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SHORT COMMUNICATIONS

# On Homogenization for Piecewise Locally Periodic Operators

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**Abstract.** We discuss homogenization of a strongly elliptic operator $\mathcal{A}^\varepsilon = -\operatorname{div} A(x, x/\varepsilon_\#)\nabla$ on a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with either Dirichlet or Neumann boundary condition. The function $A$ is piecewise Lipschitz in the first variable and periodic in the second one, and the function $\varepsilon_\#$ is identically equal to $\varepsilon_i(\varepsilon)$ on each piece $\Omega_i$, with $\varepsilon_i(\varepsilon) \to 0$ as $\varepsilon \to 0$. For $\mu$ in a resolvent set, we show that the resolvent $(\mathcal{A}^\varepsilon - \mu)^{-1}$ converges, as $\varepsilon \to 0$, in the operator norm on $L_2(\Omega)^n$ to the resolvent $(\mathcal{A}^0 - \mu)^{-1}$ of the effective operator at the rate $\varepsilon_\vee$, where $\varepsilon_\vee$ stands for the largest of $\varepsilon_i(\varepsilon)$. We also obtain an approximation for the resolvent in the operator norm from $L_2(\Omega)^n$ to $H^1(\Omega)^n$ with error of order $\varepsilon_\vee^{1/2}$.

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## 1. INTRODUCTION

Consider an elliptic operator - div $A^\varepsilon\nabla$ on a domain $\Omega$ with Dirichlet or Neumann boundary condition. Here we assume that the parameter $\varepsilon$ is small and positive and the coefficient $A^\varepsilon$ is locally periodic, i.e., $A^\varepsilon(x) = A(x, x/\varepsilon)$ with $A$ depending smoothly on the "slow" variable $x$ and periodically on the "fast" variable $x/\varepsilon$. One can think of $A^\varepsilon$ as a rapidly oscillating nearly periodic function with slowly changing amplitude. As is well known, for a given $f \in L_2(\Omega)$, the solution of, e.g., the Dirichlet problem

$$\begin{aligned} -\operatorname{div} A^\varepsilon \nabla u_\varepsilon &= f \qquad \text{in } \Omega,\\ u_\varepsilon &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

converges, as $\varepsilon \to 0$, to the solution $u_0$ of a similar problem

$$\begin{aligned} -\operatorname{div} A^0 \nabla u_0 &= f \qquad \text{in } \Omega,\\ u_0 &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where the coefficient $A^0$ is no longer oscillating; see, e.g., [2], [1], and [11]. The classical results yield the strong convergence of $u_\varepsilon$ to $u_0$ in $L_2(\Omega)$; in other words, the inverse of - div $A^\varepsilon\nabla$ converges to the inverse of - div $A^0\nabla$ in the strong operator topology on $L_2(\Omega)$. More recent studies following the pioneering works [3], [4], and [12], reveal that it converges in fact in the uniform operator topology on $L_2(\Omega)$ at the rate $\varepsilon$ provided that $A$ is Lipschitz in the slow variable; see [6], where this was proved for the first time but with a worse rate, and [8] and [9]. In [7] and [10], the smoothness of $A$ was relaxed to the assumption that $A$ is Hölder continuous of order $s \in [0, 1]$ in the slow variable, which led to the convergence at the slower rate $\varepsilon^s$ (or just convergence if $s = 0$). In this paper, we are interested in similar results for the case in which the coefficient $A^\varepsilon$ loses the continuity in the slow variable, thus becoming piecewise locally periodic. In addition, we allow $A^\varepsilon$ to have different periodic structures in each of the pieces. Although the main focus of this paper is the (piecewise) locally periodic case, the last assumption makes our results interesting even in the most heavily studied case of purely periodic operators, when the coefficients do not depend on the slow variable.

It is worth noting that, while we discuss only the Dirichlet and the Neumann problems on $C^{1,1}$ domains with piecewise Lipschitz coefficients, our results carry over to problems on Lipschitz domains with fairly general conditions and at least piecewise uniformly continuous coefficients (although the rates can change). We refer to [10] for details.

