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Reflexivity for spaces with extended norm

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Abstract — An analogue of reflexivity in asymmetric cone spaces is introduced and studied. Some classical results known for ordinary normalized spaces are carried over to the case of essentially asymmetric spaces.

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1. INTRODUCTION

The definition of a reflexive space is extended from the classical case of normed linear spaces to the asymmetric cone-spaces setting. Approximative properties of sets in asymmetric cone-spaces are studied with the main emphasis on existence of nearest points in cone-subspaces. Many problems of geometric approximation theory are closely related to reflexivity type properties. For some results on linear spaces with asymmetric norm, see [1]–[6]. For some results and problems of geometric approximation theory in asymmetric spaces, see [7]–[24].

Let B be a convex subset of a linear space X over $\mathbb{R}$ with the following properties:

1) $0 \in B$;

2) any line $\ell = \{te \mid t \in \overline{\mathbb{R}} := [-\infty, +\infty]\}$ $(e \neq 0)$ passing through the origin intersects B in an extended closed interval; that is, $B \cap \ell = \{te \mid t \in [\alpha, \beta] \subset \overline{\mathbb{R}}\}$. Here, we assume that $\alpha = -\infty$ $(\beta = +\infty)$ if $\ell \cap B$ is not upper (lower) bounded. With B we will associate the extended Minkowski functional $p_B : X \to [0, +\infty]$ by setting, for all $x \in X \setminus \{0\}$,

$$p_B(x) := \inf\{t \in \overline{\mathbb{R}}_+ := [0, +\infty] \mid tB \ni x\}.$$

If x does not lie in tB for any finite $t \geqslant 0$, then we define $p_B(x) = +\infty$. We also set $p_B(0) = 0$.

Now the formula

$$\| \cdot | := p_B(\,\cdot\,)$$

defines on X an *extended* (or *generalized*) asymmetric seminorm. Let $\mathbf{Z} := \{e \in X \mid e \neq 0,\ \|e| < +\infty\}$. The cone-space $\mathbf{K} = \mathbf{K}(X)$ for a seminorm $\| \cdot |$ is defined by $\mathbf{K} = \{te \mid t \geqslant 0,\ e \in \mathbf{Z}\}$. So, $\mathbf{K} = \mathbf{Z} \cup \{0\}$. Let $\mathbf{L} = \mathbf{L}(\mathbf{K})$ be the maximal linear manifold consisting of the vectors te $(t \in \mathbb{R},\ \pm e \in \mathbf{Z})$. If $\|e| > 0$ for any nonzero $e \in \mathbf{Z}$, then the extended seminorm $\| \cdot |$ is called the *extended norm.*

By definition, we set

$$\mathring{B} := \{x \in X \mid \|x| < 1\}, \quad \mathring{B}(x_0, r) = x_0 + r\mathring{B} = \{x \in X \mid \|x - x_0| < r\} \quad (x_0 \in X,\ r > 0).$$

Similarly, given $x_0 \in X,\ r \geqslant 0$, we define

$$B(x_0, r) := x_0 + rB = \{x \in X \mid \|x - x_0| \leqslant r\},$$
$$S(x, r) := B(x_0, r) \setminus \mathring{B}(x_0, r),$$

For brevity, we write $S := S(0, 1)$ (the unit sphere). Note that if $\| \cdot |$ is a norm, the ball $B(0, 1)$ consists of intervals of the form $[0, x]$, where $x \in S$. If $\| \cdot |$ is a seminorm, then the ball $B(0, 1)$ contains in addition all the rays $\{te \mid t \geqslant 0\}$, where $e \in \mathbf{K}$ is an arbitrary nonzero point such that $\|e| = 0$.

Definition 1.1. A set $A \subset \mathbf{K}$ is called *bounded* if there exists a ball $B(0, R)$ containing the set A. A sequence in a cone-space is called *bounded* if it lies in a bounded set, and, therefore, in some ball $B(0, R)$.

The (natural) topology on X is generated by the subbase of the balls $\{\mathring{B}(x_0, r)\}_{x,r}$.

Let $\mathbf{L}^*$ be the set of all bounded (the definition is given below) linear functionals, and let $S^*(\mathbf{L})$ be the set of all linear norm-one functionals from space $\mathbf{L}^*$ (see [5], [6]).

Given $x \in X$ and a nonempty set $A \subset X$, $A \cap \mathbf{K}_x \neq \varnothing$, where $\mathbf{K}_x = (\mathbf{K} + x)$, we define

$$\varrho(x, A) = \inf\{\|y - x| \mid y \in A\}$$

(an analogue of the distance from a point x to a set A)

In the case $A \cap \mathbf{K}_x = \varnothing$, we set $\varrho(x, A) = +\infty$.

The extended norm (seminorm) $\|\cdot| : X \to \overline{\mathbb{R}}$ on a linear space X has the following properties:

1. $\|\alpha x| = \alpha\|x|$ for all $\alpha \geqslant 0$ and $x \in X$;
2. $\|x + y| \leqslant \|x| + \|y|$ for all $x, y \in X$;
3. $\|x| \geqslant 0$ for all $x \in X$, and $\|x| = 0 \Leftrightarrow x = 0$ (for a seminorm, the last condition is replaced by $\|0\,| = 0$).

From an asymmetric norm (seminorm) we define the symmetric generalized norm (seminormy) by

$$\|\cdot\|^{\mathrm{sym}} := \max\{\|\cdot|, \|-\cdot|\} : X \to \overline{\mathbb{R}}.$$

If the norm $\|\cdot|$ is not equivalent to the symmetrization norm $\|\cdot\|^{\mathrm{sym}}$, then we say that $(X, \|\cdot|)$ is an *essentially asymmetric space*.

A linear functional $x^* : X \to \overline{\mathbb{R}}$ is *bounded* if $x^*(x) \leqslant C\|x|$ (for some $C \geqslant 0$) for all $x \in X$. In the actual fact, this inequality can be considered only for $x \in \mathbf{K}$. The asymmetric norm (seminorm) $\|x^*|_*$ of a bounded linear functional x^* (as defined in [5], [6]) is the smallest constant C for which the above inequality is satisfied for all $x \in \mathbf{K}$.

Let X° be the set of all linear functionals on X, and let $\mathbf{K}^* \subset X^\circ$ be the cone of all bounded linear functionals on X. The cone-space $\mathbf{K}^*$ is equipped with the asymmetric norm (seminorm) $\|\cdot|_*$, which is extended to a generalized norm on X° via $\|x^*|_* := +\infty$ for all $X^\circ \setminus \mathbf{K}^*$. Indeed, this norm (seminorm) is the Minkowski functional of the set of all linear functionals on X upper bounded by 1. The cone-space $\mathbf{K}^*$ will be called the dual space to $\mathbf{K}$. Here it is worth pointing out that the dual space to an essentially asymmetric linear space is not a linear space (see [1]).

The second dual cone $\mathbf{K}^{**} := (\mathbf{K}^*)^*$, its dual, etc., are defined similarly.

As supports of extended norms $\|\cdot|$, $\|\cdot|_*$ and $\|\cdot|_{**}$ we may consider, respectively, in place of the linear spaces X, X° and $X^{\circ\circ} := (X^\circ)^\circ$, the linear hulls of the cones $\mathbf{K}$, $\mathbf{K}^*$ and $\mathbf{K}^{**}$ or any linear spaces which contain these cones. If necessary, the corresponding extended (generalized) norms are extended by $+\infty$. Any point $x \in \mathbf{K}$ can be looked upon as an element of the second dual cone if x is evaluated on any x^* from the cone-space $\mathbf{K}^*$ by

$$x(x^*) := x^*(x).$$

The duality between x and x^* (that is, the action of a point onto a different one) is conveniently written as

$$(x, x^*).$$

This correspondence between elements of a cone-space $\mathbf{K}$ and the cone-space $\mathbf{K}^{**}$ will be called the *natural embedding*

$$\mathfrak{J} : \mathbf{K} \to \mathbf{K}^{**}$$

of a cone-space $\mathbf{K}$ into the cone-space $\mathbf{K}^{**}$. By Remark 2.3 (see below), the norm $\|\mathfrak{J}(x)|_{**}$ coincides with the norm $\|x|$ for all $x \in \mathbf{K}$. In this way, the elements $\mathfrak{J}(x)$ and x are identified under the natural embedding $\mathfrak{J}$.

Remark 1.1. A linear functional $x^* : X \to \overline{\mathbb{R}}$ (and, in particular, that from Remark 2.3) can be considered as an affine functional on $\mathbf{K}$ (that is, convex and concave functional on $\mathbf{K}$) whose restriction to $\mathbf{L}$ is a linear functional. Its (semi) norm (as a affine functional), as defined by

$$\sup_{x \in B} x^*(x),$$

coincides with its asymmetric (semi)norm. This means that the cone-space $\mathbf{K}^*$ can be interpreted as a family of affine functionals on X with finite norm $\sup_{x \in B} x^*(x)$ whose restrictions to the maximal linear manifold $\mathbf{L}(\mathbf{K})$ are linear functionals. In particular, such affine functionals vanish at some nonzero point. Moreover, these bounded functionals on the cone $\mathbf{K}$ do not take the values $+\infty$, but can assume the values $-\infty$ on $\mathbf{K}$. In order to define the extended distance between such affine functionals, it suffices to extend them as linear functionals to the linear hull of the cone-space $\mathbf{K}$ (this extension is defined uniquely), and then evaluate the extended norm of its difference.

Definition 1.2. A sequence $\{x_n\} \subset X$ is a *Cauchy sequence* (an *inverse Cauchy sequence*) if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_m - x_n| < \varepsilon$ ($\|x_n - x_m| < \varepsilon$) for all $m \geqslant n \geqslant N$.

An asymmetric space $X = (X, \|\cdot|)$ is *right-* (*left-*) *complete* if, for any Cauchy sequence $\{x_n\} \subset X$, there exists a point $x \in X$ such that $\|x - x_n| \to 0$ ($\|x_n - x| \to 0$) as $n \to \infty$. A right complete space will be simply called a complete space.

An asymmetric space $X = (X, \|\cdot|)$ is called *inversely right-* (*left-*) *complete* if, for any inverse Cauchy sequence $\{x_n\} \subset X$, there exists a point $x \in X$ such that $\|x - x_n| \to 0$ ($\|x_n - x| \to 0$) as $n \to \infty$. An inversely right-complete space will be simply called an *inversely complete space*.

This definition of completeness applies to spaces with usual asymmetric norms or seminorms, and also to those with extended norms or seminorms. This definition also extends verbatim to spaces with asymmetric metric or semimetric.

Example 1.1. Let $M(Q)$ be the linear space of all functions $f : Q \to \overline{\mathbb{R}}$ equipped with the extended seminorm

$$\|f| = \|f|_{M(Q)} := \sup_{t\in Q} f_+(t), \qquad \text{where} \qquad f_+(t) := \max\{f(t), 0\}.$$

We claim that $M(Q)$ is a right-complete space. Indeed, let $\{f_k\}$ be a Cauchy sequence in $M(Q)$. There exists a subsequence $\{\varphi_k := f_{n_k}\}$ ($n_1 = 1$) for which the series $\sum_{k=1}^{\infty} \|\varphi_{k+1} - \varphi_k|$ is convergent. One of the limits of this sequence (and, therefore, of the original sequence) is the function

$$\varphi_1(t) + \sum_{k=1}^{\infty} (\varphi_{k+1} - \varphi_k)_+.$$

As another limit we can also consider the function $\limsup_n f_n(t)$ or its upper envelope $\sup_n f_n(t)$.

Remark 1.2. The cone-space $\mathbf{K}^*$ is right-complete.

Proof. Let $\{x_n^*\} \subset \mathbf{K}^*$ be a Cauchy sequence. There exists a subsequence $\{x_{n_k}^*\}$ such that the series $\sum_{k=0}^{\infty} \|x_{n_{k+1}}^* - x_{n_k}^*|$ is convergent. Setting $x_{n_0}^* = 0$, we get that the series $\sum_{k=0}^{\infty}(x_{n_{k+1}}^* - x_{n_k}^*)$ converges in the space X° pointwisely to a linear functional $s : X \to \overline{\mathbb{R}}$ whose norm is upper bounded by the sum of the series $\sum_{k=0}^{\infty} \|x_{n_{k+1}}^* - x_{n_k}^*|$. So, any bounded functional s which is the right limit of the partial sums $s_K := \sum_{k=0}^{K}(x_{n_{k+1}}^* - x_{n_k}^*) = x_{n_{K+1}}^*$ is also the right limit of the original Cauchy sequence. This proves the claim in the remark. ☐

The next result shows that any asymmetric space can be isometrically embedded into $M(Q)$ for some Q.

