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On a Nonlinear Nonlocal Parabolic Problem

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Abstract. We consider the Cauchy problem for a parabolic equation with a placian or a general second-order quasilinear equation with boundary conditions of the Bitsadze–Samarskii type. We prove that at least one generalized solution of such problem exists.

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INTRODUCTION

Nonlocal elliptic boundary value problems have been considered since the 1930s of the XXth century, see Carleman [1]. In the 50–60s of the XXth century, abstract nonlocal problems were studied by Vishik [2], Browder [3], etc. The theory of nonlocal boundary value problems has applications to physics, engineering, biology, e.a. In 1969, Bitsadze and Samarskii considered an elliptic equation with nonlocal conditions connecting the values of unknown function on a boundary with its values on some manifold inside a domain, see [4]. Since the 70s, linear nonlocal parabolic problems with the Bitsadze–Samarskii type of nonlocality have also been studied, see, for example, [5, 6] and the bibliography there. Note that, for a long time, only some special cases of the Bitsadze–Samarskii boundary value problems were considered. In 1980, Samarskii described the question of solvability for nonlocal elliptic problems as "unsolved one", see [7]. A method for studying linear elliptic boundary value problems with such nonlocal conditions was developed in the 80-90s, see [8–12]. The nonlinear elliptic boundary value problems with such nonlocal conditions were studied in [13]. The linear parabolic boundary value problems with such nonlocal conditions were studied in [14]. In this paper, we consider nonlinear nonlocal parabolic problems. Note also that nonlocal parabolic problems have important applications to Feller's semigroup theory, see [15–17].

1. STATEMENT OF THE PROBLEM

Let $Q \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial Q \in \mathbb{C}^\infty$ or a cylinder $(0,d) \times G$, where $G \subset \mathbb{R}^{n-1}$ is a bounded domain (with boundary $\partial G \in \mathbb{C}^\infty$ if $n \geqslant 3$), $\Delta_p w = -\sum\limits_{1\leqslant i\leqslant n} \partial_i \left(|\partial_i w|^{p-2}\partial_i w\right)$. We assume that

$$2 \leqslant p < \infty, \quad 1/p + 1/q = 1, \quad f \in L_q(0,T; W_q^{-1}(Q)), \quad \psi \in L_2(Q).$$

All functions are real–valued. In the cylinder $\Omega_T = Q \times (0,T)$, we consider the differential equation

$$\partial_t w(x,t) + \Delta_p w(x,t) = f(x,t) \quad ((x,t) \in \Omega_T) \tag{1.1}$$

with initial condition

$$w(x,0) = \psi(x) \qquad (x \in Q) \tag{1.2}$$

and with nonlocal boundary conditions

$$\left.\begin{aligned} w|_{\Gamma_{rl}^T} &= \sum_{j=1}^{J_0} \gamma_{lj}^r w|_{\Gamma_{rj}^T} && (r \in B,\ l = J_0+1,\dots,J),\\ w|_{\Gamma_{rl}^T} &= 0 && (r \notin B,\ l = 1,\dots,J), \end{aligned}\right\} \tag{1.3}$$

where the set $\Gamma^T = \{\Gamma_{rl}^T\}$ is defined as follows.

Let $\mathscr{M}$ be a finite set of vectors $h \in \mathbb{Z}^n$, and let M be the additive group generated by $\mathscr{M}$. Denote by Q_r the open connected components of the set $Q \setminus (\bigcup\limits_{h\in M} (\partial Q + h))$. The set Q_r is called a *subdomain*. The family $\mathscr{R}$ of all subdomains Q_r $(r = 1, 2, \dots)$ is called a *partition of the domain* Q. It is easy to see that the set $\mathscr{R}$ is at most countable,

$$\bigcup_r \partial Q_r = \Big(\bigcup_{h \in M} (\partial Q + h) \Big) \cap \overline{Q}, \quad \text{and} \quad \bigcup_r \overline{Q}_r = \overline{Q}.$$

As is known, see Lemma 7.1 of [9, Ch.II,§7], for any subdomain Q_{r_1} and an arbitrary vector $h \in M$, either there is a subdomain Q_{r_2} such that $Q_{r_2} = Q_{r_1} + h$ or $Q_{r_1} + h \subset \mathbb{R}^n \setminus \overline{Q}$. Thus, the family $\mathscr{R}$ can be divided into disjoint classes as follows: subdomains $Q_{r_1}, Q_{r_2} \in \mathscr{R}$ belong to the same class if $Q_{r_2} = Q_{r_1} + h$ for some $h \in M$. We denote the subdomains Q_r by Q_{sl}, where s is the class number and l is the subdomain number in the sth class. Obviously, each class consists of a finite number $N = N(s)$ of subdomains Q_{sl}, and $N(s) \leqslant ([\mathrm{diam} Q] + 1)^n$. The set of classes can be finite or countable (see examples in Section 7, Ch. II of [9]).