## 2. PROBLEM FORMULATION

For simplicity, we assume that there is only one interface on which $A$ is not Lipschitz in the slow variable. Thus, let $\Omega$ be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$, which is divided into two subdomains by a $(d-1)$-dimensional $C^{1,1}$ closed surface $\Gamma$ in $\Omega$. These subdomains are denoted by $\Omega_1$ (the inner one) and $\Omega_2$ (the outer one);

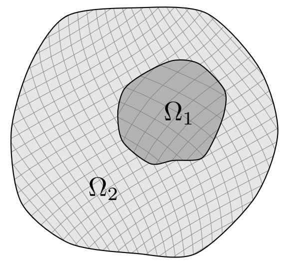


**Fig. 1.**

see Fig. 1. Let $Q$ stand for the unit cube in $\mathbb{R}^d$ centered at the origin. We introduce the set $A$ of functions $A_{kl}\colon \Omega\times\mathbb{R}^d \to \mathbb{C}^{n\times n}$ satisfying $A\in C^{0,1}(\bar\Omega_i;\tilde L_\infty(Q))$ for any $i$, i.e., $A|_{\Omega_i\times\mathbb{R}^d}$ is Lipschitz in the first variable and periodic (with respect to the lattice $\mathbb{Z}^d$) in the other. The scale of the periodic structure on each $\Omega_i$ is described by a function $\varepsilon_i\colon \varepsilon\mapsto\varepsilon_i(\varepsilon)$ tending to 0 as $\varepsilon$ tends to 0. We set $A^\varepsilon_{kl}(x) = A_{kl}(x, x/\varepsilon_\#(x,\varepsilon))$, where $\varepsilon_\#(x,\varepsilon) = \varepsilon_i(\varepsilon)$ for $x\in\Omega_i$.

Next, let $\mathcal{H}^1(\Omega)$ be either the complex Sobolev space $H^1(\Omega)$ or the subspace $\mathring{H}^1(\Omega)$ of all functions in $H^1(\Omega)$ that vanish on $\partial\Omega$. The former corresponds to the case of the Neumann problem, and the latter to the Dirichlet problem. The dual of $\mathcal{H}^1(\Omega)$ is denoted by $\mathcal{H}^{-1}(\Omega)$.

Now we define the matrix operator $\mathcal{A}^\varepsilon\colon \mathcal{H}^1(\Omega)^n\to\mathcal{H}^{-1}(\Omega)^n$ by

$$\mathcal{A}^\varepsilon = -\operatorname{div} A^\varepsilon\nabla = -\sum_{k,l=1}^d \partial_k A^\varepsilon_{kl}\partial_l.$$

We suppose that $\mathcal{A}^\varepsilon$ is strongly elliptic and coercive uniformly in $\varepsilon$ for $\varepsilon\in\mathcal{E}=(0,\varepsilon_0]$, i.e., there are $c_A>0$ and $C_A<\infty$ such that, for any $\varepsilon\in\mathcal{E}$,

$$\operatorname{Re}(\mathcal{A}^\varepsilon u,u)_{L_2(\Omega)} \geqslant c_A\|\nabla u\|^2_{L_2(\Omega)} - C_A\|u\|^2_{L_2(\Omega)}, \qquad u\in\mathcal{H}^1(\Omega)^n.$$

This implies that $\mathcal{A}^\varepsilon$ is $m$-sectorial and, for any $\mu$ outside the corresponding sector $\mathcal{S}\subset\mathbb{C}$, the resolvent $(\mathcal{A}^\varepsilon-\mu)^{-1}$ is bounded uniformly in $\varepsilon\in\mathcal{E}$.

For such a $\mu$, we want to approximate the resolvent in the operator norms from $L_2(\Omega)^n$ to the spaces $L_2(\Omega)^n$ and $H^1(\Omega)^n$. As usual, these approximations are described in terms of the effective operator and a corrector, which we proceed to define.

## 3. THE EFFECTIVE OPERATOR AND A CORRECTOR

First, we need to introduce an auxiliary function, the solution of the so-called cell problem. For $x\in\Omega$ and $\xi\in\mathbb{C}^{d\times n}$, let us look at the problem

$$-\operatorname{div} A(x,\cdot)(\nabla N_\xi(x,\cdot)+\xi) = 0,$$
$$\int_Q N_\xi(x,y)\,dy = 0,$$

on the cube $Q$ with periodic boundary conditions. It follows from the coercivity of $\mathcal{A}^\varepsilon$ that this problem is strongly elliptic, and there is a unique vector-valued solution in the periodic Sobolev space $\tilde H^1(Q)^n$. Next, $N_\xi$ is linear in $\xi$, so the mapping $\xi\mapsto N_\xi$ acts as the multiplication by a function. One can easily check that this function, denoted by $N$, is as regular in the first variable as the function $A$ is, and, therefore, $N\in C^{0,1}(\bar\Omega_i;\tilde H^1(Q))$ for each $i\in\{1,2\}$.

