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# Trace Formulas for Schrödinger Operators on a Lattice

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**Abstract** – We consider Schrödinger operators with complex decaying potentials (in general, not of trace class) on a lattice. We determine trace formulas in terms of the eigenvalues and the singular measure and some integrals of a Fredholm determinant. The proof is based on estimates of the free resolvent and analysis of functions in Hardy spaces. Moreover, we obtain an estimate for eigenvalues and singular measure in terms of potentials.

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## 1. INTRODUCTION AND MAIN RESULTS

### *1.1. Introduction*

We consider Schrödinger operators $H = \Delta + V$ on the lattice $\mathbb{Z}^d, d \geqslant 3$, where $\Delta$ is the discrete Laplacian on $\ell^2(\mathbb{Z}^d)$ given by

$$\big(\Delta f\big)(n) = \frac{1}{2} \sum_{|n-m|=1} f_m, \qquad n = (n_j)_1^d \in \mathbb{Z}^d, \tag{1.1}$$

and $f = (f_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. We assume that the potential $V$ is complex valued and satisfies the condition

$$(Vf)(n) = V_n f_n, \qquad V \in \ell^p(\mathbb{Z}^d), \qquad \begin{cases} 1 \leqslant p < \frac{6}{5} & \text{if } \ d = 3, \\ 1 \leqslant p < \frac{4}{3} & \text{if } \ d \geqslant 4. \end{cases} \tag{1.2}$$

Here $\ell^p(\mathbb{Z}^d), p \geqslant 1$, is the space of sequences $f = (f_n)_{n\in\mathbb{Z}^d}$ equipped with the norm given by

$$\|f\|_p = \|f\|_{\ell^p(\mathbb{Z}^d)} = \begin{cases} \sup_{n\in\mathbb{Z}^d} |f_n|, & p = \infty, \\ \big(\sum_{n\in\mathbb{Z}^d} |f_n|^p\big)^{\frac{1}{p}}, & p \in [1, \infty). \end{cases}$$

It is known that the spectrum of the Laplacian $\Delta$ is absolutely continuous and is equal to

$$\sigma(\Delta) = \sigma_{\rm ac}(\Delta) = [-d, d].$$

Note that, if $V$ satisfies (1.2), then $V$ is a compact operator and the essential spectrum of the Schrödinger operator $H$ is given by $\sigma_{\rm ess}(H) = [-d, d]$. The operator $H$ has $N \leqslant \infty$ eigenvalues $\{\lambda_j, j = 1, \ldots, N\}$ outside the interval $[-d, d]$. Here and below, every eigenvalue is counted according to its algebraic multiplicity. Denote the set of these eigenvalues by $\sigma_d$.

We discuss trace formulas for operators $H$ with complex potentials. Recall that, in general, a trace formula is an identity connecting the integral of the potential and various sums of eigenvalues and integrals of coefficients of the $S$-matrix of the Schrödinger operator (or other spectral characteristics). We briefly describe results about trace formulas.

Real potentials:

• In 1960, Buslaev and Faddeev [6] determined the classical results about trace formulas for Schrödinger operators with decaying potentials on half-line.

• There are a lot of results about the one-dimensional case, applied to integrable nonlinear equations, see, e.g., the paper of Faddeev and Zakharov [8] about the KdV equation, see also [20] and the references therein.

• The multidimensional case was studied in [5], see also [12, 41, 42] and the references therein. Trace formulas for Stark operators and magnetic Schrödinger operators were discussed in [32, 31].

• Trace formulas for Schrödinger operators in terms of resonances were discussed in the case of $\mathbb{R}^3$ in [17] and in the case of $\mathbb{R}_+$ in [27].

• The trace formulas for Schrödinger operators with real periodic potentials on the real line were obtained in [19, 25]. They were used to obtain two-sided estimates for the potential in terms of gap lengths (or the action variables for KdV) in [26] via the conformal mapping theory for the quasimomentum.

• The trace formulas for multidimensional Schrödinger operators on the lattice $\mathbb{Z}^d$ with real decaying potentials were obtained in [16].

Unfortunately, we know only few papers about the trace formulas for Schrödinger operators with complex-valued potentials decaying at infinity:

• Trace formulas for Schrödinger operators with complex potentials were considered in the case of $\mathbb{R}_+$ in [23] and in the case of $\mathbb{R}^3$ in [24].

• Trace formulas for Schrödinger operators with complex potentials were considered on the lattice $\mathbb{Z}^d$ in [38, 39] and in [28] for $|V|^{\frac{2}{3}} \in \ell^1(\mathbb{Z}^d)$.

We shortly discuss results about spectral properties of discrete self-adjoint Schrödinger operators. There are a lot of papers about self-adjoint Schrödinger operators on periodic graphs and in particular on the lattice $\mathbb{Z}^d$. Most of the results were obtained for $\mathbb{Z}^1$, see, for example, [46]. Schrödinger operators with decaying potentials on the lattice $\mathbb{Z}^d, d \geqslant 2$, have been considered by Boutet de Monvel–Sahbani [4], Isozaki–Korotyaev [16], Isozaki–Morioka [18], Kopylova [22], Korotyaev–Moller [30], Korotyaev–Slousch [35], Rosenblum–Solomjak [43], Shaban–Vainberg [44], and see the references therein. The scattering on other graphs was discussed by Ando [1], Korotyaev–Saburova [33] and Parra–Richard [40]. Note that estimates of negative eigenvalues and their number were discussed in [35, 43]. However, the methods of [35, 43] do not work for complex-valued potentials. For the nonself-adjoint operators, we mention [3, 13] on $\mathbb{Z}$ and [28] on $\mathbb{Z}^d, d \geqslant 3$, and see the references therein.

Define the cut domain $\Lambda = \mathbb{C} \setminus [-d, d]$. Introduce the free resolvent $R_0(\lambda) = (\Delta - \lambda)^{-1},\ \lambda \in \Lambda$. We assume that a potential $V$ satisfies 1.2 and define the regularized determinant $\psi$ by

$$\psi(\lambda) = \det\Big[(I + VR_0(\lambda))e^{-VR_0(\lambda)}\Big], \qquad \lambda \in \Lambda = \mathbb{C} \setminus [-d, d]. \tag{1.3}$$

It is similar to the continuous case of [24] for Schrödinger operators with complex potentials on $\mathbb{R}^3$, where the corresponding trace formulas were discussed. The function $\psi$ is a suitable regularization of the undefined determinant $\det(I + VR_0(\lambda))$, since $V$ is not a trace class operator, see [14]. The function $\psi$ is the basic function in our study.

Our main goal is to determine trace formulas for the Schrödinger operators $H = \Delta + V$ on the lattice, when, in general, a perturbation $V$ is not a trace class operator. We discuss the modified Fredholm determinant $\psi$, defined by 1.3, in the upper half plane $\mathbb{C}_+$. In this case, we obtain the trace formulas, for example, 1.14. Here the first term is $B_0^+ = 2\sum_{\lambda_j \in \mathbb{C}_+} \operatorname{Im} \lambda_j \geqslant 0$, and the second term is $\nu^+(\mathbb{R})$, where $\nu^+$ is a singular compactly supported measure. The third term is the integral of $\log|\psi|$ on the real line. Moreover, we determine similar trace formulas with eigenvalues in the lower half-plane $\mathbb{C}_-$ with the term $B_0^- = 2\sum_{\lambda_j \in \mathbb{C}_-} |\operatorname{Im} \lambda_j| \geqslant 0$,... Moreover, we determine another trace formula, associated with some domain on the complex plane.

In the proof, we combine the technique of [24] for Schrödinger operators on $\mathbb{R}^3$, the free resolvent estimates of [30], and classical results about Hardy spaces on the half plane and, in particular, we use the so-called canonical factorization of analytic functions in Hardy spaces via its inner and outer factors. This gives us new trace formulas for discrete Schrödinger operators $H = \Delta + V$ on the lattice, where the potential $V$ is complex and satisfies the condition 1.2 and which are not trace class operators, in general. Moreover, using the conformal mapping of $\Lambda = \mathbb{C} \setminus [-d, d]$ onto the unit disk and using the Hardy spaces in the disk, we improve results of [28], where the potentials are considered under the weaker condition $|V|^{\frac{2}{3}} \in \ell^1(\mathbb{Z}^d)$; moreover, we specify constants in different estimates. This gives us a new class of trace formulas for Schrödinger operators with complex-valued potentials on the lattice for which there exists an additional term associated with singular measure.

### *1.2. Main Results*

We describe basic properties of the function $\psi$.

**Theorem 1.1.** *Let $V$ satisfy* 1.2, *and let a constant $C_*$ be defined by* 2.26. *Then the modified determinant $\psi$ is analytic in the domain $\Lambda = \mathbb{C} \setminus [-d, d]$, is Hölder continuous up to the boundary, and satisfies*

$$\sup_{\lambda \in \Lambda} |\psi(\lambda)| \leqslant e^{C_*^2 \|V\|_p^2/2}. \tag{1.4}$$

*Moreover, the function* $\log\psi(\lambda)$ *is analytic in the domain* $\{|\lambda| > d + \|V\|\}$ *and has the Taylor series*

$$\log\psi(\lambda) = -\sum_{n\geqslant 2}\frac{Q_{n-1}}{\lambda^n} = -\frac{Q_1}{\lambda^2} - \frac{Q_2}{\lambda^3} - \frac{Q_3}{\lambda^4} - \cdots,$$
$$Q_1 = \frac{1}{2}\operatorname{Tr} V^2, \quad Q_{n-1} = \frac{1}{n}\operatorname{Tr}\left(H^n - H_0^n - nH_0^{n-1}V\right), \quad n \geqslant 3. \tag{1.5}$$

**Remark 1.** In the proof of 1.4, we use the following fact from [30]: an operator-valued function $\lambda \to Y(\lambda) = |V|^{\frac{1}{2}}R_0(\lambda)|V^{\frac{1}{2}}$, where $|V|^{\frac{1}{2}}V^{\frac{1}{2}} = V$, acting from $\Lambda$ to $\mathcal{B}_2$ (i.e., of the Hilbert–Schmidt class) is analytic in the domain $\Lambda$, is Hölder continuous up to the boundary, and satisfies $\|Y(\lambda)\|_{\mathcal{B}_2} \leqslant C_*\|V\|_p$. Note there are estimates of $\sup_{\lambda\in\Lambda}\|Y(\lambda)\|$ in [16, 45].