Introduce the set $\mathscr{K}$ given by the formula

$$\mathscr{K} = \bigcup_{h_1, h_2 \in M} \left\{ \overline{Q} \cap (\partial Q + h_1) \cap \overline{[(\partial Q + h_2) \setminus (\partial Q + h_1)]} \right\}, \tag{1.4}$$

Let Γ_ρ denote the open connected (in the topology of ∂Q) components of the set $\partial Q \setminus \mathscr{K}$. The following result was obtained in [9, §7].

Lemma 1. *If $(\Gamma_\rho + h) \cap \overline{Q} \neq \emptyset$ for some $h \in M$, then either $\Gamma_\rho + h \subset Q$ or there exists a $\Gamma_r \subset \partial Q \setminus \mathscr{K}$ such that $\Gamma_\rho + h = \Gamma_r$.*

According to this property, the sets $\left\{ \Gamma_\rho + h : \Gamma_\rho + h \subset \overline{Q},\ \rho = 1, 2, \dots,\ h \in M \right\}$ can be divided into classes. The sets $\Gamma_{\rho_1} + h_1$ and $\Gamma_{\rho_2} + h_2$ belong to the same class if

1) there is a vector $h \in M$ such that $\Gamma_{\rho_1} + h_1 = \Gamma_{\rho_2} + h_2 + h$;

2) for any $\Gamma_{\rho_1} + h_1, \Gamma_{\rho_2} + h_2 \subset \partial Q$ the normals to ∂Q at the points $x \in \Gamma_{\rho_1} + h_1$ and $x - h \in \Gamma_{\rho_2} + h_2$ have the same direction.

Let the set $\Gamma_\rho + h$ be denoted by Γ_{rj}, where r is the index of a class and j is the index of an element in that class $(1 \leqslant j \leqslant J = J(r))$. Without lost of generality, we assume that $\Gamma_{r1}, \dots, \Gamma_{rJ_0} \subset Q$ and $\Gamma_{r,J_0+1}, \dots, \Gamma_{rJ} \subset \partial Q$ $(0 \leqslant J_0 = J_0(r) < J(r))$.

It is well known (see [9, §7]) that this partition has the following properties.

Lemma 2. *For any $\Gamma_{rj} \subset \partial Q$, there exists a subdomain Q_{sl} such that $\Gamma_{rj} \subset \partial Q_{sl}$. Moreover, the inclusion $\Gamma_{rj} \subset \partial Q_{sl}$ implies $\Gamma_{rj} \cap \partial Q_{s_1 l_1} = \emptyset$ if $(s_1, l_1) \neq (s, l)$.*

Lemma 3. *For every $r = 1, 2, \dots$, there exists a unique index $s = s(r)$ such that $N(s) = J(r)$ and $\Gamma_{rl} \subset \partial Q_{sl}$ $(l = 1, \dots, N(s))$ (up to reindexing).*

Write $\Gamma^T_{rl} := \Gamma_{rl} \times (0, T)$.

Example 1. We consider problem (1.1), (1.2) in the rectangular parallelepiped $\Omega_T = (0, 2) \times (0, 1) \times (0, T)$ with the Bitsadze–Samarskii nonlocal boundary conditions

$$\left. \begin{array}{ll} w(x_1, 0, t) = w(x_1, 1, t) = 0 & (0 \leqslant x_1 \leqslant 2; 0 < t < T), \\ w(x, t)|_{x_1=0} = \gamma_1 w(x, t)|_{x_1=1}, & (0 < x_2 < 1; 0 < t < T) \\ w(x, t)|_{x_1=2} = \gamma_2 w(x, t)|_{x_1=1} & (0 < x_2 < 1; 0 < t < T) \end{array} \right\}. \tag{1.5}$$

Here we have 4 classes of sets:

1) $\Gamma^T_{11} = \{1\} \times (0, 1) \times (0, T), \Gamma^T_{12} = \{0\} \times (0, 1) \times (0, T)$; 2) $\Gamma^T_{21} = \{1\} \times (0, 1) \times (0, T), \Gamma^T_{22} = \{2\} \times (0, 1) \times (0, T)$;

3) $\Gamma^T_{31} = (0, 1) \times \{0\} \times (0, T), \Gamma^T_{32} = (1, 2) \times \{0\} \times (0, T)$; 4) $\Gamma^T_{41} = (0, 1) \times \{1\} \times (0, T), \Gamma^T_{42} = (1, 2) \times \{1\} \times (0, T)$.