The effective operator $\mathcal{A}^0\colon \mathcal{H}^1(\Omega)^n\to\mathcal{H}^{-1}(\Omega)^n$ is given by

$$\mathcal{A}^0 = -\operatorname{div} A^0\nabla,$$

where

$$A^0(x) = \int_Q A(x,y)(\nabla N(x,y)+I)\,dy.$$

It turns out that $\mathcal{A}^0-\mu$ is an isomorphism whenever $\mathcal{A}^\varepsilon-\mu$ is. By the regularity of $A$ and $N$, we see that $A^0\in C^{0,1}(\bar\Omega_i)$ for any $i$, and then, according to the usual elliptic regularity theory, the resolvent $(\mathcal{A}^0-\mu)^{-1}$ maps $L_2(\Omega)^n$ to $\mathcal{H}^1(\Omega)^n\cap H^2(\Omega_i)^n$ continuously.

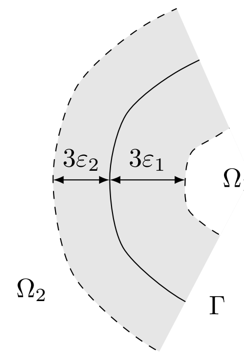


**Fig. 2.**

To define the corrector, we fix, for $i \in \{1,2\}$, an extension operator $\mathcal{E}_i$ taking $H^1(\Omega_i)$ and $H^2(\Omega_i)$ into $H^1(\mathbb{R}^d)$ and $H^2(\mathbb{R}^d)$, respectively, and extend the functions $A|_{\Omega_i}$ and $N|_{\Omega_i}$ to Lipschitz mappings $A_i$ and $N_i$ on the entire $\mathbb{R}^d$. Then introduce the operator $\mathcal{K}_\mu^\varepsilon\colon L_2(\Omega)^n \to L_2(\Omega)^n$ by

$$\mathcal{K}_\mu^\varepsilon f(x) = \int_Q N_i(x+\varepsilon_i(\varepsilon)z, x/\varepsilon_i(\varepsilon)) D\mathcal{E}_i(\mathcal{A}^0-\mu)^{-1} f(x+\varepsilon_i(\varepsilon)z)\, dz$$

for $x \in \Omega_i$. It can be readily seen that $\mathcal{K}_\mu^\varepsilon$ is bounded as an operator from $L_2(\Omega)^n$ to each $H^1(\Omega_i)^n$ but, generally, not to $H^1(\Omega)^n$, which is not sufficient for our purposes. Thus, let $\rho_\varepsilon \in C^{0,1}(\bar{\Omega})$ be a cutoff function with support in the two-sided (nonsymmetric) $3\varepsilon_\#$-neighborhood of $\Gamma$ (see Fig. 2) such that $\rho_\varepsilon$ is identically 1 in the $2\varepsilon_\#$-neighborhood of $\Gamma$ and $\|\nabla\rho_\varepsilon\|_{L_\infty(\Omega_i)} \leqslant C\varepsilon_i(\varepsilon)^{-1}$. Now, if $\chi_\varepsilon = 1-\rho_\varepsilon$, then the range of $\varepsilon_\#\chi_\varepsilon\mathcal{K}_\mu^\varepsilon$ lies in $H^1(\Omega)^n$, and this is the operator that we shall use as a corrector for an approximation in the operator norm from $L_2(\Omega)^n$ to $H^1(\Omega)^n$.

We note that our corrector is basically just a classical corrector regularized with the Steklov smoothing and the cutoff function. From another point of view, it can be thought of as the sum of the regularized corrector $\varepsilon_\#\mathcal{K}_\mu^\varepsilon$, which first appeared in the locally periodic settings in [6] (and is good enough in that context), and a boundary-layer correction term $-\rho_\varepsilon\varepsilon_\#\mathcal{K}_\mu^\varepsilon$.