Theorem 1.1. *Let $\|\cdot|$ be an extended norm or seminorm on a linear space X, let $Q \subset X$ be a nonempty subset. Consider the mapping $\mathfrak{M} : Q \to M(Q)$ associating with each $x \in Q$ the function $\varphi_x(y) := \|x - y|$. Then $\|u - v| = \|\mathfrak{M}(u) - \mathfrak{M}(v)|_{M(Q)}$ for all $u, v \in Q$.*

Proof. For any $w \in Q$, we have $\|u - w| - \|v - w| \leqslant \|u - v|$. This inequality becomes an equality at the point $w = v$. Hence $\sup_Q(\mathfrak{M}(u) - \mathfrak{M}(v))_+ = \|u - v|$, proving the claim. ☐

In view of Theorem 1.1, we can always assume (up to a natural isometry) that Q is a subset of $M(Q)$. In this case, for each Cauchy sequence in Q, we may consider its upper envelop as a limit. In this way, by augmenting Q with all these limits, we get the right completion of the set Q. This method of completion can be called the method of right completion for an asymmetric metric.

Nevertheless, a different construction can be conveniently applied in the case of linear asymmetric spaces Y. Namely, consider $\mathbf{K}^* = Y^*$, and, further, $\mathbf{K}^{**}$. By Remark 1.2, the cone-space $\mathbf{K}^{**}$ is a right-complete cone-space. Hence the right closure of $\mathfrak{J}(Y)$ in $\mathbf{K}^{**}$ is a right-complete cone. This closure can be conveniently considered as the right completion (or, for simplicity, the *right closure*) of the space Y, which we identify with $\mathfrak{J}(Y)$. Of course, with the right completion of $\mathbf{K}$, some bounded affine functionals (on the cone space $\mathbf{K}^*$) are added, these functionals can take the values $-\infty$ on $\mathbf{K}^*$.

For various results on reflexivity in asymmetric spaces, see Cobzaş [5]. In the present paper, we will consider give the definition of reflexivity in the case where a cone-space $\mathbf{K}$ is contained in the right closure of $\mathbf{L}(\mathbf{K})$. With this approach, some classical results for usual normed spaces can be carried over to the case asymmetric spaces. In particular, from Theorems 3.1–5.2 (see below) it will follow that the 1-regular cone-space (see § 2 below) are more appropriate for this aim.

Let us now briefly discuss the problem of extension (or, more precisely, recovery) of an affine functional $f:\mathbf{K}\to\mathbb{R}$ that vanishes at the origin to the linear hull $\mathfrak{L}$ of the cone $\mathbf{K}$. On the cone $-\mathbf{K}$ this functional can be defined by

$$f(-x):=-f(x),\quad x\in\mathbf{K}.$$

Hence any finite linear combination $\sum_\alpha c_\alpha x_\alpha$ of vectors $\{x_\alpha\}$ can be written as

$$\sum_{\alpha:c_\alpha\geqslant 0}c_\alpha x_\alpha+\sum_{\alpha:c_\alpha<0}c_\alpha x_\alpha,$$

where the first sum is $x\in\mathbf{K}$, and the second sum is $y\in-\mathbf{K}$. This defines the linear functional f on $\mathfrak{L}$ by

$$f\Big(\sum_\alpha c_\alpha x_\alpha\Big):=f(x)+f(y).$$

Next, this functional can be extended also to X if on the algebraic complement $\mathfrak{L}$ to X it is defined to be, say, zero.

Note also that if a linear functional $f:X\to\overline{\mathbb{R}}$ is nonpositive on $\mathbf{K}$, then its seminorm $\|f|_*:=\sup_{x\in B}f(x)=\sup_{x\in B}f_+(x)$ is zero. So, if our aim is to generate a norm from a seminorm, we need at least to identify all such functionals with the zero functional.

We claim that if a functional f assumes positive values, then from the set $\mathbf{K}_+:=\{x\in\mathbf{K}\mid f(x)\geqslant 0\}$ one can uniquely recover the linear functional f on $\mathfrak{L}$ which extends f to $\mathbf{K}_+$. If $\mathbf{K}=\mathbf{K}_+$, then this extension is obtained by the above scheme. If the set $\mathbf{K}_-:=\{x\in\mathbf{K}\mid f(x)<0\}=\mathbf{K}\backslash\mathbf{K}_+$ is nonempty, then by convexity of $\mathbf{K}_-$ and $\mathbf{K}_+$, these sets can be separated by a linear functional $\varphi:\mathfrak{L}\to\mathbb{R}$ so that φ would be positive at each point from $\mathbf{K}_+^*:=\{x\in\mathbf{K}\mid f(x)>0\}$ and negative on $\mathbf{K}_-$ (see, for example, [27]). It is easily seen that the set $\mathbf{K}_0:=\{x\in\mathbf{K}\mid\varphi(x)=0\}=\operatorname{Ker}\varphi\cap\mathbf{K}$ is uniquely determined from f and is independent of a separating functional φ. Let P_+ be the convex hull of $\mathbf{K}_+\cup(-\mathbf{K}_-)\cup\mathbf{K}_0\cup(-\mathbf{K}_0)$. Then the functional f is uniquely extended to the cone P_+, and $f\geqslant 0$ on P_+. The functional f is uniquely extended to the cone P_-, which is the convex hull of the set $\mathbf{K}_-\cup(-\mathbf{K}_+)\cup\mathbf{K}_0\cup(-\mathbf{K}_0)$. Note that $P_-=-P_+$, and $f\leqslant 0$ on P_-. Next, f is uniquely extended to the linear hull of the cones P_+ and P_-, which coincides with $\mathfrak{L}$ (this fact is proved by the same arguments). Further, $f\geqslant 0$ ($f\leqslant 0$) on P_+ (P_-), that is, $\operatorname{Ker}f$ separates these cones, and, therefore, $\mathbf{K}_0:=\{x\in\mathbf{K}\mid f(x)=0\}$. To verify that f on $\mathbf{K}_+$ uniquely determines f on $\mathfrak{L}$, we assume that there exists a linear functional f_1 which coincides with f on $\mathbf{K}_+$ and which is different from f on the nonempty set $\mathbf{K}_-$. We have $\mathbf{K}_0=\operatorname{Ker}f\cap\mathbf{K}=\operatorname{Ker}f_1\cap\mathbf{K}$. For arbitrary points $x\in\mathbf{K}_+$ and $y\in\mathbf{K}_-$ such that $f(y)\neq f_1(y)$, there exists a point $z\in(x,y)\cap\mathbf{K}_0$. The functional $f-f_1$ vanishes at the points x and z, and, therefore, on the interval $[x,y]$, which contradicts the choice of the point y. So, any affine functionals which coincide on $\mathbf{K}_+$ are also equal on $\mathbf{K}$, hence, by unique extendability of these functionals from $\mathbf{K}$ to $\mathfrak{L}$, we find that $f\equiv f_1$ on $\mathfrak{L}$.

From the arguments of the preceding paragraph we get the following fact. If there exists a point $x\in\mathbf{K}$ at which $f(x)>0$, then, from the known nonnegative part $f_+(\,\cdot\,)$ of f on $\mathbf{K}$, the linear functional f is uniquely recovered on $\mathfrak{L}$. We also note that if $f\leqslant 0$ on $\mathbf{K}$, then $f_+\equiv 0$, and so from f_+ we recover the zero functional. This corresponds to $\|f|_*=0$, which should lead to no confusion if such nonpositive functionals (on $\mathbf{K}$) are identified with the zero functional.

2. DEFINITION OF REFLEXIVITY FOR SPACES WITH EXTENDED NORM OR SEMINORM

Definition 2.1. If a cone-space $\mathbf{K}$ coincides with $\mathbf{K}^{**}$ under the natural embedding $\mathfrak{J}$, which associates with each bounded functional x^{**} on X° a unique element $x\in\mathbf{K}$ such that

$$x^*(x)=x^{**}(x^*)\text{ for all }x^*\in\mathbf{K}^*,$$

then the cone-space $\mathbf{K}$ will be called *reflexive*; the class of all reflexive cone-spaces will be denoted by (Rf).

Remark 2.1. Note that if $\mathbf{K}\in$ (Rf), then the maximal linear manifolds $\mathbf{L}=\mathbf{L}(\mathbf{K})$ and $\mathbf{L}^{**}:=\mathbf{L}(\mathbf{K}^{**})$ are identified under the natural embedding $\mathbf{K}^{**}$ on $\mathbf{K}$.

Definition 2.2. Let Y be a linear space with (usual) asymmetric norm $\|\cdot|$. Given a linear space $X\supset Y$, we extend this norm (keeping the same notation) by defining it to be $+\infty$ on $X\setminus Y$. Let $\mathbf{K}^*=Y^*$ be the dual cone to $\mathbf{K}:=Y$ and $\mathbf{K}^{**}$ be the second dual cone. If the maximal linear manifold $\mathbf{L}(\mathbf{K}^{**})$ is identified with the space Y under the natural embedding, then Y will be called a *twice predual* space.

Remark 2.2. Let Y be twice predual relative to $X \supset Y$. Then, by identifying the spaces Y and $\mathfrak{J}(Y)$ and replacing $X \supset Y$ by $X^{\circ\circ} \supset Y$, we can extend the norm (seminorm) $\|\cdot|$ using the (seminorm) $\|\cdot|_{**}$, and consider the cone $\mathbf{K}^{**}$ as the cone-space $\mathbf{K}$. If here the right closure of the space $\mathfrak{J}(Y) \equiv Y$ contains $\mathbf{K}^{**}$, then the dual cone $\mathbf{K}^{*}$ consists of the extensions of the same linear functionals from Y^{*} (they are uniquely defined from their values on Y) to the cone-space $\mathbf{K} := \mathbf{K}^{**}$ with preservation of the original norm (seminorm); here $\mathbf{L}(\mathbf{K})$ is identified (or, as we will say, coincides) with Y. In this case, the cone-space $\mathbf{K}$ is naturally referred to as the reflexive extension of the twice predual space Y.

Of course, if Y is a complete symmetrizable space, than it is prereflexive if and only Y is reflexive.

Definition 2.3. Let Y be a twice predual linear asymmetric space for which there exists a reflexive cone-space $\mathbf{K}$ such that $Y = \mathbf{L}(\mathbf{K})$, $\mathfrak{J}(Y) = \mathbf{L}(\mathbf{K}^{**})$, and the norm on $\mathbf{K}$ is an extension of the norm on Y. Such a Y will be called a *prereflexive space.*

Remark 2.3. Let $\|\cdot|$ be an extended norm (seminorm) on an asymmetric space X. Then, for any $x_0 \in S$, there exists a linear functional x^* such that $x^*(x) \leqslant \|x|$ for all $x \in X$ and

$$x^*(x_0) = \|x_0| = 1.$$

Such functional x^* will be said to be a support functional to the unit sphere S at a point $x_0 \in S$. The asymmetric norm of any such linear functional 1 (see [5], [6]). We also note that, for each $x \in \mathbf{K}$ there exists a norm-one linear functional x^* such that $x^*(x) = \|x|$.

The next result, which is a direct analogue of the corresponding result for reflexive normed spaces, means that any bounded linear functional in a reflexive cone-space attains its norm.

Theorem 2.1. *Let $\mathbf{K} = \mathbf{K}(X)$ be a reflexive cone-space. Then, for any (norm-one) functional $x^* \in S^*$, there is a point $x_0 \in S$ such that*

$$x^*(x_0) = \|x_0| = 1.$$

Proof. By applying Remark 2.3 to space X° with extended norm $\|\cdot|_{*}$, we find a norm-one functional $x^{**} \in \mathbf{K}^{**}$ on X° such that

$$x^{**}(x^*) = 1.$$

By reflexivity of the cone-space $\mathbf{K}$, there exists $x_0 \in S$ such that $1 = x^{**}(x^*) = x^*(x_0)$. This proves the theorem. □

3. THE WEAK* AND REGULAR WEAK*-TOPOLOGY. COMPACTNESS OF BALLS IN REGULAR CONE-SPACES.

In what follows, we will consider the cone-spaces $\mathbf{K}$ such that the right closure of $\mathbf{L}(\mathbf{K})$ contains $\mathbf{K}$. The cone-space $\mathbf{K}^*$ and $\mathbf{K}^{**}$ will also be assumed to satisfy the same constraint.

Definition 3.1. Given $R \geqslant 1$, we say that a cone-space $\mathbf{K}$ is *R-regular* if the unit ball $B(0,1)$ of the cone-space $\mathbf{K}$ is contained in the right completion of $B(0,R) \cap \mathbf{L}$. This property means that $\mathbf{K}$ is a right-complete cone-space and there exists R such that, for any $x \in B \subset \mathbf{K}$, there exists a sequence $\{x_n\} \subset \mathbf{L}$ such that $\|x_n| \leqslant R$ $(n \in \mathbb{N})$ and $\|x - x_n| \to 0$ as $n \to \infty$.