We define the Hardy space in a domain $\mathscr{D}$, where $\mathscr{D}$ is the half-plane or the disk. We say a function $F$ belongs to the Hardy space $\mathscr{H} = \mathscr{H}_\infty(\mathscr{D})$ if $F$ is analytic in $\mathscr{D}$ and satisfies

$$\|F\|_{\mathscr{H}} := \sup_{\lambda\in\mathscr{D}}|F(\lambda)| < \infty.$$

Theorem 1.1 shows that the function $\psi^+ := \psi|_{\mathbb{C}_+}$ belongs to the Hardy space $\mathscr{H}_\infty(\mathbb{C}_+)$. In order to study the zeros of $\psi$ in the half-plane $\mathbb{C}_+ = \{\operatorname{Im} z > 0\}$, we define the Blaschke product by

$$B^+(\lambda) = \prod_{\lambda_j\in\mathbb{C}_+}\left(\frac{\lambda-\lambda_j}{\lambda-\overline{\lambda}_j}\right), \qquad \lambda\in\mathbb{C}_+. \tag{1.6}$$

Here the Blaschke product $B^+(\lambda)$ converges absolutely for $\lambda\in\mathbb{C}_+$, since, due to 1.4, 1.5, all zeros are uniformly bounded, and $|B^+(\lambda)| \leqslant 1$ for all $\lambda\in\mathbb{C}_+$ (see, e.g., [21] or [12]). The Blaschke product $B^+$ has an analytic continuation from $\mathbb{C}_+$ to the domain $\{|\lambda| > r_o\}$, where $r_o = \sup|\lambda_j|$, and has the following Taylor series:

$$\log B^+(\lambda) = -i\frac{B_0^+}{\lambda} - i\frac{B_1^+}{2\lambda^2} - i\frac{B_2^+}{3\lambda^3} - \cdots \qquad \text{as} \qquad |\lambda| > r_o, \tag{1.7}$$

where $B_0^+ = 2\sum_{\lambda_j\in\mathbb{C}_+}\operatorname{Im}\lambda_j$ and $B_n^+ = 2\sum_{\lambda_j\in\mathbb{C}_+}\operatorname{Im}\lambda_j^{n+1}$ for all $n\geqslant 1$, see more in Section 3. We describe the canonical factorization of $\psi^+$ in the domain $\mathbb{C}_+$.

**Theorem 1.2.** *Let a potential* $V$ *satisfy* 1.2. *Then* $\psi^+ := \psi|_{\mathbb{C}_+}$ *has a canonical factorization in* $\mathbb{C}_+$ *given by*

$$\psi^+ = \psi_{in}^+\psi_{out}^+, \qquad \psi_{in}^+ = B^+e^{-iK^+}, \qquad \psi_{out}^+ = e^{iM^+},$$
$$K^+(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{d\nu^+(t)}{\lambda-t}, \qquad M^+(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{\log|\psi^+(t)|}{\lambda-t}dt, \quad \lambda\in\mathbb{C}_+. \tag{1.8}$$

• *Here* $d\nu^+(t)\geqslant 0$ *is a singular compactly supported measure on* $\mathbb{R}$ *and, for some* $r_*>0$, *it satisfies*

$$\nu^+(\mathbb{R}) = \int_{\mathbb{R}}d\nu^+(t) < \infty, \qquad \operatorname{supp}\nu^+ \subset \{\lambda\in\mathbb{R} : \psi^+(\lambda) = 0\} \subset [-r_*, r_*]. \tag{1.9}$$

• *The function* $K^+$ *has an analytic continuation from* $\mathbb{C}_+$ *to the domain* $\mathbb{C}\setminus[-r_*, r_*]$ *and has the following Taylor series*:

$$K^+(\lambda) = \sum_{j=0}^{\infty}\frac{K_j^+}{\lambda^{j+1}}, \qquad K_j^+ = \frac{1}{\pi}\int_{\mathbb{R}}t^j d\nu^+(t). \tag{1.10}$$

• $h^+ := \log|\psi^+(\cdot + i0)| \in L^1(\mathbb{R})$, *and the function* $M^+$ *satisfies*

$$M^+(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{\log|\psi^+(t+i0)|}{\lambda-t}dt = \frac{\mathcal{J}_0^+}{\lambda} + \frac{\mathcal{J}_1^+ - iI_1^+}{\lambda^2} + \cdots, \tag{1.11}$$

*as* $\operatorname{Im}\lambda\to\infty$, *where*

$$I_j^+ = \operatorname{Re}Q_j, \quad h_j^+ = t^{j+1}(h^+ - P_j^+), \quad P_j^+ = -\frac{I_0^+}{t} - \frac{I_1^+}{t^2} - \cdots - \frac{I_j^+}{t^{j+1}},$$
$$\mathcal{J}_0^+ = \frac{1}{\pi}\int_{\mathbb{R}}h^+(t)dt, \quad \mathcal{J}_1^+ = \text{v.p.}\frac{1}{\pi}\int_{\mathbb{R}}th^+(t)dt, \quad \mathcal{J}_j^+ = \text{v.p.}\frac{1}{\pi}\int_{\mathbb{R}}h_{j-1}^+(t)dt, \tag{1.12}$$

$j = 2, 3, \ldots$ *Here all integrals in* (1.12) *converge, since* $\psi$ *satisfies* (1.5).

**Remark 2.** In the proof of the theorem, we use results of [24] about the asymptotics 1.11.

We present our main result about trace formulas. Recall that $\sigma_d = \{\lambda_j, j = 1, \ldots\}$.

**Theorem 1.3.** *Let a potential* $V$ *satisfy* 1.2. *Then the following trace formula holds true*:

$$\operatorname{Tr}\Big(R(\lambda) - R_0(\lambda) + R_0(\lambda)VR_0(\lambda)\Big) = \frac{i}{\pi}\int_{\mathbb{R}} \frac{d\mu^+(t)}{(t-\lambda)^2} - \sum_{\lambda_j \in \mathbb{C}_+} \frac{2i \operatorname{Im} \lambda_j}{(\lambda-\lambda_j)(\lambda-\overline{\lambda}_j)}, \tag{1.13}$$

*for any* $\lambda \in \mathbb{C}_+ \setminus \sigma_d$, *where the measure is* $d\mu^+(t) = \log|\psi^+(t)|dt - d\nu^+(t)$, *and the series converges uniformly in every bounded disk in* $\mathbb{C}_+ \setminus \sigma_d$. *Moreover, trace formulas hold true*:

$$B_0^+ + \frac{\nu^+(\mathbb{R})}{\pi} = \frac{1}{\pi}\int_{\mathbb{R}} \log|\psi^+(t)|dt, \tag{1.14}$$

$$\frac{B_1^+}{2} + K_1^+ = \operatorname{Im} Q_1 + \mathcal{J}_1^+, \qquad \frac{B_j^+}{j+1} + K_j^+ = \operatorname{Im} Q_j + \mathcal{J}_j^+, \quad j = 2, 3, \ldots \tag{1.15}$$

**Remark 3.** (1) The measure $d\mu^+(t)$ in (1.13) is some analog of the spectral shift function for complex potentials [36] (see also a recent paper [38] and the references therein).

(2) The trace formula 1.14 has the term $\nu^+(\mathbb{R})$, which is absent for real potentials. There is an open problem: when this term is absent (or exists) for specific complex potentials?

We have discussed the properties of the function $\psi$ in $\mathbb{C}_+$. We can use similar arguments to study properties of $\psi$ in $\mathbb{C}_-$. We define the function $\psi^-(\zeta) := \psi(-\zeta), \zeta \in \mathbb{C}_+$. The function $\psi^-$ is analytic in $\mathbb{C}_+$ and has zeros $\zeta_j = -\lambda_j \in \mathbb{C}_+$, where $\lambda_j \in \mathbb{C}_-$ are zeros of $\psi$ in $\mathbb{C}_-$. Define the Blaschke product by $B^-(\zeta) = \prod_{\zeta_j \in \mathbb{C}_+} \frac{\zeta-\zeta_j}{\zeta-\overline{\zeta}_j}$ for $\zeta \in \mathbb{C}_+$. The function $B^-$ has an analytic continuation from $\mathbb{C}_+$ to the domain $\{|\zeta| > r_o\}$ and has the following Taylor series:

$$B^-(\zeta) = -i\frac{B_0^-}{\zeta} - i\frac{B_1^-}{2\zeta^2} - i\frac{B_2^-}{3\zeta^3} - \cdots \qquad \text{as} \qquad |\lambda| > r_o = \sup|\lambda_j|, \tag{1.16}$$

where $B_0^- = 2\sum_{\lambda_j \in \mathbb{C}_-} |\operatorname{Im} \lambda_j|, \ldots$ Thus, we obtain a similar canonical factorization of $\psi^-$.