## 4. MAIN RESULTS

**Theorem 4.1.** *For any $\varepsilon \in \mathcal{E}$ and $f \in L_2(\Omega)^n$, we have*

$$\|(\mathcal{A}^\varepsilon-\mu)^{-1}f - (\mathcal{A}^0-\mu)^{-1}f\|_{L_2(\Omega)} \leqslant C\varepsilon_\vee \|f\|_{L_2(\Omega)}, \tag{4.1}$$

$$\|\nabla(\mathcal{A}^\varepsilon-\mu)^{-1}f - \nabla(\mathcal{A}^0-\mu)^{-1}f - \varepsilon_\#\nabla\chi_\varepsilon\mathcal{K}_\mu^\varepsilon f\|_{L_2(\Omega)} \leqslant C\varepsilon_\vee^{1/2} \|f\|_{L_2(\Omega)}, \tag{4.2}$$

*where $\varepsilon_\vee$ is the largest of $\varepsilon_i(\varepsilon)$. The constants can be written down explicitly in terms of $d$, $n$, $\mu$, the $C^{1,1}$ structures of $\Omega$ and $\Gamma$, the $C^{0,1}$ norms of $A_i$, and the constants $c_A$ and $C_A$.*

For locally periodic operators, the second estimate is improved as soon as we step away from the boundary [10, Corollary 6.5]. This is also true now if, in addition, we require that $A$ is Lipschitz on the corresponding subset and the periodic structure does not change there, e.g., if the subset intersects $\Gamma$. In the latter case, the extensions used in $\mathcal{K}_\mu^\varepsilon$ can be chosen (and we actually do so) in such a way that $\operatorname{ran}\mathcal{K}_\mu^\varepsilon \subset H^1(\Omega)^n$.

**Theorem 4.2.** *Suppose that $A$ is Lipschitz in the first variable on an open set $\Sigma$ with $\bar{\Sigma} \subset \Omega$ and that $\varepsilon_\#$ is constant on $\Sigma$. Then, for any $\varepsilon \in \mathcal{E}$ and $f \in L_2(\Omega)^n$,*

$$\|\nabla(\mathcal{A}^\varepsilon-\mu)^{-1}f - \nabla(\mathcal{A}^0-\mu)^{-1}f - \varepsilon_\#\nabla\mathcal{K}_\mu^\varepsilon f\|_{L_2(\Sigma)} \leqslant C\varepsilon_\vee \|f\|_{L_2(\Omega)}. \tag{4.3}$$

*The constants can be written down explicitly in terms of $d$, $n$, $\mu$, the $C^{1,1}$ structures of $\Omega$ and $\Gamma$, the $C^{0,1}$ norms of $A_i$, and the constants $c_A$ and $C_A$.*

## 5. SCHEME OF THE PROOF

Our approach here is an extension of that in [10]. The core of the proof is a suitable operator identity for the difference of $(\mathcal{A}^\varepsilon-\mu)^{-1}$ and the first-order approximation composed of $(\mathcal{A}^0-\mu)^{-1}$ and $\varepsilon_\#\chi_\varepsilon\mathcal{K}^\varepsilon_\mu$.

For $f\in L_2(\Omega)^n$ and $f^+\in (H^1(\Omega)^n)^*$, we set $u_0=(\mathcal{A}^0-\mu)^{-1}f$, $U_\varepsilon=\mathcal{K}^\varepsilon_\mu f$, and $u^+_\varepsilon=((\mathcal{A}^\varepsilon-\mu)^+)^{-1}f^+$, where $(\mathcal{A}^\varepsilon-\mu)^+$ stands for the adjoint of $\mathcal{A}^\varepsilon-\mu$. Then

$$((\mathcal{A}^\varepsilon-\mu)^{-1}f-(\mathcal{A}^0-\mu)^{-1}f-\varepsilon_\#\chi_\varepsilon\mathcal{K}^\varepsilon_\mu f,f^+)_{L_2(\Omega)^n} = (f,u^+_\varepsilon)_{L_2(\Omega)^n}-(u_0,f^+)_{L_2(\Omega)^n}-(\varepsilon_\#\chi_\varepsilon U_\varepsilon,f^+)_{L_2(\Omega)^n}. \quad (5.1)$$