A cone-space $\mathbf{K}$ will be said to be *regular* if $\mathbf{K}$ is a right-complete cone-space, and, for each point from $\mathbf{K}$, there exists a bounded sequence from $\mathbf{L}(\mathbf{K})$ converging to this point (in this case, we will also say the right closure of $\mathbf{L}(\mathbf{K})$ contains $\mathbf{K}$).

A cone-space $\mathbf{K}$ is called *superregular* if both $\mathbf{K}$ and its dual are 1-regular.

Remark 3.1. Let $a \in \mathbf{K}$ be an arbitrary point and $\{a_k\} \subset \mathbf{L}$ be an arbitrary sequence. Then any is a Cauchy sequence $\{a_k\}$ is bounded starting from some number. Moreover, if $\|a - a_k| \to 0$ as $k \to \infty$, then, for all $f \in \mathbf{L}(\mathbf{K}^*)$,

$$\liminf_{n\to\infty} f(a_n) \leqslant f(a) \leqslant \limsup_{n\to\infty} f(a_n).$$

of Remark 3.1. Indeed, starting from some number n_0 we have $\|a_n - a_{n_0}| \leqslant 1$. Hence $\|a_n| \leqslant \|a_{n_0}| + \|a_n - a_{n_0}| \leqslant \|a_{n_0}| + 1$. Since $f(a) - f(a_k) \leqslant \|f|_*\|a - a_k| \to 0$, $k \to \infty$, we have $f(a) \leqslant \limsup_{n\to\infty} f(a_n)$. For all $f \in \mathbf{L}(\mathbf{K}^*)$, replacing f by $-f$, we get $-f(a) \leqslant -\liminf_{n\to\infty} f(a_n)$, that is, $\liminf_{n\to\infty} f(a_n) \leqslant f(a)$. □

Using an analogue of the Baire theorem it can be shown that the function $R^*(a) := \limsup_{n\to\infty} f(a_n)$ is locally upper bounded, and the function $R_*(a) := \liminf_{n\to\infty} f(a_n)$ is lower bounded. Moreover, if $\|a_n| \leqslant C$ ($n \in \mathbb{N}$), then $f(a) \leqslant R^*(a) \leqslant C\|f|_*$.

Recall that the asymmetric norm of a linear functional $f : \mathbf{K} \to \mathbb{R}$ is defined by

$$\|f|_* := \sup_B f(x) = \sup_B f_+(x), \quad \text{where} \quad B \subset \mathbf{K} \quad \text{is the unit ball.}$$

Remark 3.2. From Remark 3.1 it follows that if $\mathbf{K}$ is a 1-regular cone-space, then

$$f(a) \leqslant \sup_{x\in B\cap\mathbf{L}} f(x) \quad \text{for any} \quad a \in B.$$

So, any linear functional f attains its norm on $B \cap \mathbf{L}$.

On the set of all bounded linear functionals (with finite norm) consider the pointwise right convergence of a net $\{f_\alpha\}$, that is, $\|f(x) - f_\alpha(x)|_\mathbb{R} \to 0$, where $\|t| := \|t\|_\mathbb{R} = t_+$ is an asymmetric seminorm on $\mathbb{R}$, $x \in \mathbf{K}$. This convergence is generated by the restriction to $\mathbf{K}^*$ of the subbase of neighborhoods of f

$$O^-_{x,\varepsilon}(f) := \{\varphi \mid f(x) - \varphi(x) < \varepsilon\}.$$

Note that

$$O^-_{x,\varepsilon}(f) := f + O^-_{x,\varepsilon}(0).$$

For an arbitrary finite set of points $\{x_1, \dots, x_n\}$ and arbitrary $\varepsilon > 0$, consider the base of neighborhoods:

$$O^-_{x_1,\dots,x_n,\varepsilon}(f) := \bigcap_{k=1}^n O^-_{x_k,\varepsilon}(f) = \{\varphi \mid f(x_j) - \varphi(x_j) < \varepsilon, \quad j = \overline{1,k}\},$$

which defines the topology of right pointwise convergence in $\mathbf{K}^*$. Note that if $\{\pm x_1, \dots, \pm x_n\} \subset \mathbf{L} := \mathbf{L}(\mathbf{K})$, then the neighborhood $O^-_{\pm x_1,\dots,\pm x_n,\varepsilon}(f)$ reduces to

$$O_{x_1,\dots,x_n,\varepsilon}(f) := \{\varphi \mid |f(x_j) - \varphi(x_j)| < \varepsilon, \quad j = \overline{1,k}\}.$$

So, the topology of pointwise convergence generates on $\mathbf{L}$ the usual pointwise convergence of nets $\{f_\alpha\}$, that is, $|f(x) - f_\alpha(x)| \to 0$ for all $x \in \mathbf{L}$.

Remark 3.3. Let $B^* = \{f \in \mathbf{K}^* \mid \|f|_* \leqslant 1\}$, and $\{f_\alpha\}_{\alpha\in\mathscr{A}}$ be a net in B^*. Then there exists a subnet $\{f_\beta\}_{\beta\in\mathscr{B}}$ converging in the usual sense on $\mathbf{L}$ such that $|f(x) - f_\alpha(x)| \to 0$, where f is a linear functional whose norm is majorized by 1 on $\mathbf{L}$. This result follows from w*-compactness of the unit ball in $\mathbf{L}^*$ (see [5], [6]).

Let $g(x) = \limsup_\beta f_\beta(x) := \lim_\beta \sup_{\gamma\geqslant\beta} f_\gamma(x)$ for all $x \in \mathbf{K}$. Then g is convex and finite on $\mathbf{K}$. From Theorem 4.25 in [27], it follows that the set $\mathscr{P}(g)$ of all linear functionals $f \leqslant g$ is nonempty. Moreover, $-f(x) = f(-x) \leqslant g(-x) = -\liminf_\beta f_\beta(x)$ for all $x \in \mathbf{L}$, that is,

$$\liminf_\beta f_\beta(x) \leqslant f(x) \leqslant \limsup_\beta f_\beta(x) \text{ on } \mathbf{L}.$$

Hence $\lim_\beta f_\beta(x) = f(x)$ on $\mathbf{L}$ by the choice of the net $\{f_\beta\}_{\beta\in\mathscr{B}}$. It is easily seen that $\|f|_* \leqslant 1$ for all $f \in \mathscr{P}(g)$. Moreover, for each $x \in \mathbf{K}$, $f(x)$ is a $\|\cdot|_\mathbb{R}$-limit point of the net $\{f_\beta(x)\}_\beta$. We claim that $f \in B^*$ is a limit point in the topology of right pointwise convergence. Indeed, for any $\varepsilon > 0$, we can choose a subnet $\{f_\gamma\}_{\gamma\in\Gamma}$ such that, for any fixed $x_k \in \mathbf{K}$ and all $\gamma \in \Gamma$, we have $f_\gamma(x_k) \geqslant g(x_k) - \varepsilon$, $k = \overline{1,n}$. So, $f(x_k) \leqslant f_\gamma(x_k) + \varepsilon$ for all $k = \overline{1,n}$ and $\gamma \in \Gamma$. Hence the neighborhood $O^-_{x_1,\dots,x_n,\varepsilon}(f)$ contains points of the original net. As a result, f is a limit point in the topology of right pointwise convergence. We also have $f(x) \leqslant g(x) = \limsup_\beta f_\beta(x) \leqslant \limsup_\beta \|x| = \|x|$ for all $x \in \mathbf{K}$, and, therefore, $\|f|_* \leqslant 1$.

Note that the restrictions of the neighborhoods $O_{x_1,\dots,x_n,\varepsilon}(f)$ ($\{x_1, \dots, x_n\} \subset \mathbf{L}$) on $\mathbf{K}^*$ generates a formally weaker topology then the topology of right pointwise convergence, and hence the ball B^* is compact also in this topology, which will be called the weak* (or w*-) topology in $\mathbf{K}^*$. So, *the w*-topology coincides with the topology of pointwise convergence on* $\mathbf{L}$.

The following result is a direct consequence of Remark 3.3.

Corollary 3.1. *The unit ball B^* is compact in the topology of weak* right pointwise convergence, and also in the topology of weak* convergence in the cone-space $\mathbf{K}^*$.*

Corollary 3.1 implies that the balls $B^*(f,r) := rB^* + f$ $(f \in \mathbf{K}^*)$ are w*-compact. However, in view of Remark 3.2, these balls are w*-closed only if the cone-space $\mathbf{K}$ is 1-regular. This follows from the fact that if $f \in \mathbf{K}^* \setminus B^*$, then there exists $x \in \mathbf{L}(\mathbf{K}) \cap B$ such that $f(x) > 1$, whereas $\varphi(x) \leqslant 1$ for all $\varphi \in B^*$. Note that these balls are of course closed in the topology of weak* right pointwise convergence.

Another important property is also worth pointing out. For each $a \in \mathbf{L}$ and any $\alpha \in \mathbb{R}$,

$$\{y^* \in \mathbf{K}^* \mid (a, y^*) \geqslant \alpha\} \text{ and } \{y^* \in \mathbf{K}^* \mid (a, y^*) \leqslant \alpha\} \text{ is closed relative to } \mathbf{K}^*,$$

and

$$\{y^* \in \mathbf{K}^* \mid (a, y^*) > \alpha\} \text{ and } \{y^* \in \mathbf{K}^* \mid (a, y^*) < \alpha\} \text{ is open relative to } \mathbf{K}^*.$$

Similarly, we define the w*-topology on $\mathbf{K}^{**}$ generated by the subbase of the restrictions of the neighborhoods

$$O_{x_1^*,\dots,x_n^*,\varepsilon}(x^{**}) := \{y^{**} \in X^{\circ\circ} \mid |(x_j^*, y^{**} - x^{**})| < \varepsilon,\ j = \overline{1,n}\} \ \text{ to the set } \mathbf{K}^{**},$$

where $\{x_j^*\}_{j=1}^n \subset \mathbf{L}(\mathbf{K}^*)$, $x^{**} \in \mathbf{K}^{**}$. In this case, the unit ball B^{**} in $\mathbf{K}^{**}$ is w*- compact.

As a simple corollary, we get

Theorem 3.1. *A cone-space is reflexive if and only if its unit ball is weakly compact in the topology of right pointwise convergence. Moreover, if the ball of a cone-space $\mathbf{K}$ is weakly compact and $\mathbf{K}^*$ is 1-regular, then $\mathbf{K}$ is reflexive.*

Proof. Let $\mathbf{K}$ be reflexive. Being dual to $\mathbf{K}^*$, the unit ball of the cone-space $\mathbf{K}^{**}$ is compact in the topology of right pointwise convergence on $\mathbf{K}^*$, which implies weak compactness of the ball of the cone-space $\mathbf{K}$ in the topology of right pointwise convergence (we assume that this ball coincides with the unit ball $\mathbf{K}^{**}$ under the natural embedding).

Let the unit ball B of the cone-space $\mathbf{K}$ be weakly compact in the topology of right pointwise convergence. Under the natural embedding, this ball coincides with its w*- closure in $\mathbf{K}^{**}$ in the topology of right pointwise convergence, that is, with the ball B^{**}. Hence $\mathbf{K}$ is reflexive.

If the ball B of a cone-space $\mathbf{K}$ is weakly compact (in the w*-topology on $\mathbf{K}^*$) and $\mathbf{K}^*$ is 1-regular, then the closure in the w*-topology on $\mathbf{K}^*$ of the ball B (under the natural embedding) is contained in the unit ball B^{**} of the cone-space $\mathbf{K}^{**}$. It follows that the closure in the w*-topology on $\mathbf{K}^*$ of the ball B (under the natural embedding) is the unit ball B^{**} (see Theorem 4.2 below). This proves the theorem. ◻

Remark 3.4. In a reflexive cone-space $\mathbf{K}$ with 1-regular dual $\mathbf{K}^*$, the unit ball $B \subset \mathbf{K}$ is compact in the topology of weak pointwise convergence (that is, in the topology of w*-pointwise convergence of functionals from $\mathbf{K}^{**}$ on $\mathbf{K}^*$) if and only if it is compact in the topology of weak right pointwise convergence (that is, in the topology of weak right pointwise convergence of functionals from $\mathbf{K}^{**}$ on $\mathbf{K}^*$).

Remark 3.5. Let the right closure of $\mathbf{L} = \mathbf{L}(\mathbf{K})$ contain $\mathbf{K}$, let z^* be a w*-limit point for a sequence $\{z_n^*\}$. Then z^* is also a w*-limit point for any sequence $\{x_n^*\}$ such that $\|z_n^* - x_n^*|_* \to 0$ as $n \to \infty$.

