\xi_n^2)^3}\Big[(2\xi_n-2\xi_n^3)c(dx_n)B_0^2(x_0)c(dx_n)+(1-3\xi_n^2)c(dx_n)B_0^2(x_0)c(\xi')\\&+(1-3\xi_n^2)c(\xi')B_0^2(x_0)c(dx_n)-4\xi_nc(\xi')B_0^2(x_0)c(\xi')+(3\xi_n^2-1)\partial_{x_n}[c(\xi')]\\&-4\xi_nc(\xi')c(dx_n)\partial_{x_n}[c(\xi')]+2h'(0)c(\xi')+2h'(0)\xi_nc(dx_n)\Big]\\&+6\xi_nh'(0)\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^4},\end{aligned} \tag{134}$$

$$\begin{aligned}N_3&=\partial_{\xi_n}\frac{c(\xi)[i\sum_{q=1}^n c(e_q)\widehat{c}(\nabla^L_{e_q}V)](x_0)c(\xi)}{|\xi|^4}\\&=\frac{c(dx_n)[i\sum_{q=1}^n c(e_q)\widehat{c}(\nabla^L_{e_q}V)](x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)[i\sum_{q=1}^n c(e_q)\widehat{c}(\nabla^L_{e_q}V)](x_0)c(dx_n)}{|\xi|^4}\\&\quad-\frac{4\xi_nc(\xi)[i\sum_{q=1}^n c(e_q)\widehat{c}(\nabla^L_{e_q}V)](x_0)c(\xi)}{|\xi|^4},\end{aligned} \tag{135}$$

and

$$
\begin{aligned}
N_4 =&\bar{t}\partial_{\xi_n}\frac{c(\xi)[i\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)](x_0)c(\xi)}{|\xi|^4}\\
=&\frac{\bar{t}c(dx_n)[i\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)](x_0)c(\xi)}{|\xi|^4}+\frac{\bar{t}c(\xi)[i\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)](x_0)c(dx_n)}{|\xi|^4}\\
&-\frac{4\bar{t}\xi_n c(\xi)[i\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)](x_0)c(\xi)}{|\xi|^4}.
\end{aligned}
\tag{136}
$$

Also, straightforward computations yield

$$
\begin{aligned}
&\operatorname{tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times N_1](x_0)|_{|\xi'|=1}\\
&\quad=\frac{1}{(\xi-i)(\xi+i)^3}\operatorname{tr}[c(\xi')b_0^1(x_0)]|V|^2-\frac{i}{(\xi-i)(\xi+i)^3}\operatorname{tr}[c(dx_n)b_0^1(x_0)]|V|^2\\
&\quad=\frac{1}{(\xi-i)(\xi+i)^3}\operatorname{tr}[c(\xi')b_0^1(x_0)]|V|^2,
\end{aligned}
\tag{137}
$$

$$
\operatorname{tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times N_2](x_0)|_{|\xi'|=1}=\frac{6h'(0)(2\xi_n^3-3\xi_n^2-6\xi_n+1)}{(\xi-i)^4(\xi+i)^3},
\tag{138}
$$

$$
\begin{aligned}
\operatorname{tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times N_3](x_0)|_{|\xi'|=1}=&\frac{1}{2(\xi-i)^3(\xi+i)^2}\operatorname{tr}\left[\widehat{c}(V)c(\xi')c(dx_n)\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)c(\xi)\right]\\
&-\frac{i}{2(\xi-i)^3(\xi+i)^2}\operatorname{tr}\left[\widehat{c}(V)\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)c(\xi)\right]\\
&+\frac{(1-4\xi_n)}{2(\xi-i)^3(\xi+i)^2}\operatorname{tr}\left[\widehat{c}(V)c(\xi')c(\xi)\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)c(\xi)\right]\\
&+\frac{(1-4\xi_n)i}{2(\xi-i)^3(\xi+i)^2}\operatorname{tr}\left[\widehat{c}(V)c(dx_n)c(\xi)\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)c(\xi)\right],
\end{aligned}
\tag{139}
$$

and

$$
\begin{aligned}
\operatorname{tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times N_4](x_0)|_{|\xi'|=1}=&\frac{\bar{t}}{2(\xi-i)^3(\xi+i)^2}\operatorname{tr}\left[\widehat{c}(V)c(\xi')c(dx_n)\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)c(\xi)\right]\\
&-\frac{\bar{t}i}{2(\xi-i)^3(\xi+i)^2}\operatorname{tr}\left[\widehat{c}(V)\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)c(\xi)\right]\\
&+\frac{\bar{t}(1-4\xi_n)}{2(\xi-i)^3(\xi+i)^2}\operatorname{tr}\left[\widehat{c}(V)c(\xi')c(\xi)\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)c(\xi)\right]\\
&+\frac{\bar{t}(1-4\xi_n)i}{2(\xi-i)^3(\xi+i)^2}\operatorname{tr}\left[\widehat{c}(V)c(dx_n)c(\xi)\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)c(\xi)\right].
\end{aligned}
\tag{140}
$$