**Corollary 1.4.** *Let a potential* $V$ *satisfy* 1.2. *Then* $\psi^-(\zeta) := \psi(-\zeta), \zeta \in \mathbb{C}_+$, *has a canonical factorization in* $\mathbb{C}_+$ *given by*

$$\psi^- = B^- e^{-iK^- + iM^-}, \qquad K^-(\zeta) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\nu^-(t)}{\zeta - t}, \quad M^-(\zeta) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\log|\psi^-(t+i0)|}{\zeta - t} dt, \tag{1.17}$$

$\zeta \in \mathbb{C}_+$. *Here* $d\nu^-(t) \geqslant 0$ *is a singular compactly supported measure on* $\mathbb{R}$, *and it satisfies* $\int_{\mathbb{R}} d\nu^-(t) < \infty$. *Moreover, the following trace formula holds true*:

$$B_0^- + \frac{\nu^-(\mathbb{R})}{\pi} = \frac{1}{\pi}\int_{\mathbb{R}} \log|\psi^-(t+i0)|dt. \tag{1.18}$$

We estimate the distance $\varrho_j := \operatorname{dist}\{\lambda_j, \sigma(\Delta)\}$ and the singular measure in terms of potentials.

**Theorem 1.5.** *Let* $V$ *satisfy* 1.2, *and let* $\varrho_j = \operatorname{dist}\{\lambda_j, \sigma(\Delta)\}$. *Then the following estimate holds true*:

$$\frac{\nu^+(\mathbb{R})}{2\pi} + \frac{\nu^-(\mathbb{R})}{2\pi} + \sum \varrho_j \leqslant 4(1 + (d+1)C_*^2)\|V\|_p^2. \tag{1.19}$$

**Remark 4.** (1) We briefly describe the proof. In order to obtain estimates for the global distance $\sum \varrho_j$, we need to consider additional conformal mappings. The simple conformal mapping $k(\lambda) = \sqrt{\lambda^2 - d^2}$ transforms the domain $\Lambda$ to the cut domain $\mathbb{K} = \mathbb{C} \setminus [-id, id]$. The function $\lambda(k) = \sqrt{k^2 + d^2}$ is the inverse mapping. Define the function $\tilde{\psi}(k) = \psi(\lambda(k))$ for $k \in \mathbb{K}$. The function $\tilde{\psi}(k)$ is analytic in $\mathbb{K}_+ = \{\operatorname{Re} k > 0\}$ and belongs to $\mathscr{H}_\infty(\mathbb{K}_+)$. For this function $\tilde{\psi}(k)$, in the half plane $\mathbb{K}_+$, we can use the above results and obtain the new trace formulas for the $k$-plane. In this case, instead of $\operatorname{Im} \lambda_j$, we have $\operatorname{Re} k(\lambda_j)$. Note that, using the simple estimate $|\operatorname{Im} \lambda_j| + |\operatorname{Re} k(\lambda_j)| \geqslant \varrho_j = \operatorname{dist}\{\lambda_j, \sigma(\Delta)\}$, we can estimate the global distance $\sum \varrho_j$ plus the singular measure in terms of $\|V\|_p^2$. We discuss this case in Section 3.

(2) There are many papers concerning the eigenvalues of Schrödinger operators in $\mathbb{R}^d$ with complex-valued potentials decaying at infinity, and we mention only recent results. Bounds on sums of powers of eigenvalues were obtained in [11, 9, 34]; see the references therein. Note that, in [9], the author estimates the sum of the distances between the complex eigenvalues and the continuous spectrum $[0, \infty)$ in terms of $L^p$-norms of the potentials.

We briefly describe the plan of the paper. In Section 2 we present the main properties of the Fredholm determinant. In Section 3, we prove the main theorems. In Section 4, we prove trace formulas for the disk.

## 2. FREDHOLM DETERMINANTS

### *2.1. Trace Class Operators*

Let $\mathcal{B}$ denote the class of bounded operators. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the trace and the Hilbert–Schmidt classes equipped with the norms $\|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{\mathcal{B}_2}$, respectively. Recall some well-known facts; see, e.g., [14].

• Let $A, B \in \mathcal{B}$ and $X, AB, BA \in \mathcal{B}_1$. Then

$$\operatorname{Tr} AB = \operatorname{Tr} BA, \tag{2.1}$$

$$\det(I + AB) = \det(I + BA), \tag{2.2}$$

$$|\det(I + X)| \leqslant e^{\|X\|_{\mathcal{B}_1}}. \tag{2.3}$$

• Let an operator-valued function $\Omega : \mathscr{D} \to \mathcal{B}_1$ be analytic in some domain $\mathscr{D} \subset \mathbb{C}$. Then the function $F(\lambda) = \det(I+\Omega(\lambda))$ is analytic in $\mathscr{D}$. If, in addition, $(I+\Omega(\lambda))^{-1} \in \mathcal{B}$ for any $\lambda \in \mathscr{D}$, then the function $F(\lambda)$ satisfies

$$F'(\lambda) = F(\lambda)\mathrm{Tr}\ \Omega(\lambda)^{-1}\Omega'(\lambda). \tag{2.4}$$

• In the case of $A \in \mathcal{B}_2$, the modified determinant $\det_2(I + A)$ is defined by

$$\det{}_2(I + A) = \det\Big((I + A)e^{-A}\Big). \tag{2.5}$$

The modified determinant satisfies (see (2.2) in Chapter IV, [14])

$$|\det{}_2(I + A)| \leqslant e^{\frac{1}{2}\|A\|^2_{\mathcal{B}_2}}, \tag{2.6}$$

and $I + A$ is invertible if and only if $\det_2(I + A) \neq 0$.

### *2.2. Krein's Results*

Recall the famous results of Krein about the trace formulas for bounded self-adjoint operators $H = H_o + V, H_o$ on a Hilbert space $\mathcal{H}$, where $V$ is a trace class operator. Define the determinant

$$D(\lambda) = \det(I + V(H_o - \lambda)^{-1}), \quad \lambda \in \mathbb{C}_\pm.$$

In this case, the function $D(\lambda), \lambda \in \mathbb{C}_\pm$, is analytic on $\mathbb{C} \setminus [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$. Let us recall the basic properties of $\xi(\lambda)$ from [36, 37, 2]:

(1) *The determinant $D(\lambda)$ has the following form*:

$$\log D(\lambda) = \int_{\mathbb{R}} \frac{\xi(t)}{t - \lambda} dt, \quad \lambda \in \mathbb{C}_+, \tag{2.7}$$

*where the branch of* $\log D(\lambda)$ *is chosen so that* $\log D(\lambda) = o(1)$ *as* $|\lambda| \to \infty$, *and* $\xi(t) \in L^1(\mathbb{R})$. *We have*

$$\xi(t) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \arg D(t + i\varepsilon) \qquad a.e.\ t \in \mathbb{R}, \tag{2.8}$$

*where the limit on the right-hand side exists for a.e.* $t \in \mathbb{R}$. *The support satisfies* $\operatorname{supp}\xi \subset [\alpha - \|V\|, \beta + \|V\|]$, *where* $\sigma(H_o) \subset [\alpha, \beta]$, *and we have*

$$\int_{\mathbb{R}} |\xi(\lambda)| d\lambda \leqslant \|V\|_{\mathcal{B}_1}, \tag{2.9}$$

$$\int_{\mathbb{R}} \xi(\lambda) d\lambda = \mathrm{Tr}\,(V). \tag{2.10}$$

(2) *The relation to the S-matrix is*

$$\det \mathcal{S}(\lambda) = e^{-2\pi i \xi(\lambda)} \quad \textit{for a.e. } \lambda \in \sigma_{ac}(H_o). \tag{2.11}$$

(3) *If $h$ has $N_- \geqslant 0$ negative eigenvalues and $N_+ \geqslant 0$ positive eigenvalues, then*

$$-N_- \leqslant \xi(\lambda) \leqslant N_+ \quad \textit{for a.e.} \quad \lambda \in \mathbb{R}. \tag{2.12}$$

(4) *Suppose that $H_o$ has no eigenvalues in an interval $(a, b) \subset \mathbb{R}$. Assume that $\lambda_0 \in (a, b)$ is an isolated eigenvalue of finite multiplicity $d_0$ of $H$. Then $\xi(\lambda)$ takes an integer value $n_-$ ($n_+$) on the interval $(a, \lambda_0)$ (on the interval $(\lambda_0, b)$). Moreover, we have*

$$\xi(\lambda_0 + 0) - \xi(\lambda_0 - 0) = -d_0. \tag{2.13}$$

(5) *If $V \geqslant 0$ (or $V \leqslant 0$), then $\xi(\lambda) \geqslant 0$ (or $\xi(\lambda) \leqslant 0$) for all $\lambda \in \mathbb{R}$.*
(6) *If the perturbation $V$ has rank $N < \infty$ then $-N \leqslant \xi(\lambda) \leqslant N$ for all $\lambda \in \mathbb{R}$.*
(7) *The following identity holds true (see e.g., [16]) :*

$$\log D(\lambda) = -\sum_{n \geqslant 1} \frac{F_n}{n\lambda^n}, \tag{2.14}$$

*where the right-hand side is uniformly convergent on $\{|\lambda| > r_0\}$ for $r_0 > 0$ large enough and*

$$\begin{aligned} F_n &= n \int_{\mathbb{R}} \xi(t) t^{n-1} dt = \mathrm{Tr}\,(H^n - H_0^n), \quad n \geqslant 1, \\ F_1 &= \mathrm{Tr}\ V = \int_{\mathbb{R}} \xi(t) dt, \quad F_2 = \mathrm{Tr}\,(2H_o V + V^2) = 2\int_{\mathbb{R}} t\xi(t) dt, \dots \end{aligned} \tag{2.15}$$