Proof. Consider arbitrary $\{a_i\}_{i=1}^k \subset \mathbf{L}$ and $\varepsilon > 0$. Let $M > 0$ be such that $\|a_i| \leqslant M$, $i = \overline{1,k}$. The neighborhood $O_{\varepsilon/2,a_1,\dots,a_k}(z^*)$ contains infinitely many terms of the sequence $\{z_n^*\}$. Since $\|z_n^* - x_n^*|_* < \frac{\varepsilon}{2M}$ starting from some number N, and since

$$(a_i, z^* - x_n^*) = (a_i, z^* - z_n^*) + (a_i, z_n^* - x_n^*) < \varepsilon/2 + \|a_i|\|z_n^* - x_n^*|_* < \varepsilon \quad (i = \overline{1,k}),$$

it follows that the neighborhood $O^-_{\varepsilon,a_1,\dots,a_k}(z^*)$ contains infinitely many terms of the sequence $\{x_n^*\}$. Therefore, z^* is a w*-limit point for $\{x_n^*\}$. ◻

For regular cone-spaces $\mathbf{K}$, the *regular w*-topology* is introduced as follows. Consider the system of neighborhoods

$${}^rO^-_{x_1,\dots,x_n}(0) := \{\varphi \in \mathbf{K}^* \mid -\varphi \in \mathbf{K}^*,\ -\varphi(x_k) < \varepsilon,\ k = \overline{1,n}\};$$
$${}^rO^-_{x_1,\dots,x_n}(f) := \{\varphi \in \mathbf{K}^* \mid f - \varphi \in \mathbf{K}^*, (f-\varphi)(x_k) < \varepsilon,\ k = \overline{1,n}\}, \quad f \in \mathbf{K}^*.$$

Theorem 3.2. *Let $\mathbf{K}^*$ be a 1-regular cone-space. Then the ball B^* is a regular w*-compact set.*

Proof. The set $BL^* := B^* \cap \mathbf{L}(\mathbf{K}^*)$ is dense in the (strong) topology of right convergence in the ball B^*, and hence is dense in the regular w*-topology. We claim that any net $\{\varphi_\alpha\}_{\alpha\in\mathscr{A}} \subset BL^*$ has a subnet converging to some point $f \in B^*$. By the ordinary w*-compactness of the ball B^*, there exists a subnet $\{\varphi_\beta\}_{\beta\in\mathscr{B}}$ converging to $f \in B^*$ in the w*-topology. Since $\varphi_\beta \in \mathbf{L}(\mathbf{K}^*)$, we find that $f, \varphi_\beta, f-\varphi_\beta \in \mathbf{K}^*$. Therefore, the subnet $\{\varphi_\beta\}_{\beta\in\mathscr{B}}$ converges to some f in the regular w*-topology.

Consider an arbitrary open (in the regular w*-topology) covering $\{G_\lambda\}_{\lambda\in\Lambda}$ of B^*. In each element of the covering G_λ, for each $g \in G_\lambda$, we embed a closed neighborhood

$$V_{\alpha_g}(g) = {}^rV^-_{x_1,\dots,x_n,\varepsilon_g}(g) := \{\varphi \in \mathbf{K}^* \mid g-\varphi \in \mathbf{K}^*, (g-\varphi)(x_k) \leqslant \varepsilon_g,\ k=\overline{1,n}\}$$

with an appropriate family $\alpha_g = (x_1,\dots,x_n,\varepsilon_g)$ such that $x_1,\dots,x_n \in \mathbf{L}(\mathbf{K})$ and $\varepsilon_g > 0$. We claim that there exists a finite system of neighborhoods

$$O_{\alpha_{g_j}}(g_j) = {}^rO^-_{x_1,\dots,x_n,\varepsilon_{g_j}}(g_j) := \{\varphi \in \mathbf{K}^* \mid g_j-\varphi \in \mathbf{K}^*, (g_j-\varphi)(x_k) < \varepsilon_{g_j},\ k=\overline{1,n}\},$$

$j=\overline{1,N}$, which cover the set BL^*. Assume on the contrary that there is no such finite system. Then with each set $\beta := \underline{(\alpha_{g_1},\dots,\alpha_{g_N})}$ we associate a point $\psi_\beta \in BL^*$ not lying in the union of the neighborhoods $O_{\alpha_{g_j}}(g_j)$, $j=\overline{1,N}$. On the index set $\{\beta\}$ we introduce the partial order

$$\beta_1 \leqslant \beta_2 \Leftrightarrow \beta_1 \subset \beta_2.$$

By the above, there exists a subnet $\{\psi_{\beta_\gamma}\}$ of $\{\psi_\beta\}$ converging to some point $f \in B^*$ in the regular w*-topology. Hence $O_{\alpha_f}(f)$ contains the terms of the subnet $\{\psi_{\beta_\gamma}\}$ for all γ exceeding some γ_0, which contradicts the construction of this subnet. Thus, we have shown that there exists a finite set of neighborhoods

$$O_{\alpha_{g_j}}(g_j) = {}^rO^-_{x_1,\dots,x_n,\varepsilon_{g_j}}(g_j) := \{\varphi \in \mathbf{K}^* \mid g_j-\varphi \in \mathbf{K}^*, (g_j-\varphi)(x_k) < \varepsilon_{g_j},\ k=\overline{1,n}\},$$

$j=\overline{1,N}$, which covers the set BL^*.

If the corresponding family of closed neighborhoods $V_{\alpha_{g_j}}(g_j)$, $j=\overline{1,N}$, covers the entire set B^*, then the family consisting of the points of the original covering G_{λ_j} such that $V_{\alpha_{g_j}}(g_j) \subset G_{\lambda_j}$, is a finite subcovering of the original covering. Hence, B^* is a compact set in the regular w*-topology. Suppose that there exists a point $g_0 \in B^*$ not lying in the union of the sets $V_{\alpha_{g_j}}(g_j)$, $j=\overline{1,N}$. If $g_j - g_0 \notin K^*$, then we choose $y_j \in \mathbf{L}(\mathbf{K})$ so as to have $(g_j-g_0)(y_j) > 1$; otherwise we choose $y_j \in \mathbf{L}(\mathbf{K})$ arbitrarily. Hence there exists a neighborhood ${}^rO^-_{x_1,\dots,x_n,y_1,\dots,y_n,\varepsilon_0}(g_0)$ disjoint from any neighborhood from the family $O_{\alpha_{g_j}}(g_j) \subset V_{\alpha_{g_j}}(g_j)$, $j=\overline{1,N}$, and which contains the set BL^*, however, this cannot be the case (see the first paragraph of the proof). This proves the theorem. ☐

Recall (see p. 405) that if $\mathbf{K}$ is also a 1-regular cone-space, then B^* is closed in the w*-topology, and, therefore, closed and in the regular w*-topology (which is stronger).

Remark 3.6. A similar argument shows that the right closure N_R of the set $BL^*(R) := B^*(0,R)\cap\mathbf{L}(\mathbf{K}^*)$ is a regular w*- compact set in $\mathbf{K}^*$. It follows that if the cone-space $\mathbf{K}^*$ is R-regular and if the ball B^* is closed relative to the regular w*-topology, then it is a regular w*- compact set in $\mathbf{K}^*$. This condition is satisfied, in particular, if the cone-space $\mathbf{K}$ is 1-regular. Indeed, the required compactness follows from the inclusion $B^* \subset N_R$ and since the ball B^* is closed in the w*-topology, and, therefore, in the regular w*-topology, which is a stronger topology. That the ball B^* is closed in the w*-topology was pointed out on p. 405.

Note that the regular w*-topology is in fact associated with a concrete cone. Hence with each cone $\mathbf{K}^*_f := f + \mathbf{K}$, $f \in \mathbf{K}^*$, we will have to associate the cone-specific regular w*-topology, which will be called the regular weak* topology with respect to the cone $\mathbf{K}^*_f$. This topology is generated by the neighborhoods

$${}^rO^-_{x_1,\dots,x_n}(f) := \{\varphi \in \mathbf{K}^* \mid f-\varphi \in \mathbf{K}^*,\ f-\varphi(x_k) < \varepsilon,\ k=\overline{1,n}\};$$

$${}^rO^-_{x_1,\dots,x_n}(f+g) := \{\varphi \in \mathbf{K}^* \mid (f+g)-\varphi \in \mathbf{K}^*, ((f+g)-\varphi)(x_k) < \varepsilon,\ k=\overline{1,n}\},\ g \in \mathbf{K}^*.$$

In particular, the above topology is the regular weak* topology with respect to the cone $\mathbf{K}^*_0 = \mathbf{K}^*$. Indeed, for $f, g \in \mathbf{K}^*$, any open set in the regular weak* topology with respect to the cone $\mathbf{K}^*_f$ is a translation by $\varphi := f - g \in X^\circ$ of some open set in the regular weak* topology with respect to the cone $\mathbf{K}^*_g$, and vice versa. So, by analogy, it will be convenient to carry over this topology also to the cones $\mathbf{K}_f$ for $f \in X^\circ$ (that is, f not necessarily lies in the cone $\mathbf{K}^*$). However, it should be pointed out that if $f \in \mathbf{L}(\mathbf{K}^*)$, then $M \subset \mathbf{K}^*$ is compact relative to the regular w*-topology with respect to the cone $\mathbf{K}^*$ if and only if its compact relative to the cone $\mathbf{K}^*_f$. A similar property also holds for the cones $\mathbf{K}^*_f$ and $\mathbf{K}^*_g$, $f,g\in X^\circ$ if $f-g\in\mathbf{L}(\mathbf{K}^*)$. In the more general setting, M is compact in the regular w*-topology with respect to the cone $\mathbf{K}^*_f$ if and only if $M_g:=M+g$ are compact in the regular w*-topology with respect to the one $\mathbf{K}^*_{f+g}$, $f,g\in X^\circ$.

Corollary 3.2. *Let a cone-space* $\mathbf{K}$ *be* 1*-regular. and the cone-space* $\mathbf{K}^*$ *be regular. Then there exists a nondegenerate ball* $B^*(g,r)$ *compact in the regular w***-topology.*

Proof. Assume that the right closure $N_n=N_{R_n}$ of the set $BL^*(R):=B^*(0,R)\cap\mathbf{L}(\mathbf{K}^*)$ is a regular w*-compact set in $\mathbf{K}^*$ $(n\in\mathbb{N})$, where $R_n\uparrow+\infty$ as $n\to\infty$. As in Remark 3.6, it can be shown that the sets N_n, $n\in\mathbb{N}$, are compact in the regular w*-topology. Since the cone-space $\mathbf{K}^*$ is regular, it can be covered by a nested system of compact sets in the regular w*-topology $\{N_n\}_{n\in\mathbb{N}}$. Since $\mathbf{K}^*$ is right complete, an application of the Baire theorem (see [5, § 1.2]) shows that there exists a nondegenerate ball $B^*(g,r)$ lying in some set N_{n_0}. Since the cone-space $\mathbf{K}$ is 1-regular, the ball $B^*(g,r)$ is a closed set in the regular w*-topology, and, therefore, is a compact set in the regular w*-topology. This proves Corollary 3.2. ☐

Definition 3.2. A cone-space $\mathbf{K}$ is called *left-regular* if, for any point $x\in\mathbf{K}$ and arbitrary $\varepsilon>0$, there exists $\varphi\in\mathbf{L}(\mathbf{K})$ such that $\|\varphi-x|<\varepsilon$. A cone-space $\mathbf{K}$ is *strongly regular* if it is simultaneously regular and left-regular.

Corollary 3.3. *Let a cone-space* $\mathbf{K}$ *be* 1*-regular and the cone-space* $\mathbf{K}^*$ *be regular. Then the ball* B^* *is compact in the regular w***-topology.*

Proof. By Corollary 3.2, there exists a nondegenerate ball $B^*(g,r)$ which is compact in the regular w*-topology. There exists a point $\varphi\in\mathbf{L}(\mathbf{K})$ such that $\|\varphi-g|<r/2$. Hence the ball $B^*(\varphi,r/2)$ is contained in the ball $B^*(g,r)$, and hence, since $B^*(\varphi,r/2)$ is closed, is compact in the regular w*-topology. Hence the ball $B^*(0,r/2)=B^*(\varphi,r/2)+(-\varphi)$ is also compact in this topology, because $-\varphi\in\mathbf{L}(\mathbf{K})$. It follows that the ball $B^*=\frac{2}{r}B^*(0,r/2)$ is compact in the regular w*-topology. This proves Corollary 3.3. ☐

The cone-spaces $\mathbf{K}$ feature one more of weak* convergence property. This is the property of pointwise convergence on the cone-space $\mathbf{K}$ for all functionals $\mathbf{L}(\mathbf{K}^*)$. Indeed, consider an arbitrary point $a\in\mathbf{K}$ and an arbitrary sequence $\{a_k\}\subset\mathbf{L}$ such that $\|a-a_k|\to0$ as $k\to\infty$. Assume that, for arbitrary $\varepsilon>0$, there exists k_0 such that $\|a-a_{k_0}|<\varepsilon$. Then, for any functional $\varphi\in\mathbf{L}(\mathbf{K}^*)$ such that $|\varphi(a_{k_0})|<\varepsilon$, we have

$$\varphi(a)=\varphi(a-a_{k_0})+\varphi(a_{k_0})<\|\varphi|_*\|a-a_{k_0}|+\varepsilon\leqslant\varepsilon(1+\|\varphi|_*)$$

and, similarly,

$$-\varphi(a)=-\varphi(a-a_{k_0})-\varphi(a_{k_0})<\|-\varphi|_*\|a-a_{k_0}|+\varepsilon\leqslant\varepsilon(1+\|-\varphi|_*).$$

Hence

$$|\varphi(a)|<\varepsilon(1+\|\varphi|_*+\|-\varphi|_*).$$

For an arbitrary functionals $f\in\mathbf{K}^*$, since $f(a)-f(a_k)\leqslant\|f|_*\|a-a_k|\to0$, we have $\|f(a)-f(a_k)|_{\mathbb{R}}\to0$ $(k\to\infty)$, that is, the value at a point a is defined as the limit of the sequence $\{f(a_k)\}$ in the space $(\mathbb{R},\|\cdot|_{\mathbb{R}})$.