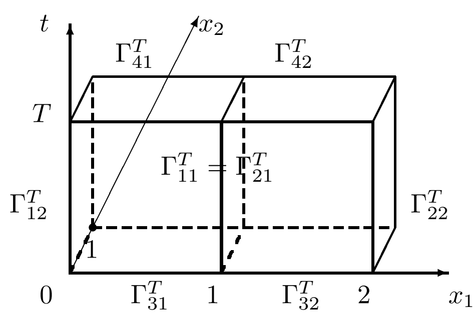

Fig. 1.

Our nonlocal boundary conditions (1.5) can be written as follows

$$\left.\begin{array}{ll} w|_{\Gamma^T_{rl}} = 0 & (r = 3, 4; l = 1, 2), \\ w|_{\Gamma^T_{12}} = \gamma_1 w|_{\Gamma^T_{11}}, & \\ w|_{\Gamma^T_{22}} = \gamma_2 w|_{\Gamma^T_{21}} & \end{array}\right\}. \tag{1.6}$$

2. ISOMORPHISM OF SPACES

We assume that the following conditions hold.

Condition 1. The set $\mathscr{K}$ given by formula (1.4) satisfies condition

$$\mathrm{mes}_{n-1}(\mathscr{K} \cap \partial Q) = 0. \tag{2.1}$$

Condition 2. For any subdomain Q_{sl} $(s = 1, 2, \ldots, l = 1, \ldots, N(s))$ and an arbitrary $\varepsilon > 0$, there exists some open set $G_{sl} \subset Q_{sl}$ with boundary $\partial G_{sl} \in C^1$ such that $\mathrm{mes}_n(Q_{sl} \setminus G_{sl}) < \varepsilon$ and $\mathrm{mes}_{n-1}(\partial G_{sl} \Delta \partial Q_{sl}) < \varepsilon$.

We consider our problem in the Sobolev space $L_p(0, T; W^1_p(Q))$. This is the set of functions $u \in L_p(\Omega_T)$ having all generalized derivatives $\partial_i u$ in $L_p(\Omega_T)$. It is well known that this space is reflexive and Banach. Let

$$L_p(0, T; W^1_{p,\gamma}(Q)) := \{w \in L_p(0, T; W^1_p(Q)) : w \,\text{satisfies (1.3)}\}, \tag{2.2}$$

$$L_p(0, T; \mathring{W}^1_p(Q)) := \{u \in L_p(0, T; W^1_p(Q)) : u|_{x\in\partial Q} = 0 \,\text{for almost all}\, t \in (0, T)\}, \tag{2.3}$$

where $\|u\|^p_{L_p(0,T;\mathring{W}^1_p(Q))} := \sum\limits_{1\leqslant i\leqslant n} \int\limits_0^T \int\limits_Q |\partial_i u(x, t)|^p \, dx \, dt$.

Consider a collection of real constant coefficients $\{a_h \in \mathbb{R} : h \in \mathscr{M}\}$. Define the difference operator $R : L_p(0, T; \mathbb{R}^n) \to L_p(0, T; \mathbb{R}^n)$ by the formula

$$Ru(x, t) = \sum_{h\in\mathscr{M}} a_h u(x + h, t). \tag{2.4}$$

We define the operator $R_Q = P_Q R I_Q : L_p(\Omega_T) \to L_p(\Omega_T)$, where $I_Q : L_p(\Omega_T) \to L_p(\mathbb{R}^n \times (0, T))$ is the operator of extention for functions from $L_p(\Omega_T)$ by zero to $(\mathbb{R}^n \setminus Q) \times (0, T)$ and we denote by P_Q : $L_p(\mathbb{R}^n \times (0, T)) \to L_p(\Omega_T)$ is the operator of restriction for functions from $L_p(\mathbb{R}^n \times (0, T))$ to Ω_T.

As mentioned above, the difference operator $Ru(x) = \sum\limits_{h\in\mathscr{M}} a_h u(x + h)$ as well as the operator $R_Q = P_Q R I_Q : L_p(Q) \to L_p(Q)$ were studied earlier, see [8, 9, 12] for $p = 2$ and [19] for $1 < p < \infty$. To simplify the record here and below, we will denote by R_Q both the operators:

$$R_Q : L_p(\Omega_T) \to L_p(\Omega_T) \quad \text{and} \quad R_Q : L_p(Q) \to L_p(Q).$$