Using the definition of $u_0$ and $u^+_\varepsilon$ again, we see that

$$(f,u^+_\varepsilon)_{L_2(\Omega)^n}-(u_0,f^+)_{L_2(\Omega)^n}=(A^0Du_0,Du^+_\varepsilon)_{L_2(\Omega)^n}-(A^\varepsilon Du_0,Du^+_\varepsilon)_{L_2(\Omega)^n}.$$

As for the last term in (5.1), we introduce a cutoff function $\rho'_\varepsilon\in C^{0,1}(\bar\Omega)$ that vanishes outside the $3\varepsilon_2$-neighborhood of $\partial\Omega$ and is identically 1 in the $2\varepsilon_2$-neighborhood of $\partial\Omega$ with $\|\nabla\rho'_\varepsilon\|_{L_\infty(\Omega_2)}\leqslant C\varepsilon_2(\varepsilon)^{-1}$. If $\xi_\varepsilon=\rho_\varepsilon+\rho'_\varepsilon$ and $\eta_\varepsilon=1-\xi_\varepsilon$, then $\varepsilon_\#\eta_\varepsilon U_\varepsilon$ belongs to $\mathcal{H}^1(\Omega)^n$, and we may write

$$(\varepsilon_\#\eta_\varepsilon U_\varepsilon,f^+)_{L_2(\Omega)^n}=(A^\varepsilon D\varepsilon_\#\eta_\varepsilon U_\varepsilon,Du^+_\varepsilon)_{L_2(\Omega)^n}-\mu(\varepsilon_\#\eta_\varepsilon U_\varepsilon,u^+_\varepsilon)_{L_2(\Omega)^n}.$$

Thus,

$$\begin{aligned}((\mathcal{A}^\varepsilon-\mu)^{-1}f-(\mathcal{A}^0-\mu)^{-1}f-\varepsilon_\#\chi_\varepsilon\mathcal{K}^\varepsilon_\mu f,f^+)_{L_2(\Omega)^n}&\\ =(\eta_\varepsilon A^0Du_0,Du^+_\varepsilon)_{L_2(\Omega)^n}-(\eta_\varepsilon A^\varepsilon D(u_0+\varepsilon_\#U_\varepsilon),Du^+_\varepsilon)_{L_2(\Omega)^n}+\mu(\varepsilon_\#\eta_\varepsilon U_\varepsilon,u^+_\varepsilon)_{L_2(\Omega)^n}&\\ +(\xi_\varepsilon(A^0-A^\varepsilon)Du_0,Du^+_\varepsilon)_{L_2(\Omega)^n}+(\varepsilon_\#A^\varepsilon D\xi_\varepsilon\cdot U_\varepsilon,Du^+_\varepsilon)_{L_2(\Omega)^n}-(\varepsilon_\#\rho'_\varepsilon U_\varepsilon,f^+)_{L_2(\Omega)^n}.&\end{aligned}$$

After rather intricate and lengthy calculations to appropriately extract those terms that live near either the surface $\Gamma$ or the boundary $\partial\Omega$, we arrive at an identity of the form

$$(\mathcal{A}^\varepsilon-\mu)^{-1}-(\mathcal{A}^0-\mu)^{-1}-\varepsilon_\#\chi_\varepsilon\nabla\mathcal{K}^\varepsilon_\mu|_{L_2(\Omega)^n}=\mathcal{I}^\varepsilon_\mu+\mathcal{B}^\varepsilon_\mu, \quad (5.2)$$

where $\mathcal{I}^\varepsilon_\mu$ and $\mathcal{B}^\varepsilon_\mu$ are the “interior” and the “boundary” parts (cf. [10, (8.11)]).

The terms in $\mathcal{B}^\varepsilon_\mu$ involve integration over the $5\varepsilon_\#$-neighborhoods of $\Gamma$ and $\partial\Omega$ and are handled with the following lemma (see [5, Lemma 5.1]).