4. JAMES REFLEXIVITY THEOREM

In what follows, for completeness of presentation, we will carry over some standard facts from the theory of normed linear space to the asymmetric setting.

Theorem 4.1. *Let* $A\subset\mathbf{K}^*$ *be a nonempty convex w***-closed set. Then, for any functional* $f\in\mathbf{K}^*\setminus A$*, there exists a point* $y\in\mathbf{L}=\mathbf{L}(\mathbf{K})$ *separating* f *and* A*, that is,*

$$y(f)=f(y)>\sup_{g\in A}g(y)=\sup_{g\in A}y(g).$$

Proof. We will assume that $\mathbf{K}$ is naturally embedded into $\mathbf{K}^{**}$. Since A is w*- closed, there exists a neighborhood $U=\{y^*\mid|y^*(y_i)|<\varepsilon,\,i=\overline{1,N}\}$, of the origin, $\{y_i\}_{i=1}^N\subset\mathbf{L}$, such that $(f+U)\cap A=\varnothing$. Since U is symmetric, it follows that f does not lie in the w*-open convex set $A+U=A-U$, and, therefore, by the Hahn–Banach theorem, there exists a linear functional $F\in\mathbf{K}^{**}$ such that $F(f)>\sup_{v\in A+U}F(g)\geqslant\sup_{u\in U}F(u)+F(g_0)$ for any $g_0\in A$. We claim that the functional F is an action of some $y\in\mathbf{L}$ which is a linear combination of y_i, $i=\overline{1,N}$. Indeed, $P:=\sup_U F\leqslant F(f)-F(g_0)$, where g_0 is some fixed point from A, is finite, and hence, for any $\pm y^* \in \bigcap_{i=1}^N \operatorname{Ker} y_i =: T_0$ and arbitrary $t > 0$, we have $F(\pm t y^*) \leqslant P$. Hence $\pm F(y^*) \leqslant P/t \to 0$ $(t \to +\infty)$, that is, $F(y^*) = 0$. Let T be the linear hull T_0 in the space X°, and $\{z_i^*\}_{i=1}^N \subset \mathbf{K}^*$ be a linearly independent system of functionals such that $T \cap Z = \{0\}$, where Z is the linear hull of this system. We can extend F to the linear manifold $T \oplus Z$ as a linear functional. Note that F is an annihilator of T. The linear space Z contains a set of elements $\{\overline{z}_i^*\}_{i=1}^N$ such that $y_j(\overline{z}_i^*) = \delta_{ji}$ (δ_{ji} is the Kronecker delta). Hence $F - \sum_{i=1}^N F(\overline{z}_i^*) y_i$ vanishes on $T \oplus Z$, and, therefore, $F = \sum_{i=1}^N F(\overline{z}_i^*) y_i$ on $\mathbf{K}^* \subset T \oplus Z$. Hence F is a linear combination of y_i $(i = \overline{1, N})$, that is, a point in the space $\mathbf{K}$. This proves the theorem. □

Theorem 4.2 (Goldstein). *If K^* is a 1-regular cone-space, then the w*-closure of the unit ball B of the cone-space $\mathbf{K}$ as a subset of $\mathbf{K}^{**}$ under the natural embedding (relative to $\mathbf{K}^*$) is the unit ball B^{**} in the cone-space $\mathbf{K}^{**}$.*

Proof. The w*-closure A of the set B (as a subset of B^{**} under the natural embedding) is contained in B^{**}. Let $Y = \mathbf{K}^*$. If $A \neq B^{**}$, then for some $f \in B^{**} \setminus A \subset Y^* \setminus A$ in view of Theorem 4.1, there would exist $y \in Y = \mathbf{K}^*$ (more precisely, $y \in \mathbf{L}(\mathbf{K}^*)$) such that $f(y) > \sup_{g \in A} g(y)$.

Note that $\|y|_* \neq 0$, since otherwise we would have $f(y) \leqslant \|f|_{**} \|y|_* = 0$, which, however, is impossible, because $\sup_{g \in A} g(y) \geqslant 0$. Since $B \subset A$ and since $\|y|_* = \sup_{g \in B} y(g) > 0$, we find that the right-hand side of the inequality $f(y) > \sup_{g \in A} g(y)$ is positive, and so, if necessary, we could multiply y by a positive number so that the inequality would remain valid and its right-hand side would be equal to 1. So, we may assume without loss of generality that $f(y) > \sup_{g \in A} g(y) = 1$. Since the supremum here is 1 and $B \subset A$, we get $\|y| \leqslant 1$, but $f(y) > 1$ for $\|f| \leqslant 1$, which cannot be the case. This proves the theorem. □

Remark 4.1. Note that a version of the Goldstein theorem on the density of the ball B of a linear asymmetric space in the ball B^{**} is Theorem 2.5.26 of [5]. As in Theorem 4.2, it can be shown that the ball $B \subset \mathbf{K}$ is dense in B^{**} relative to the weak topology of right pointwise convergence generated by the neighborhoods $O^-_{x_1^*, \dots, x_n^*, \varepsilon}(f)$ $(f \in \mathbf{K}^{**}, x_k^* \in \mathbf{K}^*, k = \overline{1, n})$. This follows from the fact that each functional $f \in \mathbf{K}^{**}$ separating this neighborhood from some set can be written as $\sum_{k=1}^n \alpha_k x_k^*$ (Kuhn–Tucker's theorem; see [26]), where $\alpha_k \geqslant 0$ $(k = \overline{1, n})$. Therefore, $f \in \mathbf{K}^*$. Next, we proceed as in the proof of Theorem 4.2 starting from the second paragraph.

Since the ball B^{**} is closed in the weak topology of right pointwise convergence, we have the following result.

The closure of the unit ball B of a cone-space $\mathbf{K}$ in $\mathbf{K}^{**}$ (under the natural embedding) relative to the topology of weak right pointwise convergence is the unit ball B^{**} of the cone-space $\mathbf{K}^{**}$.

Given a nonempty set E, consider the asymmetric space

$$m(E) = \{x = (x_\alpha)_\alpha \mid \|x| := \sup_\alpha (x_\alpha)_+ < +\infty\},$$

here $x : E \to \mathbb{R}$ and $x_\alpha = x(\alpha)$ $(\alpha \in E)$, $(x_\alpha)_+ := \max\{x_\alpha, 0\}$. For a bounded sequence $\{x_n\} \subset m(E)$ (a sequence is bounded if there is a number uniformly majorizing all coordinates of this sequence), consider the point $\limsup_{n\to\infty} x_n = \big(\limsup_{n\to\infty} x_n(\alpha)\big)_\alpha \in m(E)$. We say that a point $x = (x_\alpha)_\alpha \in m(E)$ attains its supremum on E if there exists an index β such that $x_\beta = \sup_E x := \sup_{\alpha \in E} x_\alpha$.

Below, by $\operatorname{conv} M$ and $\overline{\operatorname{conv}}\, M$ we denote, respectively,, the convex hull of a set M and its right closure. For an arbitrary sequence $\{x_n\}$, the set of all elements of the form $\sum_n \lambda_n x_n$ will be denoted by $\operatorname{conv}\{x_n\}$, where $\{\lambda_n\}$ is an arbitrary sequence of nonnegative numbers with $\sum_{n=1}^\infty \lambda_n = 1$.

Lemma 4.1 (Simon's inequality). *Let E be a nonempty set and let $\{x_n\}$ be a bounded sequence in $m(E)$. Assume that, for any sequence of nonnegative numbers $\{\lambda_n\}$ with $\sum_{n=1}^\infty \lambda_n = 1$, the vector $\sum_n \lambda_n x_n$ attains its supremum on E. Then*

$$\sup_E \Big(\limsup_{n\to\infty} x_n\Big) \geqslant \inf_{x \in \operatorname{conv}\{x_n\}} \sup_E x.$$

Proof. For each $k \in \mathbb{N}$, let C_k be the set of all sums of the form $\sum_{m=k}^\infty \lambda_m x_m$, where $\{\lambda_m\} \subset \mathbb{R}_+$ is such that $\sum_{m=k}^\infty \lambda_m = 1$. Note that, for each α,

$$\limsup_{k\to\infty} \sum_{m=k}^\infty \lambda_m x_m(\alpha) \leqslant \sum_{m=k}^\infty \lambda_m \limsup_{k\to\infty} x_k(\alpha) = \limsup_{k\to\infty} x_k(\alpha).$$

We need to show that $\sup_E \big(\limsup_{n\to\infty} x_n\big) \geqslant \inf_{x\in C_1} \sup_E x$. Consider an arbitrary $\varepsilon > 0$ and define $z_{k+1} \in C_{k+1}$ $(k \in \mathbb{Z}_+)$ inductively from the condition

$$\sup_E(2^k v_k + z_{k+1}) \leqslant \inf_{z\in C_{k+1}} \sup_E(2^k v_k + z) + \frac{\varepsilon}{2^{k+1}},$$

where $v_0 = 0$ and $v_k = \sum_{m=1}^k \frac{z_m}{2^m}$. Next, since

$$z_{k+1} = 2^{k+1}(v_{k+1} - v_k) = 2^{k+1}v_{k+1} - 2^k v_k - 2^k v_k,$$

we have $2^k v_k + z_{k+1} = 2^{k+1}v_{k+1} - 2^k v_k$. We also set $v = \sum_{m=1}^\infty \frac{z_m}{2^m}$. Since $z_m = \sum_{n=k+1}^\infty \lambda_{mn} x_n$, where $\sum_{n=k+1}^\infty \lambda_{mn} = 1$, we have $\sum_{m,n=k+1}^\infty 2^{k-m}\lambda_{mn} = 1$, and, therefore,

$$2^k v - 2^k v_k = 2^k \sum_{m=k+1}^\infty \frac{z_m}{2^m} = \sum_{m=k+1}^\infty 2^{k-m} \sum_{n=k+1}^\infty \lambda_{mn} x_n = \sum_{n=k+1}^\infty x_n \sum_{m=k+1}^\infty 2^{k-m}\lambda_{mn} \in C_{k+1}.$$

As a result,

$$\begin{aligned}\sup_E(2^{k+1}v_{k+1} - 2^k v_k) = \sup_E(2^k v_k + z_{k+1}) \leqslant \sup_E(2^k v_k + (2^k v - 2^k v_k)) + \frac{\varepsilon}{2^{k+1}} \\ = \sup_E 2^k v + \frac{\varepsilon}{2^{k+1}} = 2^k \sup_E v + \frac{\varepsilon}{2^{k+1}}.\end{aligned}$$

Since $v \in C_1$, by the hypotheses of the lemma, there exists $\alpha \in E$ such that $v(\alpha) = \sup_E v$. The equalities $\sum_{m=0}^{k-1} 2^m = 2^k - 1$ for all k imply

$$2^k v_k(\alpha) = \sum_{m=0}^{k-1}(2^{m+1}v_{m+1} - 2^m v_m)(\alpha) \leqslant (2^k - 1)\sup_E v + \varepsilon = 2^k v(\alpha) + \varepsilon - \sup_E v.$$

Hence $\sup_E v \leqslant 2^k v(\alpha) - 2^k v_k(\alpha) + \varepsilon$, and, therefore,

$$\inf_{x\in C_1} \sup_E x \leqslant \sup_E v \leqslant \limsup_{k\to\infty}(2^k v(\alpha) - 2^k v_k(\alpha)) + \varepsilon \leqslant \limsup_{k\to\infty}(x_k(\alpha)) + \varepsilon$$

(because $2^k v - 2^k v_k \in C_{k+1}$). Now the required inequality follows since ε is arbitrary. This proves the lemma. ☐

Definition 4.1. Let $\mathbf{K}$ be a cone-space, let $\mathbf{K}^*$ be the dual cone-space, and let A be a bounded subset of $\mathbf{K}^*$. A set $C \subset A$ is called a *James boundary* for A if, for each $x \in \mathbf{K}$, there exists a functional $g \in C$ such that

$$x(g) = (g, x) = g(x) = \sup_{f\in A} x(f) = \sup_{f\in A} f(x).$$

Remark 4.2. Given a cone-space $\mathbf{K}$, a bounded set $A \subset \mathbf{K}^*$, and its nonempty James boundary E, we can map $\mathbf{K}$ into a subset of the space $m(A)$ or $m(E)$ by associating with each point $x \in X$ the vector $(x_f)_{f\in A}$ or $(x_f)_{f\in E}$, respectively, where $x_f = f(x)$. Note that $\sup_A(x_f) = \sup_E(x_f)$ is attained for all $x \in \mathbf{K}$, and hence the Simons inequality can be applied in these spaces.