*2.3. Fredholm Determinant*

Consider the bounded operators $V$ and $H_0$ acting on the Hilbert space $\mathcal{H}$. Define the operator $H = H_0 + V$. Introduce the resolvents

$$R_0(\lambda) = (H_0 - \lambda)^{-1}, \quad \lambda \notin \sigma(H_0) \qquad \text{and} \qquad R(\lambda) = (H - \lambda)^{-1}, \quad \lambda \notin \sigma(H).$$

For $V \in \mathcal{B}_2$, we define the regularized determinant $\mathcal{D}$ by

$$\mathcal{D}(\lambda) = \det\Big[(I + VR_0(\lambda))e^{-VR_0(\lambda)}\Big], \quad \lambda \notin \sigma(H_0). \tag{2.16}$$

This determinant is well defined, since

$$\|VR_0(\lambda)\|_{\mathcal{B}_2} \leqslant \frac{\|V\|_{\mathcal{B}_2}}{\mathrm{dist}\{\lambda, \sigma(H_0)\}} \quad \text{for} \quad \lambda \notin \sigma(H_0). \tag{2.17}$$

If, in addition, an operator $H_0 \in \mathcal{B}$ is self-adjoint, then every zero of $\mathcal{D}(\lambda)$ outside $\sigma(H_0)$ is an eigenvalue of $H = H_0 + V$ with some algebraic multiplicity. For $\lambda \notin \sigma(H_0)$, the eigenvalue problem $(H - \lambda)u = 0$ is equivalent to $(I + (H_0 - \lambda)^{-1} V)u = 0$, which has a nontrivial solution if and only if $\mathcal{D}(\lambda) = 0$. We describe the modified determinant $\mathcal{D}(\lambda)$.

**Lemma 2.1.** *Let operators $V \in \mathcal{B}_2$ and $H_0 \in \mathcal{B}$ and the modified determinant $\mathcal{D}(\lambda)$ be defined by* 2.16. *Then $\mathcal{D}(\lambda)$ is analytic in $\{\lambda \in \mathbb{C} : |\lambda| > r_0\}$ for $r_0 = \|H_0\|$. Moreover,*

$$\mathcal{D}(\lambda) = 1 + O(1/\lambda^2) \quad \textit{as} \quad |\lambda| \to \infty, \tag{2.18}$$

$$\log \mathcal{D}(\lambda) = -\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \mathrm{Tr}\ (VR_0(\lambda))^n, \tag{2.19}$$

*where, due to* 2.18, *we take the branch of $\log \mathcal{D}$ such that $\log \mathcal{D}(\lambda) = o(1)$ as $|\lambda| \to \infty$, and*

$$\begin{aligned} \log \mathcal{D}(\lambda) &= -\sum_{n \geqslant 2} \frac{Q_{n-1}}{\lambda^n} = -\frac{Q_1}{\lambda^2} - \frac{Q_2}{\lambda^3} - \frac{Q_3}{\lambda^4} - \cdots, \\ Q_1 &= \frac{1}{2}\mathrm{Tr}\ V^2, \quad Q_{n-1} = \frac{1}{n}\mathrm{Tr}\left(H^n - H_0^n - nH_0^{n-1}V\right), \quad n \geqslant 2; \end{aligned} \tag{2.20}$$

*here the right-hand side is uniformly convergent on $\{\lambda \in \mathbb{C} : |\lambda| \geqslant r\}$ for $r = \|V\| + \|H_0\| + 1$. In particular,*

$$Q_2 = \frac{1}{3}\mathrm{Tr}\,(3V^2H_0 + V^3), \quad Q_3 = \frac{1}{4}\mathrm{Tr}\,(2VH_0VH_0 + 4V^2H_0^2 + 4V^3H_0 + V^4). \tag{2.21}$$

**Proof**. (i) The Taylor series for the entire function $e^{-w}$ and the estimate 2.17 give

$$[(I+w)e^{-w}] = (I+w)(1-w+w^2O(1)) = 1-w^2+w^2O(1) = I+w^2O(1)$$

at $w = VR_0(\lambda)$. We have, by the resolvent equation for $|\lambda| > r$,

$$R(\lambda) = R_0(\lambda) + \sum_{n=1}^{\infty}(-1)^n R_0(\lambda)\Big(VR_0(\lambda)\Big)^n = \sum_{n=0}^{\infty}(-1)^n R_0(\lambda)\Big(VR_0(\lambda)\Big)^n, \tag{2.22}$$

where the right-hand side is uniformly convergent on $\{\lambda \in \mathbb{C} : |\lambda| \geqslant r\}$. By (2.4) and (2.17), and using 2.1, we have for $|\lambda| > r$ the following equation:

$$\begin{aligned}\mathcal{D}'(\lambda) &= -\mathcal{D}(\lambda)\operatorname{Tr}\Big(VR(\lambda)V\Big(R_0(\lambda)\Big)'\Big) = -\mathcal{D}(\lambda)\operatorname{Tr}\Big(VR(\lambda)VR_0^2(\lambda)\Big)\\ &= -\mathcal{D}(\lambda)\operatorname{Tr}\Big(R_0(\lambda)VR(\lambda)VR_0(\lambda)\Big) = -\mathcal{D}(\lambda)\operatorname{Tr}\Big(R(\lambda)(VR_0(\lambda))^2\Big).\end{aligned} \tag{2.23}$$

Thus, (2.22) gives

$$(\log\mathcal{D}(\lambda))' = -\operatorname{Tr}\sum_{n=0}^{\infty}(-1)^n R_0(\lambda)\Big(VR_0(\lambda)\Big)^{n+2} = -\operatorname{Tr}\sum_{n=2}^{\infty}(-1)^n R_0(\lambda)\Big(VR_0(\lambda)\Big)^{n}.$$

Then, integrating and using

$$\frac{d}{d\lambda}\Big(\operatorname{Tr}\Big(VR_0(\lambda)\Big)^n\Big) = n\operatorname{Tr} R_0(\lambda)\Big(VR_0(\lambda)\Big)^n,$$

we obtain 2.19. The identities in 2.23 and $R = R_0 - RVR_0$ imply

$$(\log\mathcal{D}(\lambda))' = -\operatorname{Tr}\Big(R(\lambda) - R_0(\lambda) + R_0(\lambda)VR_0(\lambda)\Big) = -\operatorname{Tr}\Big(R(\lambda) - R_0(\lambda) + VR_0^2(\lambda)\Big).$$

Note that, for any bounded operator $A$ and for large $\lambda$, we have

$$(A-\lambda)^{-1} = -\frac{1}{\lambda}\sum_{n\geqslant 0}(A/\lambda)^n, \qquad |\lambda| > \|A\|,$$

where the series is absolutely convergent. Using this identity, we obtain

$$(\log\mathcal{D}(\lambda))' = \sum_{n=0}^{\infty}\frac{\operatorname{Tr}\left(H^n - H_0^n - nH_0^{n-1}V\right)}{\lambda^{n+1}} = \sum_{n=2}^{\infty}\frac{nQ_{n-1}}{\lambda^{n+1}}.$$

In view of 2.18, we have 2.20 and 2.21.

### *2.4. Estimates of Determinants*

Define the operator-valued function

$$Y(\lambda) = |V|^{\frac12}R_0(\lambda)|V|^{\frac12}e^{i\arg V}, \quad \lambda \in \Lambda,$$

where $R_0(\lambda) = (\Delta - \lambda)^{-1}$. We recall needed results from [30]:

*Let the potential $V$ satisfy* 1.2. *Then the operator-valued function $Y : \Lambda \to \mathcal{B}_2$ is analytic and Hölder continuous up to the boundary. Moreover, it satisfies*

$$\|Y(\lambda)\|_{\mathcal{B}_2} \leqslant C_*\|V\|_p \qquad \forall\ \lambda \in \Lambda, \tag{2.24}$$

$$\|Y(\lambda) - Y(\mu)\|_{\mathcal{B}_2} \leqslant C_\alpha|\lambda-\mu|^\alpha\|V\|_p \qquad \forall\ \lambda,\mu \in \overline{\mathbb{C}}_\pm, \tag{2.25}$$

*where $\alpha, C_\alpha$ are some positive constants and the constant $C_*$ is defined by*

$$C_* = p^{\frac{d(p-1)}{2p}} + c_d(3+2c)^{d-\frac{d}{p}}, \quad c_d = \begin{cases}16\\ 4\\ \frac{14\cdot 2^{\frac{d}{4}}}{d-4},\end{cases} \quad c = \begin{cases}\frac{6(p-1)}{6-5p} & if\ d=3\\ \left(\frac{5p-1}{4-3p}\right)^{\frac{5p-4}{4(p-1)}} & if\ d=4\\ \frac{3d(p-1)}{3d-(2d+1)p} & if\ d\geqslant 5.\end{cases} \tag{2.26}$$

**Proof of Theorem 1.1.** Due to results 2.24–2.25, the operator-valued function $Y : \Lambda \to \mathcal{B}_2$ is analytic on $\Lambda$ and is Hölder continuous up to the boundary. Then the determinant $\psi(\lambda)$ is analytic on $\Lambda$ and Hölder continuous up to the boundary, and 2.24, 2.6 implies 1.4, where the constant $C_*$ is defined in 2.26. The asymptotics 1.5 of the function $\psi(\lambda)$ have been proved in 2.20.