Consequently, it will be clear from the context where we consider $R_Q : L_p(0, T; \mathring{W}^1_p(Q)) \to L_p(0, T; W^1_{p,\gamma}(Q))$ and where we consider $R_Q : \mathring{W}^1_p(Q) \to W^1_{p,\gamma}(Q)$. Here we denote by $\mathring{W}^1_p(Q)$ the space of elements from Sobolev space $u \in W^1_p(Q)$ that satisfy the Dirichlet boundary condition $u|_{x\in\partial Q} = 0$; and we denote by $W^1_{p,\gamma}(Q)$ the elements from Sobolev space $u \in W^1_p(Q)$ that satisfy nonlocal boundary conditions

$$\left.\begin{array}{lll} w|_{\Gamma_{rl}} & = \sum\limits_{j=1}^{J_0} \gamma^r_{lj} w|_{\Gamma_{rj}} & (r \in B,\ l = J_0 + 1, \ldots, J), \\ w|_{\Gamma_{rl}} & = 0 & (r \notin B,\ l = 1, \ldots, J). \end{array}\right\}$$

Lemma 4. *The linear operators* $R : L_p(\mathbb{R}^n \times (0, T)) \to L_p(\mathbb{R}^n \times (0, T))$ *and* $R_Q : L_p(\Omega_T) \to L_p(\Omega_T)$ *are bounded for* $1 < p < \infty$.

This statement follows from Lemma 8.1 in [9] and Lemma 1 in [19].

Denote by $L_p\left(\bigcup\limits_l Q_{sl} \times (0, T)\right)$ the subspace of functions from $L_p(\Omega_T)$ that vanish for x not belonging to $\bigcup\limits_l Q_{sl}$ $(l = 1, \ldots, N(s))$. Introduce the bounded operator $P_s : L_p(\Omega_T) \to L_p\left(\bigcup\limits_l Q_{sl} \times (0, T)\right)$ by the formula $P_s u(x,t) = u(x,t)$ $(x \in \bigcup\limits_l Q_{sl},\, t \in (0,T))$, $P_s u(x,t) = 0$ $(x \in Q \setminus \bigcup\limits_l Q_{sl},\, t \in (0,T))$. Obviuosly, P_s is a projector to $L_p\left(\bigcup\limits_l Q_{sl} \times (0,T)\right)$. Since $\mathrm{mes}_n(\partial Q_{sl}) = 0$, we have

$$L_p(Q) = \dot{+}_s L_p\left(\bigcup_l Q_{sl}\right); \qquad L_p(\Omega_T) = \dot{+}_s L_p\left(\bigcup_l Q_{sl} \times (0,T)\right). \tag{2.5}$$

The following assertion is evident.

Lemma 5. *$L_p\left(\bigcup_l Q_{sl} \times (0,T)\right)$ is an invariant subspace of the operator R_Q.*

We define an isomorphism of reflexive Banach spaces

$$U_s : L_p\left(\bigcup_l Q_{sl} \times (0,T)\right) \to L_p^N\Big(Q_{s1} \times (0,T)\Big)$$

by the formula

$$(U_s u)_l(x,t) = u(x, t + h_{sl}) \qquad (x \in Q_{s1},\, t \in (0,T)), \tag{2.6}$$

where $l = 1, \dots, N = N(s)$ and the vector h_{sl} is such that

$$Q_{s1} + h_{sl} = Q_{sl} \qquad (h_{s1} = 0), \quad L_p^N(Q_{s1}) = \prod_l L_p(Q_{s1}), \quad L_p^N\Big(Q_{s1} \times (0,T)\Big) = \prod_l L_p\Big(Q_{s1} \times (0,T)\Big).$$

We introduce the matrices $R_s = \{r^s_{ml}\}_{1 \leqslant m,l \leqslant N(s)}$ by setting

$$r^s_{ml} = \begin{cases} a_h & (h = h_{sl} - h_{sm} \in \mathscr{M}), \\ 0 & (h_{sl} - h_{sm} \notin \mathscr{M}). \end{cases} \tag{2.7}$$

The boundedness of Q and formula (2.7) imply that a number of different matrices is finite. Let n_1 denote this number and let R_{s_ν} denote all different matrices R_s $(\nu = 1, \dots, n_1)$. We define the operator $R_{Qs} : L_p^N(Q_{s1} \times (0,T)) \to L_p^N(Q_{s1} \times (0,T))$ given by

$$R_{Qs} = U_s R_Q U_s^{-1} \tag{2.8}$$

By virtue of Lemma 8.6 from [9, Ch. II, §8], R_{Qs} is the operator of multiplication by the matrix R_s. Moreover, by virtue of Lemma 8.7 from [9, Ch. II, §8], the spectrum of R_Q is defined by the spectrums of R_{s_ν}:

$$\sigma(R_Q) = \bigcup_{1 \leqslant \nu \leqslant n_1} \sigma(R_{s_\nu}).$$

Corollary 1. *Let R_s $(s = s(r), r \in B)$ be nonsingular. Then the operator R_Q has a bounded inverse operator $R_Q^{-1} : L_p(\Omega_T) \to L_p(\Omega_T)$. Moreover, the operator $R_{Qs}^{-1} : L_p^N(Q_{s1} \times (0,T)) \to L_p^N(Q_{s1} \times (0,T))$ given by*

$$R_{Qs}^{-1} = U_s R_Q^{-1} U_s^{-1} \tag{2.9}$$

is the operator of multiplication by the matrix R_s^{-1}.