By the relation of the Clifford action and $\operatorname{tr}(AB)=\operatorname{tr}(BA)$, we have the following equalities:

$$
\begin{aligned}
&\operatorname{tr}\left[\widehat{c}(V)c(\xi')c(dx_n)\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)c(\xi')\right]=0;\quad \operatorname{tr}\left[\widehat{c}(V)\sum_{q=1}^{n}c(e_q)\widehat{c}(\nabla^L_{e_q}V)c(dx_n)\right]=0;\\
&\operatorname{tr}\left[\widehat{c}(V)c(\xi')c(dx_n)\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)c(\xi')\right]=0;\quad \operatorname{tr}\left[\widehat{c}(V)\sum_{i=1}^{n}c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)c(dx_n)\right]=0.
\end{aligned}
\tag{141}
$$

We note that $i<n,\ \int_{|\xi'|=1}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\}\sigma(\xi')=0$, so we omit some items that have no contribution for

computing Ψ_5. Therefore,

$$\begin{aligned}\Psi_5 =& - i\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\sigma_{-2}((\widetilde{D}_V^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\ =& - i\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\frac{6h'(0)(2\xi_n^3-3\xi_n^2-6\xi_n+1)}{(\xi-i)^4(\xi+i)^3}d\xi_n\sigma(\xi')dx'\\ =& - 6ih'(0)\Omega_3\frac{2\pi i}{3!}\left[\frac{2\xi_n^3-3\xi_n^2-6\xi_n+1}{(\xi_n+i)^3}\right]^{(3)}\Big|_{\xi_n=i}dx'\\ =&\frac{9(i-2)}{4}h'(0)\pi\Omega_3dx'.\end{aligned}\tag{142}$$

Now Ψ is the sum of the $\Psi_{(1,2,\dots,5)}$. Combining with the five cases, this yields

$$\Psi=\sum_{i=1}^5\Psi_i=\left(\frac{9(i-2)}{4}h'(0)-\frac{6i+5}{2}h'(0)|V|^2+\frac{1}{4}\partial_{x_n}(|V|^2)\right)\pi\Omega_3dx'.\tag{143}$$

So, by (78) and (143), we are reduced to prove the following theorem.

Theorem 3.8. *Let M be a 4-dimensional oriented compact manifold with the boundary ∂M and let the metric g^{TM} be as in Section 3, the operators $\widetilde{D}_V=\sqrt{-1}\Big(\widehat{c}(V)(d+\delta)+t\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$ and $\widetilde{D}_V^*=-\sqrt{-1}\Big((d+\delta)\widehat{c}(V)+\bar{t}\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$ be on $\widetilde{M}$ ($\widetilde{M}$ is a collar neighborhood of M); then*

$$\begin{aligned}&\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}_V^{-1}\circ\pi^+(\widetilde{D}_V^*)^{-1}]\\ =&32\pi^2\int_M|V|^{-2}\bigg(-\frac{4}{3}s+8\left|\mathrm{grad}|V|^2\right|^2\frac{1}{|V|^4}-8\sum_j g(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\frac{1}{|V|^2}\\ &-8(t^2+\bar{t}^2+t+\bar{t})\sum_i\left(e_i(|V|^2)\right)^2\frac{1}{|V|^4}+16(t^2+t\bar{t}+\bar{t}^2)\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\frac{1}{|V|^2}\\ &+16\bar{t}g^{TM}(\Delta^{TM}V,V)\frac{1}{|V|^2}\bigg)d\mathrm{Vol_M}+\int_{\partial M}\bigg(\frac{9(i-2)}{4}h'(0)-\frac{6i+5}{2}h'(0)|V|^2\\ &+\frac{1}{4}\partial_{x_n}(|V|^2)\bigg)\pi\Omega_3d\mathrm{Vol_{\partial M}}.\end{aligned}\tag{144}$$