## 3. PROOF OF TRACE FORMULAS

### 3.1. Hardy Space in the Upper Half-Plane

We describe functions in the Hardy spaces, see [12, 21]. Let $f \in \mathscr{H}_\infty$, and let all its zeros $\{\lambda_j\}$ in $\mathbb{C}_+$ be uniformly bounded by $r_o$. In this case, we can define the Blaschke product by

$$B(\lambda) = \prod_{\lambda_j \in \mathbb{C}_+} \frac{\lambda - \lambda_j}{\lambda - \overline{\lambda}_j}, \qquad \lambda \in \mathbb{C}_+. \tag{3.1}$$

Then $B \in \mathscr{H}_\infty$ with $\|B\|_{\mathscr{H}_\infty} \leqslant 1$ and, for a.e. $\lambda \in \mathbb{R}$, we have

$$\lim_{\varepsilon \to +0} B(\lambda + i\varepsilon) = B(\lambda + i0), \qquad |B(\lambda + i0)| = 1. \tag{3.2}$$

The function $\log B(\lambda)$ has an analytic continuation from $\mathbb{C}_+ \setminus \{|\lambda| < r_o\}$ to the domain $\{|\lambda| > r_o\}$, where $r_o = \sup |\lambda_j|$ and has the following Taylor expansion:

$$\log B(\lambda) = -\frac{iB_0}{\lambda} - \frac{iB_1}{2z^2} - \frac{iB_2}{3\lambda^3} - \cdots - \frac{iB_{n-1}}{n\lambda^n} - \cdots, \tag{3.3}$$

where the coefficient $B_n = 2\sum_j \operatorname{Im} \lambda_j^{n+1}, n \geqslant 0$, satisfies

$$|B_n| \leqslant 2 \sum |\operatorname{Im} \lambda_j^{n+1}| < \infty \qquad \forall\, n \geqslant 0, \tag{3.4}$$

$$|B_n| \leqslant \frac{\pi}{2}(n+1) r_o^n B_0 \qquad \forall\, n \geqslant 1. \tag{3.5}$$

We recall results about the canonical factorization in the form convenient for us from [24].

**Theorem 3.1.** *Let $f \in \mathscr{H}_\infty(\mathbb{C}_+) \cap C(\overline{\mathbb{C}}_+)$ and for any $m \geqslant 1$, $f$ satisfies*

$$f(\lambda) = \exp\left[-\frac{Q_1}{\lambda^2} - \frac{Q_2}{\lambda^3} - \frac{Q_3}{\lambda^4} - \cdots - \frac{Q_m}{\lambda^{m+1}} + \frac{O(1)}{\lambda^{m+2}}\right] \tag{3.6}$$

*as $|\lambda| \to \infty$, uniformly with respect to $\arg k \in [0, \pi]$ for some constants $Q_j \in \mathbb{C}, j \in \mathbb{N}$. Then $f$ has a canonical factorization in $\mathbb{C}_+$ given by*

$$f = Be^{-iK + iM}, \qquad K(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\nu(t)}{\lambda - t}, \qquad M(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\log|f(t)|}{\lambda - t} dt, \quad \lambda \in \mathbb{C}_+. \tag{3.7}$$

- *$B$ is the Blaschke product given by* 3.1;
- *$d\nu(t) \geqslant 0$ is some singular compactly supported measure on $\mathbb{R}$ which satisfies the following condition for some $r_c > 0$ :*

$$\nu(\mathbb{R}) = \int_{\mathbb{R}} d\nu(t) < \infty, \qquad \operatorname{supp} \nu \subset \{\lambda \in \mathbb{R} : f(\lambda) = 0\} \subset [-r_c, r_c]. \tag{3.8}$$

- *The function $K(\cdot)$ has an analytic continuation from $\mathbb{C}_+$ to the domain $\mathbb{C} \setminus [-r_c, r_c]$ and has the following Taylor series*:

$$K(\lambda) = \sum_{j=0}^{\infty} \frac{K_j}{\lambda^{j+1}}, \qquad K_j = \frac{1}{\pi}\int_{\mathbb{R}} t^j d\nu(t). \tag{3.9}$$

- *Here $h := \log|f(\cdot)| \in L^1(\mathbb{R})$, and $M(\cdot)$ satisfies the following condition as $\lambda \in \overline{\mathbb{C}}_+$, $|\lambda| \to \infty$ :*

$$M(\lambda) = \frac{\mathcal{J}_0}{\lambda} + \frac{\mathcal{J}_1 - iI_1}{\lambda^2} + \cdots + \frac{\mathcal{J}_m - iI_m}{\lambda^{m+1}} + \frac{O(1)}{\lambda^{m+2}}, \tag{3.10}$$

*uniformly with respect to $\arg \lambda \in [0, \pi]$, where the real constants $I_j$ and $\mathcal{J}_j, j \geqslant 0$, are given by*

$$\begin{gathered} \mathcal{J}_0 = \frac{1}{\pi}\int_{\mathbb{R}} h(t)dt, \qquad \mathcal{J}_j = \text{v.p.}\frac{1}{\pi}\int_{\mathbb{R}} h_{j-1}(t)dt, \quad I_j = \operatorname{Re} Q_j, \\ h_j = t^{j+1}(h(t) - P_j(t)), \quad P_j(t) = -\frac{I_0}{t} - \frac{I_1}{t^2} + \cdots - \frac{I_j}{t^{j+1}}, \end{gathered} \tag{3.11}$$

- *The following trace formulas hold true*:

$$B_j + K_j = \mathcal{J}_j + \operatorname{Im} Q_j, \qquad j = 0, 1, \ldots \tag{3.12}$$

### *3.2. Proof of Main Theorems*

We are ready to prove our main results.

**Proof of Theorem 1.2.** From Theorem 1.1, we derive that $\psi^+ \in \mathscr{H}_\infty(\mathbb{C}_+)$ and $\psi^+$ has the asymptotics 1.5. Moreover, it satisfies all conditions in Theorem 3.1. Then Theorem 3.1 gives all needed results.

Now we are ready to determine trace formulas.

**Proof of Theorem 1.3.** Differentiating the modified determinant $\psi(\lambda), \lambda \in \mathbb{C}_+$, defined by 1.3 and using 2.4, we obtain

$$\frac{\psi'(\lambda)}{\psi(\lambda)} = -\operatorname{Tr}\Big(R(\lambda) - R_0(\lambda) + R_0(\lambda)VR_0(\lambda)\Big).$$

Recall that $d\mu^+(t) = \log|\psi(t+i0)|dt - d\nu^+(t)$. From the representation 1.8, we derive

$$\frac{\psi'(\lambda)}{\psi(\lambda)} = \frac{B^+(\lambda)'}{B^+(\lambda)} - iK^+(\lambda)' + iM^+(\lambda)' = \frac{B'(\lambda)}{B(\lambda)} + \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\mu^+(t)}{(t-\lambda)^2},$$

since

$$\frac{B^+(\lambda)'}{B^+(\lambda)} = \sum_{\lambda_j \in \mathbb{C}_+} \left(\frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \overline{\lambda}_j}\right) = \sum_{\lambda_j \in \mathbb{C}_+} \frac{2i \operatorname{Im} \lambda_j}{(\lambda - \lambda_j)(\lambda - \overline{\lambda}_j)}$$

and

$$K^+(\lambda)' = \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\nu^+(t)}{(t-\lambda)^2}, \qquad M^+(\lambda)' = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\log|\psi(t+i0)|dt}{(t-\lambda)^2}.$$

Collecting all these identities, we obtain 1.13. The canonical factorization 1.8 gives $\psi^+ = B^+ e^{-iK^+ + iM^+}$. Substituting the asymptotics from Theorems 1.1 and 1.2 into this identity, we obtain

$$\psi^+(\lambda) = e^{-\left(\frac{Q_1}{\lambda^2} + \frac{Q_2}{\lambda^3} + \cdots\right)} = e^{-i\left(\frac{B_0^+}{\lambda} + \frac{B_1^+}{2\lambda^2} + \frac{B_2^+}{3\lambda^3} + \cdots\right)} e^{-i\left(\frac{K_0^+}{\lambda^1} + \frac{K_1^+}{\lambda^2} + \cdots\right)} e^{i\left(\frac{\mathcal{J}_0^+}{\lambda} + \frac{\mathcal{J}_1^+ - iI_1^+}{\lambda^2} + \frac{\mathcal{J}_2^+ - iI_2^+}{\lambda^3} + \cdots\right)},$$

which yields 1.14 and 1.15.

**Theorem 3.2.** *Let $V$ satisfy* 1.2. *Then the following estimate holds true*:

$$B_0^+ + \frac{\nu^+(\mathbb{R})}{\pi} = \int_{\mathbb{R}} \log|\psi^+(t+i0)|dt \leqslant (1 + (d+1)C_*^2)\|V\|_p^2. \tag{3.13}$$

**Proof.** Let $\omega = [-d-1, d+1]$. We have the decomposition

$$\int_{\mathbb{R}} \log|\psi^+(t+i0)|dt = X_1 + X_2, \quad X_1 = \int_\omega \log|\psi^+(t+i0)|dt,$$

and, using 2.16, 2.17, we obtain

$$X_1 \leqslant \int_\omega \frac{C_*^2}{2}\|V\|_p^2 d\lambda = (d+1)C_*^2\|V\|_p^2,$$
$$X_2 = \int_{\mathbb{R}\setminus\omega} \log|\psi^+(t)|dt \leqslant \int_{\mathbb{R}\setminus\omega} \frac{\|V\|_{\mathcal{B}_2}^2 d\lambda}{2\operatorname{dist}\{\lambda, \sigma(H_0)\}^2} = \int_d^\infty \frac{\|V\|_2^2}{(\lambda - d - 1)^2} d\lambda = \|V\|_2^2,$$

which yields 3.13.