**Lemma.** *Let $\Sigma$ be a bounded $C^{0,1}$ domain in $\mathbb{R}^d$ and $\partial\Sigma_\delta$ be the $\delta$-neighborhood of $\partial\Sigma$ in $\Sigma$. Then for any $\delta>0$ and $u\in H^1(\Sigma)$,*

$$\|u\|^2_{L_2(\partial\Sigma_\delta)}\leqslant C\delta\|u\|_{H^1(\Sigma)}\|u\|_{L_2(\Sigma)}, \quad (5.3)$$

*where the constant depends on $d$ and the $C^{0,1}$ structure of $\Sigma$.*

With this lemma, we show that

$$\|\mathcal{B}^\varepsilon_\mu f\|_{H^1(\Omega)}\leqslant C\varepsilon_\vee^{1/2}\|f\|_{L_2(\Omega)} \quad (5.4)$$

for all $f\in L_2(\Omega)^n$ (cf. [10, Lemma 8.6]).

On the other hand, the terms in $\mathcal{I}^\varepsilon_\mu$ involve the integration over an interior of $\Omega\setminus\Gamma$ away from $\Gamma$ and $\partial\Omega$, and this is where homogenization actually takes place. Using the decomposition $\eta_\varepsilon=\eta_{1,\varepsilon}+\eta_{2,\varepsilon}$, where $\eta_{i,\varepsilon}=\eta_\varepsilon|_{\Omega_i}$ is a cutoff function supported in $\Omega_i$, we split each integral in these terms into two integrals, one over $\Omega_1$ and another over $\Omega_2$. The key point here is that the coefficient $A^\varepsilon$ is locally periodic in both $\Omega_1$ and $\Omega_2$, and, therefore, all these terms can be treated in exactly the same way as in the locally periodic case; see [10, Lemma 8.3]. As a result,

$$\|\mathcal{I}^\varepsilon_\mu f\|_{H^1(\Omega)}\leqslant C\varepsilon_\vee\|f\|_{L_2(\Omega)}. \quad (5.5)$$

The bounds (5.4) and (5.5) clearly imply (4.2). Once we have the approximation (4.2), a more careful analysis of the boundary part $\mathcal{B}^\varepsilon_\mu$ also yields that

$$\|\mathcal{B}^\varepsilon_\mu f\|_{L_2(\Omega)}\leqslant C\varepsilon_\vee\|f\|_{L_2(\Omega)} \quad (5.6)$$

(cf. [10, Lemma 8.7]). Combining this with (5.5), we obtain (4.1), which completes the proof of Theorem 4.1.

Theorem 4.2 is proved similarly, except that there is no boundary part in an analog of the operator identity (5.2), since the set $\Sigma$ is away from both the interface and the boundary. Namely, instead of (5.2), we now have

$$(\mathcal{A}^\varepsilon-\mu)^{-1}\eta(\mathcal{A}^\varepsilon-\mu)\eta'\big((\mathcal{A}^\varepsilon-\mu)^{-1}-(\mathcal{A}^0-\mu)^{-1}-\varepsilon_\#\nabla\mathcal{K}_\mu^\varepsilon\big)\big|_{L_2(\Omega)^n}=\mathring{\mathcal{I}}_\mu^\varepsilon, \tag{5.7}$$

where $\eta$ and $\eta'$ are $C^{0,1}$ cutoff functions such that $\eta|_\Sigma=1$ and $\eta'|_{\operatorname{supp}\eta}=1$; see [10, (8.21)]. As before, the interior part satisfies

$$\|\mathring{\mathcal{I}}_\mu^\varepsilon f\|_{H^1(\Omega)}\leqslant C\varepsilon_\#|_\Sigma\|f\|_{L_2(\Omega)}. \tag{5.8}$$

Setting $f_\varepsilon=\eta(\mathcal{A}^\varepsilon-\mu)\eta' v_\varepsilon$ with $v_\varepsilon=u_\varepsilon-u_0-\varepsilon U_\varepsilon$, we see that $\|f_\varepsilon\|_{\mathcal{H}^{-1}(\Omega)}\leqslant C\varepsilon_\#|_\Sigma\|f\|_{L_2(\Omega)}$. Then (4.3) follows from the Caccioppoli inequality

$$\|D\eta v_\varepsilon\|_{L_2(\Omega)}\leqslant C\big(\|v_\varepsilon\|_{L_2(\Omega)}+\|f_\varepsilon\|_{\mathcal{H}^{-1}(\Omega)}\big)$$

and from the estimate (4.1) applied to $v_\varepsilon$.

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