In the following theorem, separability will be considered with respect to $\mathbf{L}(\mathbf{K})$ of some nonempty subset of E in the cone-space $Y := \mathbf{K}$. By definition, E is *separable* with respect to $\mathbf{L}(\mathbf{K})$ if there exists a countable set $D \subset \mathbf{L}(\mathbf{K})$ (here we may assume without loss of generality that $D = -D$) such that the right closure of $D = \{a_n\}$ relative to the asymmetric norm $\|\cdot|$ contains E. If the right closure of $\mathbf{L}$ contains $\mathbf{K}$, then separability of E and that of E relative to $\mathbf{L}(\mathbf{K})$ are equivalent.

In what follows, we will simply say that E is separable. In this case, each bounded subset M of the space $Y^* = \mathbf{K}^*$ can be equipped with a metric generating the pointwise convergence of functionals $y^* \in M$. Since, for arbitrary $\{a_1, \dots, a_n\} \subset D$, the w*-topology on M is generated by a countable system of seminorms

$$p_{a_1,\dots,a_n}(z^*) := \max_{k=\overline{1,n}} |(a_k, z^*)|,$$

this set is therefore metrizable (see [5]). This topology generates on $\mathbf{L}$ the pointwise convergence of functionals from $\mathbf{K}$; it is metrizable on M.

Note also that, in the separable case, the w*-topology of right pointwise convergence is generated by a countable family of asymmetric seminorms

$$p_{a_1,\dots,a_n}(z^*) := \max_{k=\overline{1,n}}(a_k, z^*),$$

which determines some countable set of neighborhoods of points $z^* \in M$. Hence, M is compact if and only if it is sequentially compact.

Theorem 4.3 (Godefroy)**.** *Let* $\mathbf{K}$ *be a cone-space, let* A *be a bounded right closed convex subset of the dual* 1*-regular cone-space* $Y = \mathbf{K}^*$*, and let* E *be a separable James boundary for* A*. Then* $A = \overline{\operatorname{conv}}\, E$*.*

Proof. Let B be the unit ball in the cone-space $\mathbf{K}$. Assume on the contrary that there exists $y_0^* \in A \setminus \overline{\operatorname{conv}}\, E$. By the Hahn–Banach theorem, there exist a a norm-one linear functional $F \in \mathbf{K}^{**}$ and numbers $\alpha, \beta : \alpha < \beta$ such that $F(y_0^*) \geqslant \beta > \alpha \geqslant F(f)$ for all $f \in E$. Consider the set $M = \{x \in B \mid x(y_0^*) = y_0^*(x) \geqslant \beta\}$. By Theorem 4.2, the functional F lies in the w*-closure of the set M in the space $\mathbf{K}^{**}$, which is the intersection of the ball B^{**} and the set $\{f \in \mathbf{K}^{**} \mid f(y_0^*) = y_0^*(f) \geqslant \beta\}$. The set E is separable in $Y = \mathbf{K}^*$, and hence on the bounded subsets of the space $Y^* = \mathbf{K}^{**}$ one can introduce a metric generating the pointwise convergence of functionals on the set E. Hence some sequence $\{x_n\} \subset M$ converges to some functional F at each point of the set E, that is, $x_n(f) \to F(f)$ for all $f \in E$, and, therefore, $\sup_E \big(\limsup_{n\to\infty} x_n\big) = \sup_E F \leqslant \alpha$ (here $\{x_n\}$ is considered in $m(E)$ (see Remark 4.2)). On the other hand, $\operatorname{conv}\{x_n\} \subset M$ and $y_0^* \in A$, and hence $\sup_E x = \sup_B x \geqslant x(y_0^*) \geqslant \beta$ for all $x \in \operatorname{conv}\{x_n\}$. By the Simons inequality (see Lemma 4.1),

$$\alpha \geqslant \sup_E \Big(\limsup_{n\to\infty} x_n\Big) \geqslant \inf_{x\in\operatorname{conv}\{x_n\}} \sup_E x \geqslant \beta,$$

contradicting the inequality $\alpha < \beta$. This proves the theorem. ☐

Remark 4.3. The closure in the weak right pointwise topology (that is, in the topology of right pointwise convergence on $\mathbf{K}^*$) of the ball B under the natural embedding coincides with B^{**}, and hence, a similar argument shows that $A = \overline{\operatorname{conv}}\, E$ without the additional requirement that the cone-space $\mathbf{K}^*$ should be 1-regular.

Theorem 4.4 (James)**.** *Let* $\mathbf{K}$ *be a cone-space.*

1) *If* $C \subset \mathbf{K}$ *is a compact set in the topology of weak right pointwise convergence* (*on* $\mathbf{K}^*$)*, then each functional* $f \in \mathbf{K}^*$ *attains its supremum* C*.*

2) *If* C *is a nonempty closed* (*relative to the topology generated by open balls*) *of convex bounded subset of a separable cone-space* $\mathbf{K}$ *for which each functional* f *from the* 1*-regular cone-space* $\mathbf{K}^*$ *attains its norm, then* C *is compact in the topology of weak right pointwise convergence, and, therefore, and in the weak topology.*

Proof. Let us prove the first assertion. If C is a compact set in the topology of weak right pointwise convergence, then, since each functional $f \in \mathbf{K}^*$ is upper semicontinuous in this topology, f assumes on C its largest value.

Let us verify the second assertion. By the assumption on norm attainability of functionals, the set B (the unit ball in $\mathbf{K}$) is a James boundary of the w*-closure of this set in the space $\mathbf{K}^{**}$, which is the unit ball B^{**} of the cone-space $\mathbf{K}^{**}$. Next, choosing $A = B^{**}$ and $E = B$, and proceeding as in the proof of Theorem 4.3, we find that $B = \overline{\operatorname{conv}}\, B = B^{**}$ (up to a natural embedding). Hence both $B = B^{**}$ and kB are compact sets in the topology of weak right pointwise convergence. Since C is bounded, we have $C \subset kB$ for some $k > 0$. Next, $\mathbf{K} = \mathbf{K}^{**}$, and hence, since C is closed, it follows from the Hahn–Banach theorem that C is closed in the topology of weak right pointwise convergence. Hence C is a compact set in the topology of weak right pointwise convergence, and, therefore, in the weak topology. This proves Theorem 4.4. ☐

Remark 4.4. A similar argument gives the following modification of the last assertion in Theorem 4.4 (without the assumption that $\mathbf{K}^*$ is 1-regular): *if* C *is a nonempty closed* (*relative to the topology generated by open balls*) *convex bounded subset of a separable cone-space* $\mathbf{K}$ *in which each functional* $f \in \mathbf{K}^*$ *attains its norm, then* C *is compact in the topology of weak right pointwise convergence.*

To prove this result, we use Remark 4.3 as in Theorem 4.4 and the fact that C is sequentially compact in the topology of weak right pointwise convergence (since E is separable).

The following result is a direct consequence of Remark 4.1, the previous theorem, and its proof.

Corollary 4.1 (James)**.** *Let a cone-space* $\mathbf{K}$ *be separable relative to* $\mathbf{L}(\mathbf{K})$*. Then* $\mathbf{K}$ *is reflexive if and only if each functional* $f \in \mathbf{K}^*$ *attains its norm.*

From Theorem 3.1 we have

Corollary 4.2 (James). *Let a cone-space* $\mathbf{K}$ *be separable relative to* $\mathbf{L}(\mathbf{K})$ *and let* $\mathbf{K}^*$ *be* 1*-regular. Then* $\mathbf{K}$ *is reflexive if and only if the unit ball in* $\mathbf{K}$ *is a compact set in the weak topology of right pointwise convergence (and so any closed convex bounded set is compact in the weak topology of right pointwise convergence).*

Corollary 4.3. *In any nonreflexive right-complete cone-space* $\mathbf{K}$ *whose right closure* $\mathbf{L}(\mathbf{K})$ *contains* $\mathbf{K}$ *and for which* $\mathbf{K}^*$ *is* 1*-regular, there exists a right-complete separable nonreflexive cone-space* $\mathbf{K}_0$.

Proof. We will assume that $\mathbf{K}$ is naturally embedded into $\mathbf{K}^{**}$. Since $\mathbf{K} \notin (\mathrm{Rf})$ and $\mathbf{K}^*$ is 1-regular, the unit ball B under the natural embedding in $\mathbf{K}^{**}$ is not a weakly compact set, and its weak closure is the ball B^{**} (under the natural embedding). In this case, each sequence in B has a limit point in B^{**}. In addition, the linear space $\mathbf{L}(\mathbf{K})$ is narrower than $\mathbf{L}(\mathbf{K}^{**})$, for otherwise the right closure $\mathbf{L}(\mathbf{K})$ would coincide with the right closure of $\mathbf{L}(\mathbf{K}^{**})$, that is, $\mathbf{K} = \mathbf{K}^{**}$. But this is impossible, since $\mathbf{K}$ is nonreflexive.

Proceeding as in Lemma 2 (see [28, Ch. 8, § 2, p. 285]), we construct, for each point $F \in \mathbf{L}(\mathbf{K}^{**}) \setminus \mathbf{K}$ such that $\|F|_{**} \leqslant 1$, a sequence $\{x_n\} \subset \mathbf{K}$ such that $\|x_n| \leqslant 1$ and for which F is a limit point. By Remark 3.5, this sequence can be taken from $\mathbf{L}(\mathbf{K})$. We can assume that, for any separable linear manifold $\mathscr{L} \subset \mathbf{L}(\mathbf{K})$, there exists $\delta = \delta(\mathscr{L}) > 0$ such that

$$\|F - y| \geqslant \delta \quad \text{for all} \quad y \in \mathscr{L}, \quad \|y| \leqslant 1,$$

because otherwise the corresponding right strongly closure $\mathscr{L}$ is a separable nonreflexive space, for which the conclusion of the corollary holds.

We construct by induction the aforementioned sequence $\{x_n\} \subset \mathbf{L}(\mathbf{K})$ whose linear hull is $\mathscr{L}$.

On p. 407, we have shown that the topology of weak* right convergence on the maximal linear hull of the dual space coincides on this hull with the weak* pointwise convergence on $Y = \mathbf{K}^*$ (in particular, on the unit ball $B^* \subset Y$). Applying this property to Y^* and Y, we find that there exists $f_1 \in B^*$ for which there exists $x_1 \in \mathbf{L}(\mathbf{K})$, $\|x_1| \leqslant 1$, such that $|(F - x_1)(f_1)| < 1$. Consider the space $Y_2 \subset \mathbf{K}^{**}$ spanned by the functionals F and $F - x_1$. There exist $\{f_1, \dots, f_{k_2}\} \subset S^*$ such that

$$\max\{G(f_m) \mid 2 \leqslant m \leqslant k_2\} \geqslant \frac{1}{2}\|G|_{**}$$

for any $G \in Y_2$.

Arguing as in the previous paragraph, we find $x_2 \in \mathbf{L}(\mathbf{K})$, $\|x_2| \leqslant 1$, such that

$$\max\{|(F - x_2)(f_m)| \mid 1 \leqslant m \leqslant k_2\} < \frac{1}{2}.$$

Consider the cone-space $Y_3 \subset \mathbf{K}^{**}$ spanned by the functionals F, $F - x_1$ and $F - x_2$, and find $\{f_{k_2}, \dots, f_{k_3}\} \subset S^*$ such that

$$\max\{G(f_m) \mid k_2 \leqslant m \leqslant k_3\} \geqslant \frac{1}{2}\|G|_{**}$$

for any $G \in Y_3$. There exists a point $x_3 \in \mathbf{L}(\mathbf{K})$ such that $\|x_3| \leqslant 1$ such that

$$\max\{|(F - x_3)(f_m)| \mid 1 \leqslant m \leqslant k_3\} < \frac{1}{3}.$$

Continuing this process, we get a sequence $\{x_n\} \subset \mathbf{L}(\mathbf{K})$ such that $\|x_n| \leqslant 1$ and satisfying

$$\max\{G(f_m) \mid k_{n-1} \leqslant m \leqslant k_n\} \geqslant \frac{1}{2}\|G|_{**}$$

for any $G \in Y_n$, where Y_n is the space spanned by the functionals F, $F - x_1$, …, $F - x_n$, and $\{f_{k_2+1}, \dots, f_{k_3}\} \subset S^*$ such that

$$\max\{|(F - x_n)(f_m)| \mid 1 \leqslant m \leqslant k_n\} < \frac{1}{n}.$$

Let $x \in \mathbf{K}^{**}$ be a limit point of the sequence $\{x_n\}$ in the weak topology. Note that if x does not lie in the right strong closure of $\mathscr{L}$, then this closure is a separable nonreflexive space, which proves the result required in the corollary.