By Lemma 3, for every $r = 1, 2, \dots$, there is a single index $s = s(r)$ such that $N(s) = J(r)$ and $\Gamma_{rl} \subset \partial Q_{sl}$, $l = 1, \dots, N(s)$, after reindexing the subdomains of the sth class. Let $R_{s(r)}$ denote the matrices obtained by renumbering the corresponding columns and rows in R_s $(s = s(r))$. Let e_j^r $(j = 1, \dots, J(r))$ be the jth row of the $J \times J_0$ matrix obtained by deleting the last $J - J_0$ columns from $R_{s(r)}$. And let R_{s0} denote the $J_0 \times J_0$ matrix obtained by deleting the last $N(s) - J_0$ rows and columns from R_s.

Definition 1. We say that the matrices R_s *correspond to the boundary conditions* (1.3) if the following condition holds:

Condition 3. There is a collection $\{a_h \in \mathbb{R} : h \in \mathscr{M}\}$ such that, for any $s = 1, 2, \dots$, the matrices R_s are nonsingular and for all $r \in B$ and $s = s(r)$ it is true that

$$e_l^r = \sum_{1 \leqslant j \leqslant J_0} \gamma_{lj}^r e_j^r \qquad (l = J_0 + 1, \dots, J). \tag{2.10}$$

Example 2. We continue to consider problem (1.1), (1.2), (1.5) in the rectangular parallelepiped $\Omega_T = (0,2) \times (0,1) \times (0,T)$, see Example 1. By nonlocal conditions (1.5), we have the set of shifts $\mathscr{M} = \{(0,0);(1,0);(-1,0)\}$; the partition of the domain $Q = (0,2) \times (0,1)$ contains two subdomains $Q_{11} = (0,1) \times (0,1)$ and $Q_{12} = (1,2) \times (0,1)$ from the same class. The set $\mathscr{K}$ consists of 6 points $\mathscr{K} = \{(i,j) : i = 0,1,2; j = 0,1\}$. Conditions 1 and 2 hold. We consider the difference operator

$$Ru(x,t) = u(x,t) + a_1 u(x_1+1, x_2, t) + a_{-1} u(x_1-1, x_2, t).$$

We define the operator $R_Q = P_Q R I_Q : L_p(\Omega_T) \to L_p(\Omega_T)$. Clearly, here $R_1 = \begin{pmatrix} 1 & a_1 \\ a_{-1} & 1 \end{pmatrix}$ and $R_{10} = 1$. For any $u \in L_p(0,T;\mathring{W}_p^1(Q))$ and $w = R_Q u$, we have

$$w|_{x_1=0} = R_Q u|_{x_1=0} = a_1 u|_{x_1=1}, \qquad w|_{x_1=2} = R_Q u|_{x_1=2} = a_{-1} u|_{x_1=1},$$
$$w|_{x_1=1} = R_Q u|_{x_1=1} = u|_{x_1=1}.$$

Thus, if $a_1 = \gamma_1$, $a_{-1} = \gamma_2$, and $u \in L_p(0,T;\mathring{W}_p^1(Q))$, then $w = R_Q u$ satisfies the nonlocal boundary conditions (1.5), i.e. $R_Q(L_p(0,T;\mathring{W}_p^1(Q))) \subset L_p(0,T;W_{p,\gamma}^1(Q))$, where $\gamma = \{\gamma_1, \gamma_2\}$.

Theorem 1. *Let Conditions* 1–3 *hold and let the corresponding matrices R_s and R_{s0} $(s = s(r), r \in B)$ be nonsingular. Then there exists a set $\gamma = \{\gamma_{lj}^r\}$ such that the operator R_Q is a continuous one-to-one mapping of $L_p(0,T;\mathring{W}_p^1(Q))$ onto $L_p(0,T;W_{p,\gamma}^1(Q))$.*

Proof. Note that the operator R_Q does not depend on t. Due to the nonsingularity of the matrices R_s and R_{s0}, the operator R_Q is a continuous one-to-one mapping of $\mathring{W}_p^1(Q)$ onto $W_{p,\gamma}^1(Q)$, see Theorem 1.1 in [13] (for $p = 2$, see Theorem 8.1 in [9] or Theorem 2.1 in [12]). Therefore, R_Q is a continuous one-to-one mapping of $L_p(0,T;\mathring{W}_p^1(Q))$ onto $L_p(0,T;W_{p,\gamma}^1(Q))$.