### *3.3. Trace Formulas for the Half-Plane* $\operatorname{Re}\lambda > 0$

In order to obtain estimates, we need to discuss the Hardy spaces for the half-plane $\pm\operatorname{Re}\lambda > 0$. We define the additional conformal mapping $k : \Lambda \to \mathbb{K}$ by

$$k(\lambda) = \sqrt{\lambda^2 - d^2}, \qquad \lambda \in \Lambda, \quad \mathbb{K} := \mathbb{C} \setminus [id, -id], \tag{3.14}$$

where the branch is defined by $k(\lambda) = \lambda - \frac{d^2}{2\lambda} + \frac{O(1)}{\lambda^3}$ as $|\lambda| \to \infty$. The function $k(\cdot)$ has the following properties:

- *The function $k(\cdot)$ is a conformal mapping from $\Lambda$ onto the spectral domain $\mathbb{K}$.*
- $k(\Lambda) = \mathbb{K}$ *and* $\lambda(\Lambda \cap \{\pm\operatorname{Re}\lambda > 0\}) = \mathbb{K}_\pm = \{\pm\operatorname{Re} k > 0\}$.

• *$\Lambda$ is the cut domain with the cut $[-d,d]$, having the upper side $[-d,d]+i0$ and the lower side $[-d,d]-i0$. The function $k(\lambda)$ takes the boundary to the sides of the cut $[0,\pm id]$ as follows: the upper side $[-d,d]+i0$ is taken onto the two-sided cut $[0,id]$ and the lower side $[-d,d]+i0$ onto the two-sided cut $[0,-id]$.*

• *The function $k(\lambda)$ takes the point $\lambda=0\pm i0$ to the point $k(\lambda)=\pm id$.*

• *The inverse mapping $\lambda:\mathbb{K}\to\Lambda$ is given by $\lambda(k)=\sqrt{k^2+d^2}$, $k\in\mathbb{K}$, and satisfies*

$$\lambda(k)=k+\frac{d^2}{2k}+\frac{O(1)}{k^3} \qquad \text{as} \quad |k|\to\infty. \tag{3.15}$$

Define the function $\tilde\psi$ on $\mathbb{K}$ by

$$\tilde\psi(k)=\tilde\psi(\lambda(k)), \qquad \lambda(k)=\sqrt{k^2+d^2}, \quad k\in\mathbb{K}. \tag{3.16}$$

The function $\tilde\psi(k)$ is analytic in $\mathbb{K}$ and has the zeros $k_j=k(\lambda_j), j=1,2,\dots$ The investigation of the function $\tilde\psi(k)$ is similar to the case of the function $\psi(\lambda)$. We define functions

$$\tilde\psi^{\pm}(\zeta):=\tilde\psi(\lambda(\mp i\zeta)), \quad \zeta\in\mathbb{C}_+. \tag{3.17}$$

From 3.16, we derive that the function $\tilde\psi^{\pm}$ belongs to the Hardy space $\mathscr{H}_\infty(\mathbb{C}_+)$. Thus, the function $\tilde\psi^{\pm}$ is analytic in the domain $\mathbb{C}_+$, is Hölder continuous up to the boundary and satisfies

$$\|\tilde\psi^{\pm}\|_{\mathscr{H}_\infty(\mathbb{C}_+)}\leqslant e^{C_*^2\|V\|_p^2/2}. \tag{3.18}$$

Moreover, the function $\log\tilde\psi(k)$ is analytic and has the following Taylor series:

$$\log\tilde\psi(k)=-\sum_{n\geqslant 2}\frac{Q_{n-1}}{\lambda^n(k)}=-\frac{\tilde Q_1}{k^2}-\frac{\tilde Q_2}{k^3}-\frac{\tilde Q_3}{k^4}-\cdots \tag{3.19}$$

in the domain $\{|k|>d+\|V\|\}$, where $\tilde Q_1=Q_1=\frac12\operatorname{Tr}V^2,\dots$ In order to study zeros of $\tilde\psi$ in $\mathbb{K}_+$, we need to define the Blaschke product $\widetilde B^{\pm}$ by

$$\tilde B^{\pm}(\zeta)=\prod_{\zeta_j=\pm ik_j,k_j\in\mathbb{K}_\pm}\left(\frac{\zeta-\zeta_j}{\zeta-\overline{\zeta}_j}\right), \qquad \zeta\in\mathbb{C}_+, \quad \zeta_j=\pm ik_j, \quad k_j=k(\lambda_j). \tag{3.20}$$

Here the Blaschke product $\tilde B^{\pm}$ converges absolutely for $\zeta\in\mathbb{C}_+$, since, due to 1.4 and 1.5, all zeros are uniformly bounded, and $|\tilde B^{\pm}(\zeta)|\leqslant 1$ for all $\zeta\in\mathbb{C}_+$ (see, e.g., [21] or [12]). The canonical factorization of $\tilde\psi^{\pm}$ is similar to the case of $\psi^{+}$ and has the following form.

**Theorem 3.3.** *Let $V$ satisfy* 1.2, *and let $\alpha=\pm$. Then $\tilde\psi^{\pm}(\zeta)=\psi(\lambda(\mp i\zeta)),\zeta\in\mathbb{C}_+$, has a canonical factorization given by*

$$\tilde\psi^{\alpha}=\tilde B^{\alpha}e^{-i\tilde K^{\alpha}+i\tilde M^{\alpha}}, \qquad \tilde K^{\alpha}(\zeta)=\frac{i}{\pi}\int_{\mathbb{R}}\frac{d\tilde\nu^{\alpha}(t)}{\zeta-t}, \quad \tilde M^{\alpha}(\zeta)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{\log|\tilde\psi^{\alpha}(t)|}{\zeta-t}dt. \tag{3.21}$$

• *$d\tilde\nu^{\alpha}(t)\geqslant 0$ is a singular compactly supported measure on $\mathbb{R}$ and, for some $r_*>0$, it satisfies*

$$\int_{\mathbb{R}}d\tilde\nu^{\alpha}(t)<\infty, \qquad \operatorname{supp}\tilde\nu^{\alpha}\subset\{z\in\mathbb{R}:\tilde\psi^{\alpha}(z)=0\}\subset[-r_*,r_*]. \tag{3.22}$$

• *The function $\tilde K^{\alpha}$ has an analytic continuation from $\mathbb{C}_+$ to the domain $\mathbb{C}\setminus[-r_*,r_*]$ and satisfies*

$$\tilde K^{\alpha}(\zeta)=\frac{\tilde\nu^{\alpha}(\mathbb{R})}{\zeta}+\frac{O(1)}{\zeta^2} \quad \text{as} \quad |\zeta|\to\infty.$$

• *Let $\tilde J_0^{\alpha}=\frac{1}{\pi}\int_{\mathbb{R}}\log|\tilde\psi^{\alpha}(t+i0)|dt$, where the function $\log|\tilde\psi^{\alpha}(t+i0)|$ belongs to $L^1(\mathbb{R})$ and $\tilde M^{\alpha}$ satisfies*

$$\tilde M^{\alpha}(\zeta)=\frac{\tilde J_0^{\alpha}}{\zeta}+\frac{O(1)}{\zeta^2} \quad \text{as } \operatorname{Im}\zeta\to+\infty. \tag{3.23}$$

• *The following trace formula holds true*:

$$\tilde B_0^{\alpha}+\frac{\tilde\nu^{\alpha}(\mathbb{R})}{\pi}=\tilde J_0^{\alpha}. \tag{3.24}$$

**Proof.** Using arguments of Theorem 1.2 and Theorem 1.3, we obtain the proof.

**Proof of Theorem 1.5.** Let a potential $V$ satisfy 1.2, and let $\alpha = \pm$. Then 1.14, 3.24 give the following trace formulas:

$$B_0^\alpha + \frac{\nu^\alpha(\mathbb{R})}{\pi} = \frac{1}{\pi}\int_{\mathbb{R}} \log|\psi^\alpha(t+i0)|d\lambda, \qquad \tilde{B}_0^\alpha + \frac{\tilde{\nu}^\alpha(\mathbb{R})}{\pi} = \frac{1}{\pi}\int_{\mathbb{R}} \log|\tilde{\psi}^\alpha(t+i0)|dt, \tag{3.25}$$

for $\alpha = \pm$. Summing, we obtain $\mathbf{B} + \mathbf{N} = \mathbf{I}$, where

$$\begin{aligned} \mathbf{B} &= \sum (|\operatorname{Im}\lambda_j| + |\operatorname{Re} k(\lambda_j)|), \\ \mathbf{N} &= \frac{1}{\pi}\Big(\nu^+(\mathbb{R}) + \nu^-(\mathbb{R}) + \tilde{\nu}^+(\mathbb{R}) + \tilde{\nu}^-(\mathbb{R})\Big), \\ \mathbf{I} &= \frac{1}{\pi}\sum_{\alpha=\pm}\int_{\mathbb{R}} \Big(\log|\psi^\alpha(t+i0)| + \log|\tilde{\psi}^\alpha(t+i0)|\Big)dt. \end{aligned} \tag{3.26}$$

From Theorem 3.2, we have the following estimate:

$$\int_{\mathbb{R}} \log|\psi^+(\lambda)|d\lambda \leqslant C_o\|V\|_p^2, \qquad C_o = 1 + (d+1)C_*^2, \tag{3.27}$$

and similar arguments yield

$$\mathbf{I} \leqslant 4C_o\|V\|_p^2. \tag{3.28}$$

From Lemma 3.4, we obtain $|\operatorname{Im}\lambda_j| + |\operatorname{Re} k(\lambda_j)| \geqslant \rho_j =$ for any $\lambda_j$, and then $\mathbf{B} \geqslant \sum \rho(\lambda_j)$. Collecting the last estimate and estimates 3.27, 3.28, we obtain 1.19.