Assume that x does not lie in the right strong closure of $\mathscr{L}$. Then $F - x \in \mathbf{K}^{**}$ and

$$\max\{(F - x)(f_m) \mid 1 \leqslant m < +\infty\} \geqslant \frac{1}{2}\|F - x|_{**}.$$

For a fixed m, we have $|(F-x_n)(f_m)|<\frac{1}{p}$ for $n\geqslant k_p\geqslant m$. Hence

$$|(F-x)(f_m)|\leqslant|(F-x_n)(f_m)|+|f_m(x_n-x)|\leqslant\frac{1}{p}+|f_m(x_n-x)|$$

for $n\geqslant k_p\geqslant m$. Since x is a limit point of the sequence $\{x_n\}$ in the weak topology, for any $N>m$ there exists $n\geqslant k_N\geqslant m$ such that $|f_m(x_n-x)|<\frac{1}{N}$. For this x_n (note that $k_N\geqslant m$) we can put $p=N$. Hence we have $|(F-x_n)(f_m)|+|f_m(x_n-x)|\leqslant\frac{2}{N}$. Since N is arbitrary, we get $(F-x)(f_m)=0$ for all $m\in\mathbb{N}$. By the assumption, there exists a point $y\in\mathscr{L}$ such that $\|x-y|<\frac{\delta}{4}\leqslant\frac{1}{4}\|F-y|$. Hence

$$\begin{aligned}0=\sup_{m\in\mathbb{N}}(F-x)(f_m)&=\sup_{m\in\mathbb{N}}\big((F-y)(f_m)-(x-y)(f_m)\big)\\ &\geqslant\sup_{m\in\mathbb{N}}\big((F-y)(f_m)-\|x-y|\big)\geqslant\frac{1}{2}\|F-y|-\frac{1}{4}\|F-y|=\frac{1}{4}\|F-y|,\end{aligned}$$

and, therefore, $0=\|F-y|\geqslant\delta$, which is impossible.

Let B_0 be the right completion of the intersection of B with the linear hull of the sequence $\{x_n\}$. Similarly, let $\mathbf{K}_0$ be the right completion (which coincides with right closure in $\mathbf{K}^{**}$) of the intersection of $\mathbf{K}$ with the linear hull of $\{x_n\}$. It is easily seen that B_0 is not a weakly compact set, since otherwise all the w*-limit points (under the natural embedding in $\mathbf{K}^{**}$) of the sequence $\{\mathfrak{J}(x_n)\}$ would lie in $\mathfrak{J}(B_0)$. Therefore, $\mathbf{K}_0\notin(\mathrm{Rf})$. This proves the corollary. □

Remark 4.5. The proof of Corollaries 4.2 and 4.3 (without the assumption that $\mathbf{K}^*$ is 1-regular) is similar. It suffices to employ the fact that the unit ball in any nonreflexive cone-space $\mathbf{K}$ is not compact in the weak topology of right pointwise convergence (see Theorem 3.1). In any nonreflexive right-complete cone-space $\mathbf{K}$ for which the right closure $\mathbf{L}(\mathbf{K})$ contains $\mathbf{K}$, there exists a right-complete separable nonreflexive cone-space $\mathbf{K}_0$.

5. NONSEPARABLE VARIANT OF THE JAMES THEOREM

Consider a sequence of linear functionals $\{\varphi_n\}$ on X bounded in the space $m(\mathbf{K})$. Let $\mathscr{L}(\{\varphi_n\})$ be the set of all linear functionals w satisfying

$$\liminf_n\varphi_n(x)\leqslant w(x)\leqslant\limsup_n\varphi_n(x)\quad\text{for all } X.$$

Note that (see [27, Theorem 4.25]) that there exists a linear function $\ell(\,\cdot\,)$ on $X\supset\mathbf{K}$ satisfying

$$\liminf_{n\to\infty}\varphi_n(x)\leqslant\ell(x)\leqslant\limsup_{n\to\infty}\varphi_n(x).$$

This inequality means that $\mathscr{L}(\{\varphi_n\})$ is nonempty.

Lemma 5.1. *Let $\mathbf{K}$ be an arbitrary cone-space, let $0<\theta<1$, and let $\{f_n\}\subset\mathbf{K}^*$, $\|f_n|\leqslant1$ $(n\in\mathbb{N})$. Suppose that $\varrho(w,\operatorname{conv}\{f_n\})\geqslant\theta$ for all $w\in\mathscr{L}(\{f_n\})$. Then, for any sequence of positive numbers $\{\lambda_n\}$, $\sum_n\lambda_n=1$, there exist a number $\alpha\in[\theta,2]$ and a sequence $\{g_n\}\subset\mathbf{K}^*$, $\|g_n|\leqslant1$, with the following properties: for each functional $w\in\mathscr{L}(\{g_n\})$,*

$$\Big\|\sum_n\lambda_n(g_n-w)\Big|=\alpha$$

and, for each functional $w\in\mathscr{L}(\{g_n\})$ and any N,

$$\Big\|\sum_{n=1}^N\lambda_n(g_n-w)\Big|<\alpha\Big(1-\theta\sum_{n=N+1}^\infty\lambda_n\Big).$$

Proof. Let $\varepsilon_n>0$ be such that

$$\sum_{n=1}^\infty\frac{\lambda_n\varepsilon_n}{(\sum_{n=N+1}^\infty\lambda_n)(\sum_{n=N}^\infty\lambda_n)}<1-\theta.$$

We construct the sequence $\{g_n\}$ by induction. Let $\psi_i^{(0)}=f_i$ $(i\in\mathbb{N})$. We set

$$\alpha_1:=\inf\{\sup\{\|g-w|\mid w\in\mathscr{L}(\{\varphi_i\})\}\},$$

where the infimum is taken over all $g \in \operatorname{conv}\{\psi_i^{(0)}\}$ and all sequences $\{\varphi_i\}$ satisfying $\varphi_k \in \operatorname{conv}\{\psi_i^{(0)}\}_{i=k}^{\infty}$ for all k. Let $g_1 \in \operatorname{conv}\{\psi_i^{(0)}\}$ and let a sequence $\{\varphi_i^{(1)}\}$ be such that $\varphi_k^{(1)} \in \operatorname{conv}\{\psi_i^{(0)}\}_{i=k}^{\infty}$ for all k and

$$\alpha_1 \leqslant \sup\{\|g_1 - w| \mid w \in \mathscr{L}(\{\varphi_i^{(1)}\})\} < \alpha_1(1+\varepsilon_1).$$

Next, we choose $w' \in \mathscr{L}(\{\varphi_i^{(1)}\})$ such that

$$\alpha_1(1-\varepsilon_1) < \|g_1 - w'| < \alpha_1(1+\varepsilon_1).$$

Let $\overline{x} \in \mathbf{K}$, $\|\overline{x}| \leqslant 1$, be such that

$$\alpha_1(1-\varepsilon_1) < (g_1 - w')(\overline{x}).$$

Since $\liminf_i \varphi_i^{(1)}(\overline{x}) \leqslant w'(\overline{x})$, there exists a subsequence $\{\psi_i^{(1)}\}$ of $\{\varphi_i^{(1)}\}$ such that, for any $w \in \mathscr{L}(\{\psi_i^{(1)}\})$,

$$\liminf_i \varphi_i^{(1)}(\overline{x}) = \lim_i \psi_i^{(1)}(\overline{x}) = w(\overline{x}) \leqslant w'(\overline{x}).$$

This means that by replacing w' by w we get

$$\alpha_1(1-\varepsilon_1) < (g_1 - w)(\overline{x}).$$

Next, we set

$$\alpha_2 := \inf\Big\{\sup\big\{\big\|\lambda_1 g_1 + \sum_{n=2}^{\infty} \lambda_n g - w\big| : w \in \mathscr{L}(\{\varphi_i\})\big\}\Big\},$$

where the infimum is taken over all $g \in \operatorname{conv}\{\psi_i^{(1)}\}_{i=2}^{\infty}$ and sequences $\{\varphi_i\}$ satisfying $\varphi_k \in \operatorname{conv}\{\psi_i^{(1)}\}_{i=k}^{\infty}$ for all k. Let $g_2 \in \operatorname{conv}\{\psi_i^{(1)}\}_{i=2}^{\infty}$ and $\{\varphi^{(2)_i}\}$ be such that $\varphi_k^{(2)} \in \operatorname{conv}\{\psi_i^{(1)}\}_{i=k}^{\infty}$ for all k, and

$$\alpha_2 \leqslant \inf\Big\{\sup\big\{\big\|\lambda_1 g_1 + \sum_{n=2}^{\infty} \lambda_n g_2 - w\big| : w \in \mathscr{L}(\{\varphi_i^{(2)}\})\big\}\Big\} < \alpha_2(1+\varepsilon_2).$$

Next, we choose $w' \in \mathscr{L}(\{\varphi_i^{(2)}\})$ such that

$$\alpha_2(1-\varepsilon_2) < \big\|\lambda_1 g_1 + \sum_{n=2}^{\infty} \lambda_n g_2 - w'\big| < \alpha_2(1+\varepsilon_2).$$

Assume that, for some $\overline{x} \in \mathbf{K}$ such that $\|\overline{x}| \leqslant 1$, the following condition is met:

$$\alpha_2(1-\varepsilon_2) < \lambda_1 g_1(\overline{x}) + \sum_{n=2}^{\infty} \lambda_n g_2(\overline{x}) - w'(\overline{x}).$$

As above, we choose a subsequence $\{\psi_i^{(2)}\}$ of $\{\varphi_i^{(2)}\}$ such that, for $w \in \mathscr{L}(\{\psi_i^{(2)}\})$,

$$\liminf_i \varphi_i^{(2)}(\overline{x}) = \lim_i \psi_i^{(2)}(\overline{x}) = w(\overline{x}) \leqslant w'(\overline{x}).$$

So, we can again replace w' by w.

Proceeding in this way, we define

$$\alpha_N := \inf\Big\{\sup\Big\{\Big\|\sum_{n=1}^{N-1} \lambda_n g_n + \sum_{n=N}^{\infty} \lambda_n g - w\Big| : w \in \mathscr{L}(\{\varphi_i\})\Big\}\Big\},$$

where the infimum is taken over all $g \in \operatorname{conv}\{\psi_i^{(i-1)}\}_{i=N}^{\infty}$ and the sequences $\{\varphi_i\}$ which satisfy the relations $\varphi_k \in \operatorname{conv}\{\psi_i^{(N-1)}\}_{i=k}^{\infty}$ for all k. Let $g_N \in \operatorname{conv}\{\psi_i^{(N-1)}\}_{i=N}^{\infty}$ and let a sequence $\{\varphi_i^{(N)}\}$ be such that $\varphi_k^{(N)} \in \operatorname{conv}\{\psi_i^{(N-1)}\}_{i=k}^{\infty}$ for all k and

$$\alpha_N \leqslant \sup\Big\{\Big\|\sum_{n=1}^{N-1} \lambda_n g_n + \sum_{n=N}^{\infty} \lambda_n g_N - w\Big| : w \in \mathscr{L}(\{\varphi_i\})\Big\} < \alpha_N(1+\varepsilon_N).$$

Next, we choose $w' \in \mathscr{L}(\{\varphi_i^{(N)}\})$ such that

$$\alpha_N(1-\varepsilon_N) < \Big\| \sum_{n=1}^{N-1} \lambda_n g_n + \sum_{n=N}^{\infty} \lambda_n g_N - w' \Big| < \alpha_N(1+\varepsilon_N).$$

Assume that, for some $\overline{x} \in \mathbf{K}$ such that $\|\overline{x}| \leqslant 1$,

$$\alpha_N(1-\varepsilon_N) < \sum_{n=1}^{N-1} \lambda_n g_n(\overline{x}) + \sum_{n=N}^{\infty} \lambda_n g_N(\overline{x}) - w'(\overline{x}).$$

Since $\liminf_i \varphi_i^{(N)}(\overline{x}) \leqslant w'(\overline{x})$, we can find a subsequence $\{\psi_i^{(N)}\}$ of $\{\varphi_i^{(N)}\}$ such that, for any $w \in \mathscr{L}(\{\psi_i^{(N)}\})$,

$$\liminf_i \varphi_i^{(N)}(\overline{x}) = \lim_i \psi_i^{(N)}(\overline{x}) = w(\overline{x}) \leqslant w'(\overline{x}).$$

So, as above, we can replace w' by w, which gives the inequality

$$\alpha_N(1-\varepsilon_N) < \sum_{n=1}^{N-1} \lambda_n g_n(\overline{x}) + \sum_{n=N}^{\infty} \lambda_n g_N(\overline{x}) - w(\overline{x}).$$

This completes the induction step.