3. OPERATOR EQUATION

Let us consider the generalized derivatives ∂_t with respect to time variable. We assume that the unbounded operator $\partial_t : L_p(0,T;W_p^1(Q)) \supset \mathscr{D}(\partial_t) \to L_q(0,T;W_q^{-1}(Q))$ has the domain

$$\mathscr{D}(\partial_t) := \{w \in L_p(0,T;W_p^1(Q)) : \partial_t w \in L_q(0,T;W_q^{-1}(Q))\}. \tag{3.1}$$

The nonlinear operator $\Delta_p : L_p(0,T;W_{p,\gamma}^1(Q)) \to L_q(0,T;W_q^{-1}(Q))$ is given by the formula

$$\langle \Delta_p w, v \rangle = \sum_{1 \leqslant i \leqslant n} \int_{\Omega_T} |\partial_i w|^{p-2} \partial_i w \, \partial_i v \, dx \, dt \qquad \forall v \in L_p(0,T;\mathring{W}_p^1(Q)).$$

We introduce the space W_γ by the formula

$$W_\gamma := \{w \in L_p(0,T;W_{p,\gamma}^1(Q)) : \partial_t w \in L_q(0,T;W_q^{-1}(Q))\}. \tag{3.2}$$

Definition 2. The function $w \in W_\gamma$ is called a *generalized solution of problem* (1.1)–(1.3) if it satisfies the operator equation

$$\partial_t w + \Delta_p w = f, \qquad w \in W_\gamma \tag{3.3}$$

with initial condition (1.2).

We assume that Conditions 1–3 hold and the corresponding matrices R_s and R_{s0} are nonsingular. By virtue of Theorem 1, there exists an isomorphism $R_Q : L_p(0,T;\mathring{W}_p^1(Q)) \to L_p(0,T;W_{p,\gamma}^1(Q))$. Thus, for any $w \in W_\gamma \subset L_p(0,T;W_{p,\gamma}^1(Q))$, there exists an unique element $u \in L_p(0,T;\mathring{W}_p^1(Q))$ such that $w = R_Q u$ and $u = R_Q^{-1} w$. We introduce the space

$$W := \{u \in L_p(0,T;\mathring{W}_p^1(Q)) : \partial_t u \in L_q(0,T;W_q^{-1}(Q))\}. \tag{3.4}$$

Lemma 6. *For any $u \in W$, $\partial_t R_Q u = R_Q \partial_t u \in L_q(0,T;W_q^{-1}(Q))$.*

Proof. The linear operator $R_Q : L_q(\Omega_T) \to L_q(\Omega_T)$ is bounded, see Lemma 4. For any $u \in W$ such that $\partial_t u \in L_q(\Omega_T)$, we have that $R_Q\partial_t u \in L_q(\Omega_T)$. Obviously, $\partial_t R_Q u = R_Q \partial_t u \in L_q(\Omega_T)$. Here we can use the density of inclution $L_q(\Omega_T) \subset L_q(0, T; W_q^{-1}(Q))$ and the closure of the graph of the bounded linear operator R_Q. I.e. for any $u \in W$, there exist the sequence $\{u_n\} \subset W$ such that $\partial_t u_n \in L_q(\Omega_T)$ and $\partial_t u_n \to \partial_t u$ in $L_q(0, T; W_q^{-1}(Q))$. Then $R_Q\partial_t u = \lim_{n\to\infty} R_Q\partial_t u_n = \lim_{n\to\infty} \partial_t R_Q u_n = \partial_t R_Q u$.

From Lemma 6 it follows that $w \in W_\gamma$ if $u \in W$, since R_Q is an isomorphism. By virtue of Theorem 1.17 in [21, Ch.4,§1], $u \in C(0, T; L_2(Q))$ and $w \in C(0, T; L_2(Q))$. Thus,

$$\varphi(x) := u(x, 0) = R_Q^{-1} w(x, 0) = R_Q^{-1}\psi(x).$$

Therefore, $u|_{t=0} = \varphi \in L_2(Q)$ and $w|_{t=0} = \psi \in L_2(Q)$ are well defined. We obtain the following result.

Theorem 2. *Let Conditions* 1–3 *hold and let the corresponding matrices* R_s *and* R_{s0} $(s = s(r), r \in B)$ *be nonsingular. If* $u \in W$ *is a solution of the operator equation*

$$\partial_t R_Q u + \Delta_p R_Q u = f, \qquad u \in W, \tag{3.5}$$

with initial condition

$$u(x, 0) = \varphi(x) = R_Q^{-1}\psi(x), \tag{3.6}$$

then $w = R_Q u$ *is a generalized solution of problem* (1.1)–(1.3).