**Lemma 3.4.** *Let $\lambda \in \Lambda = \mathbb{C}\setminus[-d,d]$, and let $k(\lambda) = \sqrt{\lambda^2 - d^2} \in \mathbb{K}$. Then*

$$|\operatorname{Im}\lambda| + |\operatorname{Re} k(\lambda)| \geqslant \varrho(\lambda) = \operatorname{dist}\{\lambda, [-d,d]\}. \tag{3.29}$$

**Proof.** Let $\lambda = \mu + i\xi \in \Lambda$. It is sufficient to consider the case $\mu, \xi > 0$. We have two cases. Firstly, let $\mu \in [0,d]$. Then we have $\operatorname{Im}\lambda = \rho(\lambda)$. Secondly, let $\mu > d$, and let $\zeta = \lambda - d = |\zeta|e^{i\varphi}$ and $\zeta + 2d = |\zeta_d|e^{i\varphi_d}$. Then we have $|\zeta_d| \geqslant |\zeta|$ and $0 \leqslant \varphi_d < \varphi$. Thus, we obtain

$$\begin{aligned} k &= \sqrt{\lambda^2 - d^2} = |\zeta|^{\frac{1}{2}} e^{i\varphi/2}\sqrt{\zeta + 2d} = |\zeta|^{\frac{1}{2}}|\zeta_d|^{\frac{1}{2}} e^{i(\varphi+\varphi_d)/2}, \\ \operatorname{Re} k &= |\zeta|^{\frac{1}{2}}|\zeta_d|^{\frac{1}{2}} \cos\frac{\varphi + \varphi_d}{2} \geqslant |\zeta|\cos\varphi = \mu - 1, \end{aligned} \tag{3.30}$$

which yields $\xi + \operatorname{Re} k \geqslant \xi + \mu - 1 \geqslant |\lambda - 1| = \varrho(\lambda)$. Thus, we have 3.29.

## 4. IDENTITIES IN THE DISK

### *4.1. Hardy Space in the Disk*

We define the disk $\mathbb{D}_r \subset \mathbb{C}$ with the radius $r > 0$ by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, and abbreviate $\mathbb{D} = \mathbb{D}_1$. Define **the new spectral variable** $z \in \mathbb{D}$ by

$$\lambda = \lambda(z) = \frac{d}{2}\Big(z + \frac{1}{z}\Big) \in \Lambda = \mathbb{C}\setminus[-d,d], \qquad z \in \mathbb{D}.$$

The function $\lambda(z)$ has the following properties.

- *The function $\lambda(z)$ is a conformal mapping from $\mathbb{D}$ onto the spectral domain $\Lambda = \mathbb{C}\setminus[-d,d]$.*
- $\lambda(\mathbb{D}) = \Lambda$ *and* $\lambda(\mathbb{D}\cap\mathbb{C}_\mp) = \mathbb{C}_\pm$.
- $\Lambda$ *is the cut domain with the cut $[-d,d]$ having the upper side $[-d,d]+i0$ and the lower side $[-d,d]-i0$. The function $\lambda(z)$ takes the upper semi-circle of the boundary onto the lower side $[-d,d]-i0$ and the lower semi-circle onto the upper side $[-d,d]+i0$.*
- *The function $\lambda(z)$ takes the point $z = 0$ to the point $\lambda = \infty$.*
- *The inverse mapping $z(\cdot) : \Lambda \to \mathbb{D}$ is defined by*

$$\begin{aligned} z(\lambda) &= \frac{1}{d}\big(\lambda - \sqrt{\lambda^2 - d^2}\big), \quad \lambda \in \Lambda, \\ z(\lambda) &= \frac{d}{2\lambda} + \frac{O(1)}{\lambda^3} \qquad \text{as} \quad |\lambda| \to \infty. \end{aligned}$$

Recall that $\mathscr{H}_\infty = \mathscr{H}_\infty(\mathbb{D})$ is the Hardy space of functions $F$ analytic in $\mathbb{D}$ and equipped with the norm $\|F\|_{\mathscr{H}_\infty} := \sup_{z\in\mathbb{D}} |F(z)| < \infty$. For $V \in \ell^2(\mathbb{Z}^d)$ (i.e., $V \in \mathcal{B}_2$) we have defined the regularized determinant $\psi(\lambda)$ in the cut domain $\Lambda$ by 1.3. We define the modified determinant $\mathfrak{f}$ in the disk $\mathbb{D}$ by

$$\mathfrak{f}(z) = \psi(\lambda(z)), \quad z \in \mathbb{D}. \tag{4.1}$$

It has $N \leqslant \infty$ zeros $\{z_j\}_{j=1}^N$ in the disk $\mathbb{D}$ such that $z_j = z(\lambda_s)$ for some $s \in \mathbb{N}$ and

$$0 < r_0 = \inf |z_j| = |z_1| \leqslant |z_2| \leqslant |z_3| \leqslant \cdots$$

**Theorem 4.1.** *Let the potential $V$ satisfy* 1.2 *and the constant $C_*$ be defined in* 2.26. *Then the modified determinant has the property $\mathfrak{f} \in \mathscr{H}_\infty(\mathbb{D})$, is Hölder continuous up to the boundary, and satisfies*

$$\|\mathfrak{f}\|_{\mathscr{H}_\infty(\mathbb{D})} \leqslant e^{C_*^2\|V\|_p^2/2}. \tag{4.2}$$

*The zeros $\{z_j\}_{j=1}^N$ of $\mathfrak{f}$ satisfy $\sum_{j=1}^N (1-|z_j|) < \infty$. Moreover, the function $\log \mathfrak{f}(z)$ is analytic in $\mathbb{D}_{r_0}$ and has the Taylor series (here $a = \frac{2}{d}$)*

$$\begin{aligned} &\log \mathfrak{f}(z) = -\mathfrak{f}_2 z^2 - \mathfrak{f}_3 z^3 - \mathfrak{f}_4 z^4 + \cdots, \qquad \text{as} \quad |z| < r_0, \\ &\mathfrak{f}_2 = \frac{a^2}{2}\operatorname{Tr} V^2, \quad \mathfrak{f}_3 = \frac{a^3}{3}\operatorname{Tr} V^3, \quad \mathfrak{f}_4 = \frac{a^4}{4}\operatorname{Tr}(V^4 + 2VH_0VH_0 + 4V^2H_0^2) - \mathfrak{f}_2, \ \ldots \end{aligned} \tag{4.3}$$

**Proof.** Recall that the determinant $\psi(\lambda), \lambda \in \Lambda$, is analytic in the domain $\Lambda$, is Hölder continuous up to the boundary, and satisfies 1.4. Then the function $\mathfrak{f}(z) = \psi(\lambda(z)), z \in \mathbb{D}$ is analytic in the domain $\mathbb{D}$, is Hölder continuous up to the boundary, and satisfies 4.2. It is well known that if $\lambda_0 \in \Lambda$ is an eigenvalue of $H$, then $z_0 = z(\lambda_0) \in \mathbb{D}$ is a zero of $\mathfrak{f}$ with the same multiplicity.

Due to Theorem 1.1, the function $\log \mathfrak{f}(z)$ is analytic in the disk $\mathbb{D}_{r_0}$ with the radius $r_0 = \inf |z_j| > 0$. Moreover, using 1.5 and the identity $\lambda = \frac{d}{2}(z + \frac{1}{z})$, we obtain the Taylor series for $|z| < r_0$ given by 4.3.

For the function $\mathfrak{f}$, we define the Blaschke product $B(z), z \in \mathbb{D}$, by: $B = 1$ if $N = 0$ and

$$B(z) = \prod_{j=1}^N \frac{|z_j|}{z_j}\frac{(z_j - z)}{(1 - \overline{z}_j z)}, \qquad \text{if} \qquad N \geqslant 1. \tag{4.4}$$

It is well known that the Blaschke product $B(z), z \in \mathbb{D}$, given by 4.4, converges absolutely for $\{|z| < 1\}$ and satisfies $B \in \mathscr{H}_\infty(\mathbb{D})$ with $\|B\|_{\mathscr{H}_\infty} \leqslant 1$, since $\mathfrak{f} \in \mathscr{H}_\infty$ (see, e.g., [21]). The Blaschke product $B$ has the standard Taylor series at $z = 0$:

$$\begin{aligned} &\log B(z) = B_0 - B_1 z - B_2 z^2 - \cdots \qquad \text{as} \qquad z \to 0, \\ &B_0 = \log B(0) < 0, \qquad B_1 = \sum_{j=1}^N \Big(\frac{1}{z_j} - \overline{z}_j\Big), \ \ldots, \qquad B_n = \frac{1}{n}\sum_{j=1}^N \Big(\frac{1}{z_j^n} - \overline{z}_j^n\Big), \ \ldots, \end{aligned} \tag{4.5}$$

where every $B_n$ satisfies $|B_n| \leqslant \frac{2}{r_0^n}\sum_{j=1}^N (1 - |z_j|)$, see, e.g., [28].
We describe the canonical representation of the determinant $\mathfrak{f}(z), z \in \mathbb{D}$.