Since $\mathscr{L}(\{g_N\}) \subset \mathscr{L}(\{\varphi_i^{(N)}\})$ for all i, and for all N, and any $w \in \mathscr{L}(\{g_N\})$, we have

$$\alpha_N(1-\varepsilon_N) < \Big\| \sum_{n=1}^{N-1} \lambda_n g_n + \sum_{n=N}^{\infty} \lambda_n g_N - w \Big| < \alpha_N(1+\varepsilon_N).$$

Note that $\|g_N| \leqslant 1$ for all n. Hence $g \in \operatorname{conv}\{g_N\}$ implies $\|g| \leqslant 1$. We have $\|w| \leqslant 1$ for $w \in \mathscr{L}(\{g_N\})$, and hence $\alpha_N \leqslant 2$. By definition of α_N it is easily seen that the sequence $\{\alpha_N\}$ is monotone increasing, and, of course, $\alpha_N \geqslant 0$ by construction and the hypotheses of the lemma. As a result, the limit $\alpha := \lim_{N\to\infty} \alpha_N$ exists and satisfies

$$0 \leqslant \Big\| \sum_N \lambda_N (g_N - w) \Big| \leqslant 2.$$

Moreover (see [25, p. 12]),

$$\Big\| \sum_{i=1}^{n} \lambda_i (g_i - w) \Big| \leqslant \Big(\sum_{i=n+1}^{\infty} \lambda_i \Big) \sum_{k=1}^{n} \frac{\lambda_k \alpha_k (1+\varepsilon_k)}{(\sum_{n=k+1}^{\infty} \lambda_n)(\sum_{n=k}^{\infty} \lambda_n)} \leqslant \alpha\Big(1 - \theta\big(\sum_{i=n+1}^{\infty} \lambda_i\big)\Big).$$

This proves the lemma.

Remark 5.1. Under the conditions of Lemma 5.1, the functional $\sum_n \lambda_n (g_n - w)$ does not attain its norm if a sequence $\{\lambda_n\}$ is chosen as follows: $\lambda_n > 0$, $\lambda_{n+1} < \Delta \lambda_n$ ($n \in \mathbb{N}$), $\sum_n \lambda_n = 1$, where $\Delta \in (0, \theta^2/2)$ is some number.

For an arbitrary $x \in \mathbf{K}$, $\|x| \leqslant 1$, the property

$$\liminf_k g_k(x) \leqslant w(x) \text{ for all } w \in \mathscr{L}(g_n)$$

implies that there exists a sufficiently large N such that

$$(g_{N+1} - w)(x) < \theta^2 - 2\Delta \leqslant \alpha\theta - 2\Delta.$$

Hence

$$\sum_n \lambda_n (g_n - w)(x) < \sum_{n=1}^{N} \lambda_n (g_n - w)(x) + \lambda_{N+1}(\alpha\theta - 2\Delta) + \sum_{n=N+2}^{\infty} \lambda_n (g_n - w)(x)$$
$$\leqslant \Big\| \sum_{n=1}^{N} \lambda_n (g_n - w)(x) \Big| + \lambda_{N+1}(\alpha\theta - 2\Delta) + 2 \sum_{n=N+2}^{\infty} \lambda_n \leqslant \alpha\Big(1 - \theta\big(\sum_{i=n+1}^{\infty} \lambda_i\big)\Big) +$$
$$\lambda_{N+1}(\alpha\theta - 2\Delta) + 2\Delta \sum_{n=N+1}^{\infty} \lambda_n = \alpha - (\alpha\theta - 2\Delta) \sum_{n=N+2}^{\infty} \lambda_n < \alpha.$$

But α is the norm of the functional $\sum_n \lambda_n (g_n - w)$, that is, this functional does no attain its norm. ☐

Corollary 5.1. *Let* $\mathbf{K}$ *be a nonreflexive cone-space whose right closure* $\mathbf{L} = \mathbf{L}(\mathbf{K})$ *contains* $\mathbf{K}$. *Then there exists a functional* $\mathbf{K}^*$ *which does not attain its norm on* $\mathbf{K}$.

Proof. We will assume that $\mathbf{K}$ is naturally embedded in $\mathbf{K}^{**}$. It suffices to consider the case of nonseparable $\mathbf{K}$. Let $\mathbf{K}_0$ be a separable nonreflexive cone-subspace which is the right closure of $\mathbf{L}_0 := \mathbf{L}(\mathbf{K}_0)$ (see Remark 4.5). If $\mathbf{L}_0$ coincides with $\mathbf{L}(\mathbf{K_0}^{**})$, then the right closures of these linear manifolds are equal (under the natural embedding $\mathbf{K}_0$ into $\mathbf{K_0}^{**}$).

Assume that $\mathbf{L}_0 := \mathbf{L}(\mathbf{K}_0)$ is different from $\mathbf{L}(\mathbf{K_0}^{**})$. Hence there exists $F \in \mathbf{L}(\mathbf{K}_0^{**})$, $\|F|_{**} = 1$, such that F does not lie in the right closure of $\mathbf{L}_0$, since otherwise $\mathbf{L}(\mathbf{K_0}^{**})$ would lie in $\mathbf{K}_0$ (under the natural embedding), which would imply $\mathbf{L}_0 = \mathbf{L}(\mathbf{K_0}^{**})$. Let B_0 be the unit ball in $\mathbf{K}_0$, and B_0^{**} be the unit ball in $\mathbf{K}_0^{**}$. There exists $\delta > 0$ such that the set $\delta B_0^{**} + \mathbf{L}_0$ contains F in its boundary. Consider the sequence $\{x_n\} \subset \mathbf{L}_0$ which contains a set symmetric about the origin and right complete in $\mathbf{K}_0$. By the proof of Corollary 4.3, it can be assumed that F is a limit point for $\{x_n\}$ in the weak topology (the w*-topology relative to $\mathbf{K}_0^*$). Let L_n be the linear hull of the vectors $\{x_k\}_{k=1}^n$, $M_n := \delta B_0 + L_n$. The cone-space $Y := W_n^*$ consisting of the support functionals to the M_n, are annihilators of the space L_n. Note that Y is the dual space to the cone-space $\mathbf{K}_0$ in which $B_0 + L_n$ is the unit ball.

Let B_n^* be the unit ball in the cone-space Y, and let B_n^{**} be the unit ball in the cone-space Y^*. Next, let $M_n^{**} := \delta B_0^{**} + L_n$, and let $F_n \in [0, F]$ lie in the boundary of M_n^{**}. On p. 407 we have shown that the topology of weak* right convergence on the maximal linear hull of the dual spaces coincides on this hull with the w*-pointwise convergence on Y (in particular, on the unit ball $B_Y \subset Y$). Applying this to Y^* and Y, we find that there exists $\psi \in B_Y$ such that $\psi(F) > \delta/2 =: \theta$. By construction, ψ vanishes on L_n.

We set $f_n := \psi$. We have $f_n(x) \to 0$ as $n \to \infty$ for all $x \in \mathbf{L}_0 = \mathbf{L}(\mathbf{K}_0)$, and the norm of the restriction to $\mathbf{L}_0$ of any functional $f \in \mathbf{K_0}^*$ such that $f \in \operatorname{conv}\{f_n\}$ is at most $\theta_0 = \theta/\|F|_{**}$.

Hence, for all $w_0 \in \mathscr{L}(\{f_n\})$, the functional w_0 is an annihilator of $\mathbf{L}_0$. Hence, for all $f \in \operatorname{conv}\{f_n\}$ and all $x \in \mathbf{L}_0$,

$$\|f - w_0|_* \geqslant (f - w_0)(x) = f(x), \text{ and hence } \quad \|f - w_0|_* \geqslant \theta_0.$$

By Lemma 5.1, for any sequence of positive numbers $\{\lambda_n\}$ such that $\sum_n \lambda_n = 1$, there exist a number $\alpha \in [\theta_0, 2]$ and a sequence $\{g_n\} \subset \mathbf{K}^*$, $\|g_n| \leqslant 1$, with the following properties: for each functional $w \in \mathscr{L}(\{g_n\})$,

$$\Big\| \sum_n \lambda_n (g_n - w) \Big| = \alpha$$

and, moreover, for each functional $w \in \mathscr{L}(\{g_n\})$ and any N,

$$\Big\| \sum_{n=1}^{N} \lambda_n (g_n - w) \Big| < \alpha \Big(1 - \theta_0 \sum_{n=N+1}^{\infty} \lambda_n \Big).$$

By Remark 5.1, there exists a functional from $\mathbf{K}^*$ onto $\mathbf{K}$ not attaining its norm. □

In the following two results, by cone-subspaces $\mathbf{K}_0$ of a cone-space $\mathbf{K}$ we will mean convex cones $\mathbf{K}_0 \subset \mathbf{K}$ lying in the right closure of $\mathbf{L}(\mathbf{K}_0)$.

Theorem 5.1. *Let* $\mathbf{K}$ *be a right-complete cone-space. Then* $\mathbf{K}$ *is reflexive if and only if so is each right-complete separable cone-subspace* $\mathbf{K}_0$.

Proof. Necessity. Let $\mathbf{K}$ be reflexive. If there exists proper separable nonreflexive cone-space $\mathbf{K}_0$, then, as in the proof of Corollary 5.1, it can be shown that there exists a functional not attaining its norm. But this contradicts Theorem 2.1. This contradiction proves that each right-complete separable cone-subspace $\mathbf{K}_0$ is reflexive.

Sufficiency. Suppose that each right-complete separable cone-subspace $\mathbf{K}_0$ is reflexive. Assume on the contrary that $\mathbf{K} \notin (\mathrm{Rf})$. Then there exists a bounded linear functional $x^* \in \mathbf{K}^*$ not attaining its norm on $\mathbf{K}$. Hence there exists a sequence $\{x_n\} \subset \mathbf{L}(\mathbf{K})$ such that $x^*(x_n) \to \|x^*|_*$ as $n \to \infty$. Let L_0 be the linear subspace spanned by $\{x_n\}$, and $\mathbf{K}_0$ be the right closure of L_0 in the cone-space $\mathbf{K}$. Then the linear functional x_0^*, which is the restriction of the functional x^* to $\mathbf{K}_0$, has the same norm on $\mathbf{K}_0$ as x^*, and this norm is not attained on $\mathbf{K}_0$. Hence by Theorem 2.1 it follows that $\mathbf{K}_0 \notin (\mathrm{Rf})$, which contradicts the assumption. This proves the theorem. □

The following result is a direct consequence of Corollary 4.3.

Theorem 5.2. *Let* $\mathbf{K}$ *be a right-complete cone-space. Then* $\mathbf{K}$ *is reflexive if and only if each right-complete cone-subspace* $\mathbf{K}_0$ *is reflexive.*

Example 5.1. Let $1 < p, q < +\infty$, $\Omega \subset \mathbb{R}^n$ be a measurable set relative to the Lebesgue measure μ, Σ be a σ-algebra of measurable subsets of Ω, $L_p := L_p(\Omega, \Sigma, \mu)$, $L_q := L_q(\Omega, \Sigma, \mu)$. Consider the linear manifold $\mathbf{L} := L_p \cap L_q$, which we equip with the asymmetric norm $\|f|_{p,q,1} := \|f_+|_{L_p} + \|f_-|_{L_q}$ or with the asymmetric norm $\|f|_{p,q,\infty} := \max\{\|f_+|_{L_p}; \|f_-|_{L_q}\}$. The right completion of the linear manifold is the cone-space $\mathbf{K}$ of measurable functions $f : \Omega \to \mathbb{R}$ with finite norm $\|f|_{p,q,r}$, where $r = 1 \vee \infty$. We denote this cone-space by $L_{p,q,r}$. The dual cone-space $\mathbf{K}^*$ is the set of all measurable functions $f : \Omega \to \mathbb{R}$ with finite norm $\|f|_{p^*,q^*,r^*}$, where $p^* = \frac{p}{p-1}$, $q^* = \frac{q}{q-1}$, $r^* = \infty$ (respectively, 1) if $r = 1$ (respectively, ∞). The second dual cone-space is $\mathbf{K}$. Note that cone-space $\mathbf{K}$ and its dual are strongly regular and 1-regular. So, the cone-space $\mathbf{K}$ is reflexive and is superregular,

The arguments in the above example show that each right-complete cone-subspace in $L_{p,q,r}$ is an existence set, that is, for any point $x \in L_{p,q,r}$, there is a $\| \cdot |_{p,q,r}$-nearest point in this cone-subspace.

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