Next, we also prove the Kastler–Kalau–Walze type theorem for 4-dimensional manifolds with boundary associated to $\widetilde{D}_V^2$. By (68) and (69), we shall compute

$$\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}_V^{-1}\circ\pi^+\widetilde{D}_V^{-1}]=\int_M\int_{|\xi'|=1}\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\sigma_{-4}(\widetilde{D}_V^{-2})]\sigma(\xi)dx+\int_{\partial M}\widetilde{\Psi},\tag{145}$$

where

$$\begin{aligned}\widetilde{\Psi}=&\int_{|\xi'|=1}\int^{+\infty}_{-\infty}\sum^{\infty}_{j,k=0}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{\xi_n}\sigma^+_r(\widetilde{D}_V^{-1})(x',0,\xi',\xi_n)\\ &\times\partial^\alpha_{x'}\partial^{j+1}_{\xi_n}\partial^k_{x_n}\sigma_l(\widetilde{D}_V^{-1})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx',\end{aligned}\tag{146}$$

and the sum is taken over $r+l-k-j-|\alpha|=-3,\quad r\leqslant-1,l\leqslant-1$.

Similarly, by Theorem 2.3, we compute the interior of $\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}_V^{-1}\circ\pi^+\widetilde{D}_V^{-1}]$, then

$$\begin{aligned}&\int_M\int_{|\xi'|=1}\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\sigma_{-4}(\widetilde{D}_V^{-2})]\sigma(\xi)dx\\ =&32\pi^2\int_M|V|^{-2}\bigg(-\frac{4}{3}s+8\left|\mathrm{grad}|V|^2\right|^2\frac{1}{|V|^4}-8\sum_j g(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\frac{1}{|V|^2}\\ &+12\sum_r\left(e_r(|V|^2)\right)^2\frac{1}{|V|^4}+16t^2\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\frac{1}{|V|^2}+8(2t-1)g^{TM}(\Delta^{TM}V,V)\frac{1}{|V|^2}\bigg)d\mathrm{Vol_M}.\end{aligned}\tag{147}$$

Now we need to compute $\int_{\partial M}\widetilde{\Psi}$. When $n=4$, by Lemma 3.7, $\sigma_{-1}(\widetilde{D}_V^{-1})=\sigma_{-1}((\widetilde{D}_V^{*})^{-1})$, and then we have the following five cases:

Case (b-I). $r=-1,\ l=-1,\ k=j=0,\ |\alpha|=1.$

$$\widetilde{\Psi}_1=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial^{\alpha}_{\xi'}\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial^{\alpha}_{x'}\partial_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})](x_0)d\xi_n\sigma(\xi')dx'=0. \tag{148}$$

Case (b-II). $r=-1,\ l=-1,\ k=|\alpha|=0,\ j=1.$

$$\begin{aligned}\widetilde{\Psi}_2&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial^{2}_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\left(\frac{(1-6i)}{2}h'(0)|V|^2+\partial_{x_n}(|V|^2)\right)\pi\Omega_3dx'.\end{aligned} \tag{149}$$

Case (b-III). $r=-1,\ l=-1,\ j=|\alpha|=0,\ k=1.$

$$\begin{aligned}\widetilde{\Psi}_3&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}(\widetilde{D}_V^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\left(-\frac{3}{2}h'(0)|V|^2-\frac{3}{4}\partial_{x_n}(|V|^2)\right)\pi\Omega_3dx'.\end{aligned} \tag{150}$$

Case (b-IV). $r=-2,\ l=-1,\ k=j=|\alpha|=0.$

$$\begin{aligned}\widetilde{\Psi}_4&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^{+}_{\xi_n}\sigma_{-2}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=-\frac{3}{2}h'(0)|V|^2\pi\Omega_3dx'.\end{aligned} \tag{151}$$

Case (b-V). $r=-1,\ l=-2,\ k=j=|\alpha|=0.$

By (146), we get

$$\widetilde{\Psi}_5=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\sigma_{-2}(\widetilde{D}_V^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{152}$$

By computations, we have

$$\pi^{+}_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})(x_0)=\frac{i\widehat{c}(V)c(\xi')-\widehat{c}(V)c(dx_n)}{2(\xi_n-i)}. \tag{153}$$