Example 3. We continue to consider the problem (1.1), (1.2), (1.5) with Bitsadze–Samarskii nonlocal boundary conditions in the rectangular parallelepiped $\Omega_T = (0, 2) \times (0, 1) \times (0, T)$. As it was proved in the example 2, the corresponding operator R_Q is defined by the matrix $R_1 = \begin{pmatrix} 1 & \gamma_1 \\ \gamma_2 & 1 \end{pmatrix}$. This matrix is nonsingular if $\gamma_1\gamma_2 \neq 1$; the matrix $R_{10} = (1)$ is nonsingular also. Thus, in order to solve of the problem (1.1), (1.2), (1.5), we can study the following problem:

$$\partial_t R_Q u(x, t) + \Delta_p R_Q u(x, t) = f(x, t) \qquad ((x, t) \in \Omega_T), \tag{3.7}$$

$$u(0, x) = \varphi(x) = R_Q^{-1}\psi(x) \qquad (x \in Q), \tag{3.8}$$

$$u(x, t) = 0 \qquad ((x, t) \in \partial Q \times (0, T)). \tag{3.9}$$

4. PROPERTIES OF DIFFERENCE OPERATORS

Here we consider the properties of the operator R_Q and its conjugate operator R_Q^*.

Lemma 7 (cf. Lemma 8.2 in [9].)**.** *The operator* $R : L_p(\mathbb{R}^n \times (0, T)) \to L_p(\mathbb{R}^n \times (0, T))$ *is bounded, and*

$$R^* u(x, t) = \sum_{h\in\mathscr{M}} a_h u(x - h, t).$$

Let $G \subset \mathbb{R}^n$ be a domain such that $G \subset Q$, and let $G_T := G \times (0, T)$. We denote by $W_p^{1,0}(G_T)$ the anisotropic Sobolev space consisting of functions $u \in L_p(G_T)$ which have all generalized derivatives $\partial_i u$ in $L_p(G_T)$ with the norm

$$\|u\|_{W_p^{1,0}(G_T)} = \left\{ \sum_{1\leqslant i\leqslant n} \int_{G_T} |\partial_i u|^p \, dt\, dx + \int_{G_T} |u|^p \, dt\, dx \right\}^{1/p}.$$

The space $W_p^{1,0}(G_T)$ can be identified with the space $L_p(0, T; W_p^1(G))$. It is also clear that

$$L_p(0, T; \mathring{W}_p^1(Q)) = \left\{ u \in W_p^{1,0}(\Omega_T) : u|_{\partial Q\times(0,T)} = 0 \right\}.$$

As it was mentioned in Section 2, the operator $R_{Qs} : L_p^N(Q_{s1} \times (0, T)) \to L_p^N(Q_{s1} \times (0, T))$ defined by the equality $R_{Qs} = U_s R_Q U_s^{-1}$ is the operator of multiplication by the matrix $R_s = \{r_{ml}^s\}$. At the same time, the operator $R_{Qs}^* : L_p^N(\Omega_{s1}) \to L_p^N(\Omega_{s1})$ defined by

$$R_{Qs}^* = U_s R_Q^* U_s^{-1}, \tag{4.1}$$

is the operator of multiplication by the transposed matrix $R_s^* = \{r_{lm}^s\}$ to the matrix $R_s = \{r_{ml}^s\}$ of order $N(s) \times N(s)$.

From Lemmas 8.13, 8.14 in [9] and Lemma 5 in [19], we obtain

Lemma 8. *The operators* $R_Q : L_p(0,T;\mathring{W}_p^1(Q)) \to L_p\left(0,T;W_p^1(Q)\right)$ *and* $R_Q^* : L_p(0,T;\mathring{W}_p^1(Q)) \to L_p\left(0,T;W_p^1(Q)\right)$ *are continuous. Moreover,*

$$\partial_i(R_Q u) = R_Q(\partial_i u), \qquad \partial_i(R_Q^* u) = R_Q^*(\partial_i u). \tag{4.2}$$

From Lemma 8.15 in [9] and Lemma 6 in [19], we obtain

Lemma 9. *For all* $u \in L_p\left(0,T;W_p^1(Q)\right)$*, we have* $R_Q u \in L_p\left(0,T;W_p^1(Q_{sl})\right)$ *and*

$$R_Q = \sum_s U_s^{-1} R_s U_s P_s, \tag{4.3}$$

$$\|R_Q u\|_{L_p\left(0,T;W_p^1(Q_{sl})\right)} \leqslant c_1 \sum_{j=1}^{N(s)} \|u\|_{L_p\left(0,T;W_p^1(Q_{sj})\right)} \quad (s = 1,2,\dots;\ l = 1,\dots,N(s)). \tag{4.4}$$