**Corollary 4.2.** *Let the potential $V$ satisfy* 1.2. *Then there exists a singular measure $\nu \geqslant 0$ on $[-\pi, \pi]$ such that the determinant $\mathfrak{f}$ has a canonical factorization for all $|z| < 1$ given by*

$$\begin{aligned} &\mathfrak{f}(z) = B(z)e^{-K(z)}e^{M(z)}, \\ &K(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\nu(t), \qquad M(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log|\mathfrak{f}(e^{it})|dt, \end{aligned} \tag{4.6}$$

*where $\log|\mathfrak{f}(e^{it})| \in L^1(\mathbb{T})$ and the measure $\nu$ satisfies*

$$\operatorname{supp}\nu \subset \{t \in [-\pi, \pi] : \mathfrak{f}(e^{it}) = 0\}. \tag{4.7}$$

*Moreover, we have the Taylor series at $z = 0$ in the disk $\mathbb{D}$ :*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) = \frac{\mu(\mathbb{T})}{2\pi} + \mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \mu_4 z^4 + \cdots, \tag{4.8}$$

*where the measure is* $d\mu(t) = \log|\mathfrak{f}(e^{it})|dt - d\nu(t)$ *and*

$$\mu(\mathbb{T}) = \int_0^{2\pi} d\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|dt - \frac{\nu(\mathbb{T})}{2\pi}, \qquad \mu_n = \frac{1}{\pi}\int_0^{2\pi} e^{-int} d\mu(t), \qquad n \in \mathbb{N}.$$

**Proof.** Theorem 4.1 implies $\mathfrak{f} \in \mathscr{H}_\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$. We now recall the canonical representation 4.6 (see, e.g., [21], p. 76). Let $\mathfrak{f} \in \mathscr{H}_\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and let $B$ be its Blaschke product. Then $\mathfrak{f}$ has the form

$$\mathfrak{f}(z) = B(z)e^{ic - K(z) + M(z)} \qquad \forall\ |z| < 1, \tag{4.9}$$

where $c$ is real constant, $\log|\mathfrak{f}(e^{it})| \in L^1(-\pi,\pi)$, and $\nu = \nu_{\mathfrak{f}} \geqslant 0$ is a singular measure on $[-\pi,\pi]$ such that $\operatorname{supp}\nu \subset \{t \in [-\pi,\pi] : \mathfrak{f}(e^{it}) = 0\}$, and here $K, M$ are given by 4.6.

In order to prove 4.6, we need to show that $e^{ic} = 1$. From 4.9 at $z = 0$, we obtain

$$1 = \mathfrak{f}(0) = B(0)e^{ic - K(0)}e^{M(0)}.$$

Since $B(0), K(0), K(0)$ and $c$ are real, we obtain $e^{ic} = 1$.

Due to the representation 4.6, the function $\mathfrak{f}_B(z) = \frac{\mathfrak{f}(z)}{B(z)}$ does not have zeros in the disk $\mathbb{D}$ and satisfies

$$\log \mathfrak{f}_B(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t), \quad z \in \mathbb{D}, \tag{4.10}$$

where the measure is $d\mu = \log|f(e^{it})|dt - d\nu(t)$. In order to show 4.13–4.16, we need the asymptotics of the Schwartz integral $\log \mathfrak{f}_B(z)$ as $z \to 0$. The following identity holds true:

$$\frac{e^{it}+z}{e^{it}-z} = 1 + \frac{2ze^{-it}}{1-ze^{-it}} = 1 + 2\sum_{n\geqslant 1}\big(ze^{-it}\big)^n = 1 + 2\big(ze^{-it}\big) + 2\big(ze^{-it}\big)^2 + \cdots \tag{4.11}$$

for all $(t,z) \in \partial\mathbb{D} \times \mathbb{D}$. Thus, 4.10, 4.11 yield the Taylor series 4.8 at $z = 0$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) = \frac{\mu(\mathbb{T})}{2\pi} + \mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \mu_4 z^4 + \cdots \qquad \text{as} \qquad |z| < 1,$$

where

$$\mu(\mathbb{T}) = \int_0^{2\pi} d\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{it})|dt - \frac{\nu(\mathbb{T})}{2\pi}, \qquad \mu_n = \frac{1}{\pi}\int_0^{2\pi} e^{-int} d\mu(t), \qquad n \geqslant 1.$$

This yields 4.8.

**Remark 5.** (1) For the canonical factorization of analytic functions, see, for example, [21].

(2) Note that, for the inner function $\mathfrak{f}_{in}(z)$ defined by $\mathfrak{f}_{in}(z) = B(z)e^{-K_\nu(z)}$, we have $|\mathfrak{f}_{in}(z)| \leqslant 1$, since $d\nu \geqslant 0$ and $\operatorname{Re}\frac{e^{it}+z}{e^{it}-z} \geqslant 0$ for all $(t,z) \in \mathbb{T} \times \mathbb{D}$.

We present our main result about trace formulas in the disk.

**Theorem 4.3.** *Let* $V$ *satisfy* 1.2. *Then the following trace formula holds true*:

$$-\operatorname{Tr}\Big(R(\lambda) - R_0(\lambda) + R_0(\lambda)VR_0(\lambda)\Big)\lambda'(z) = \sum \frac{(1-|z_j|^2)}{(z-z_j)(1-\overline{z}_j z)} + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{it}d\mu(t)}{(e^{it}-z)^2}, \tag{4.12}$$

*where* $\lambda = \frac{d}{2}(z + \frac{1}{z}) \in \mathbb{C} \setminus [-d,d]$, $z \in \mathbb{D}$, *and the measure is* $d\mu(t) = \log|\mathfrak{f}(e^{it})|dt - d\nu(t)$. *Moreover, the following identities hold*:

$$\frac{\nu(\mathbb{T})}{2\pi} - B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\mathfrak{f}(e^{it})|dt \geqslant 0, \tag{4.13}$$

$$B_1 = \sum_{j=1}^{N}\left(\frac{1}{z_j} - \overline{z}_j\right) = \frac{1}{\pi}\int_{\mathbb{T}} e^{-it} d\mu(t), \tag{4.14}$$

$$\sum_{j=1}^{N}\left(\frac{1}{z_j^2} - \overline{z}_j^2\right) = \frac{2}{d^2}\operatorname{Tr} V^2 + \frac{1}{\pi}\int_{\mathbb{T}} e^{-i2t} d\mu(t), \tag{4.15}$$

$$B_n = \mathfrak{f}_n + \frac{1}{\pi}\int_{\mathbb{T}} e^{-int} d\mu(t), \qquad n = 2, 3, ... \tag{4.16}$$

*where* $B_0 = \log B(0) = \log\Big(\prod_{j=1}^N |z_j|\Big) < 0$ *and* $B_n$ *are given by* 4.5. *In particular,*

$$\sum_{j=1}^N \Big(\operatorname{Re} k_j + i \operatorname{Im} \lambda_j\Big) = \frac{d}{2\pi}\int_{\mathbb{T}} e^{-it} d\mu(t), \tag{4.17}$$

*where* $k_j = \sqrt{\lambda_j^2 - d^2} \in \mathbb{K}$.

**Proof.** The proof of 4.12 repeats the proof of 1.13.

From 4.6, we have the identity $\log \mathfrak{f}(z) = \log B(z) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)$ for all $z \in \mathbb{D}_{r_0}$. Combining the asymptotics 4.3, 4.5 and 4.8, we obtain 4.13–4.16. In particular, we have 4.15 and $-\log B(0) = \frac{\mu(\mathbb{T})}{2\pi} \geqslant 0$.

We show 4.17. We have the following identities for $z \in \mathbb{D}$, $\lambda \in \Lambda$, and $k = \sqrt{\lambda^2 - d^2}$:

$$2\lambda = d\big(z + \tfrac{1}{z}\big), \qquad dz = \lambda - k, \qquad d(z - \tfrac{1}{z}) = -2k. \tag{4.18}$$

Let $w = \frac{d}{2}\Big(\frac{1}{z} - \overline{z}\Big)$. These identities yield

$$\begin{aligned} &2w = 2\lambda - 2d \operatorname{Re} z, \qquad \operatorname{Im} w = \operatorname{Im} \lambda, \\ &2w = 2k + 2id \operatorname{Im} z, \qquad \operatorname{Re} w = \operatorname{Re} k, \qquad w = \operatorname{Re} k + i \operatorname{Im} \lambda. \end{aligned} \tag{4.19}$$

Let $k_j = \sqrt{\lambda_j^2 - d^2}$. Then from these identities and 4.14, we obtain

$$B_1 = \sum_{j=1}^N \left(\frac{1}{z_j} - \overline{z}_j\right) = \frac{1}{\pi}\int_{\mathbb{T}} e^{-it} d\mu(t),$$

$$\frac{d}{2\pi}\int_{\mathbb{T}} e^{-it} d\mu(t) = \sum_{j=1}^N \frac{d}{2}\left(\frac{1}{z_j} - \overline{z}_j\right) = \sum_{j=1}^N (\operatorname{Re} k_j + i \operatorname{Im} \lambda_j)$$

and thus

$$\sum_{j=1}^N \operatorname{Re}\sqrt{\lambda_j^2 - d^2} = \frac{d}{2\pi}\int_{\mathbb{T}} \cos t \, d\mu(t), \qquad \sum_{j=1}^N \operatorname{Im} \lambda_j = -\frac{d}{2\pi}\int_{\mathbb{T}} \sin t \, d\mu(t),$$

which yields 4.17.

We describe estimates of eigenvalues in terms of potentials.

**Theorem 4.4.** *Let* $V$ *satisfy* 1.2. *Then we have the following estimates*:

$$\frac{\nu(\mathbb{T})}{2\pi} + \sum (1 - |z_j|) \leqslant \frac{\nu(\mathbb{T})}{2\pi} - B_0 \leqslant \frac{C_*^2}{2}\|V\|_p^2. \tag{4.20}$$

**Proof.** The simple inequality $1 - x \leqslant -\log x$ for $\forall\, x \in (0, 1]$ implies

$$-B_0 = -B(0) = -\sum \log |z_j| \geqslant \sum (1 - |z_j|). \tag{4.21}$$

The estimate 4.2 implies

$$\frac{1}{2\pi}\int_{\mathbb{T}} d\mu(t) = \frac{1}{2\pi}\int_{\mathbb{T}} \log|\mathfrak{f}(e^{it})| dt - \frac{\nu(\mathbb{T})}{2\pi} \leqslant \frac{C_*^2}{2}\|V\|_p^2 - \frac{\nu(\mathbb{T})}{2\pi}. \tag{4.22}$$

Then, substituting these estimates into the first trace formula 4.13, we obtain 4.20.

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