Since

$$\sigma_{-2}(\widetilde{D}_V^{-1})(x_0)=\frac{c(\xi)\sigma_0(\widetilde{D}_V)(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\left[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)|\xi|^2_{\partial_M}\right], \tag{154}$$

where

$$\begin{aligned}\sigma_0(\widetilde{D}_V)(x_0)&=\frac{i\widehat{c}(V)}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)\widehat{c}(e_s)\widehat{c}(e_t)-\frac{i\widehat{c}(V)}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t)+itA(x_0)\\&=i\widehat{c}(V)b_0^1(x_0)+i\widehat{c}(V)b_0^2(x_0)+itA(x_0).\end{aligned} \tag{155}$$

Then

$$\begin{aligned}
&\partial_{\xi_n}\sigma_{-2}(\widetilde{D}_V^{-1})(x_0)|_{|\xi'|=1}\\
&\quad=\partial_{\xi_n}\left\{\frac{c(\xi)[B_0^1(x_0)+B_0^2(x_0)+B_0^3(x_0)]c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\left[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)\right]\right\}\\
&\quad=\partial_{\xi_n}\frac{c(\xi)B_0^1(x_0)c(\xi)}{|\xi|^4}+\partial_{\xi_n}\left\{\frac{c(\xi)B_0^2(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\left[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)\right]\right\}\\
&\qquad+\partial_{\xi_n}\frac{c(\xi)B_0^3(x_0)c(\xi)}{|\xi|^4}:=M_1+M_2+M_3.
\end{aligned}\tag{156}$$

Similarly, an easy calculation gives

$$\begin{aligned}
\widetilde{\Psi}_5&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_V^{-1})\times\partial_{\xi_n}\sigma_{-2}(\widetilde{D}_V^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\
&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{6h'(0)(2\xi_n^3-3\xi_n^2-6\xi_n+1)}{(\xi-i)^4(\xi+i)^3}d\xi_n\sigma(\xi')dx'\\
&=-6ih'(0)\Omega_3\frac{2\pi i}{3!}\left[\frac{2\xi_n^3-3\xi_n^2-6\xi_n+1}{(\xi_n+i)^3}\right]^{(3)}\Bigg|_{\xi_n=i}dx'\\
&=\frac{9(i-2)}{4}h'(0)\pi\Omega_3dx'.
\end{aligned}\tag{157}$$

Now $\widetilde{\Psi}$ is the sum of the $\widetilde{\Psi}_{(1,2,\dots,5)}$. Combining with the five cases, this yields

$$\widetilde{\Psi}=\sum_{i=1}^{5}\widetilde{\Psi}_i=\left(\frac{9(i-2)}{4}h'(0)-\frac{6i+5}{2}h'(0)|V|^2+\frac{1}{4}\partial_{x_n}(|V|^2)\right)\pi\Omega_3dx'.\tag{158}$$

So, by (147) and (158), we are reduced to prove the following theorem.

Theorem 3.9. *Let M be a 4-dimensional oriented compact manifold with the boundary ∂M and the metric g^{TM} as in Section 3, the operator $\widetilde{D}_V=\sqrt{-1}\Big(\widehat{c}(V)(d+\delta)+t\sum_i c(e_i)\widehat{c}(\nabla^{TM}_{e_i}V)\Big)$ be on $\widetilde{M}$ ($\widetilde{M}$ is a collar neighborhood of M), then*

$$\begin{aligned}
&\widetilde{\mathrm{Wres}}[\pi^+\widetilde{D}_V^{-1}\circ\pi^+\widetilde{D}_V^{-1}]\\
&\quad=32\pi^2\int_M|V|^{-2}\bigg(-\frac{4}{3}s+8\left|\mathrm{grad}|V|^2\right|^2\frac{1}{|V|^4}-8\sum_j g(e_j,\nabla^{TM}_{e_j}\mathrm{grad}|V|^2)\frac{1}{|V|^2}\\
&\qquad+12\sum_r\left(e_r(|V|^2)\right)^2\frac{1}{|V|^4}+16t^2\sum_i\left|\nabla^{TM}_{e_i}V\right|^2\frac{1}{|V|^2}+8(2t-1)g^{TM}(\Delta^{TM}V,V)\frac{1}{|V|^2}\bigg)d\mathrm{Vol_M}\\
&\qquad+\int_{\partial M}\left(\frac{9(i-2)}{4}h'(0)-\frac{6i+5}{2}h'(0)|V|^2+\frac{1}{4}\partial_{x_n}(|V|^2)\right)\pi\Omega_3d\mathrm{Vol}_{\partial M}.
\end{aligned}\tag{159}$$

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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