If $\det R_{s_\nu} \neq 0$ $(\nu = 1,\dots,n_1)$*, then there exists an inverse operator* $R_Q^{-1} : L_p(\Omega_T) \to L_p(\Omega_T)$ *and* $R_Q^{-1}w$ *belongs to* $L_p\left(0,T;W_p^1(Q_{sl})\right)$ *for all* $w \in L_p\left(0,T;W_p^1(Q)\right)$*; moreover,*

$$R_Q^{-1} = \sum_s U_s^{-1} R_s^{-1} U_s P_s, \tag{4.5}$$

$$\|R_Q^{-1} w\|_{L_p\left(0,T;W_p^1(Q_{sl})\right)} \leqslant c_2 \sum_{j=1}^{N(s)} \|w\|_{L_p\left(0,T;W_p^1(Q_{sj})\right)} \quad (s = 1,2,\dots;\ l = 1,\dots,N(s)). \tag{4.6}$$

Here the constants $c_1, c_2 > 0$ *are independent of* s*,* u*, and* w*.*

We also consider symmetric and skew-symmetric parts of the operator R_Q:

$$R_Q^{sym} := \frac{1}{2}(R_Q + R_Q^*), \qquad R_Q^{sk} := \frac{1}{2}(R_Q - R_Q^*).$$

Obviously, R_{Qs}^{sym} and R_{Qs}^{sk} are the operators of multiplication by the matrices

$$R_s^{sym} = \frac{1}{2}(R_s + R_s^*) \qquad \text{and} \qquad R_s^{sk} = \frac{1}{2}(R_s - R_s^*),$$

respectively. For these operators, the properties from Lemmas 8 and 9 hold.

In a standard way, we define scalar products in the spaces $L_2(Q)$ and $L_2(\Omega_T)$:

$$(u,v)_{L_2(Q)} := \int_Q u(x)\,v(x)\,dx, \qquad (u,v)_{L_2(\Omega_T)} := \int_{\Omega_T} u(x,t)\,v(x,t)\,dx\,dt.$$

Definition 3. The operator $R_Q : L_2(Q) \to L_2(Q)$ is *positive definite* if there exists $c_3 > 0$ such that

$$(R_Q u, u)_{L_2(Q)} \geqslant c_3 \|u\|^2_{L_2(Q)} \qquad \forall u \in L_2(Q).$$

Remark 1. R_Q and R_Q^{sym} are positive definite if and only if $R_s^{sym} > 0$ for any s, see Lemma 8.8 in [9] or Lemma 2.8 in [12]. Moreover, if $R_s^{sym} > 0$, then R_s and R_{s0} are nonsingular.

Lemma 10. *If* $R_s^{sym} > 0$ *for any* s*, then there exists a* $c_3 > 0$ *such that*

$$\begin{aligned}(R_Q u(t), u(t))_{L_2(Q)} = \left(R_Q^{sym} u(t), u(t)\right)_{L_2(Q)} &= \sum_s \left\|\sqrt{R_{Qs}^{sym}}\,U_s P_s u(t)\right\|^2_{L_2^N(Q_{s1})} \\ &= \left\|\sqrt{R_Q^{sym}}\,u(t)\right\|^2_{L_2(Q)} \geqslant c_3 \|u(t)\|^2_{L_2(Q)} \qquad \forall u(t) \in L_2(Q).\end{aligned} \tag{4.7}$$

Proof. Clearly,

$$(R_Q u(t), u(t))_{L_2(Q)} = \left(u(t), R_Q^* u(t)\right)_{L_2(Q)} = \frac{1}{2}\left((R_Q + R_Q^*)u(t), u(t)\right)_{L_2(Q)} = \left(R_Q^{sym} u(t), u(t)\right)_{L_2(Q)}.$$

From here we obtain the first equality in (4.7). By virtue of the relation $R_s^{sym} > 0$, a positive $\sqrt{R_s^{sym}}$ exists. From formulas (2.8), (4.1), and (4.3), we derive

$$\left(R_Q^{sym}u(t), u(t)\right)_{L_2(Q)} = \sum_s \left(R_s^{sym}U_sP_su(t), U_sP_su(t)\right)_{L_2^N(Q_{s1})}$$
$$= \sum_s \left(\sqrt{R_s^{sym}}U_sP_su(t), \sqrt{R_s^{sym}}U_sP_su(t)\right)_{L_2^N(Q_{s1})} = \sum_s \left\|\sqrt{R_{Qs}^{sym}}\,U_sP_su(t)\right\|^2_{L_2^N(Q_{s1})}.$$

The symmetrical difference operator R_Q^{sym} is positive definite if and only if $R_s^{sym} > 0$ for any s, see Lemma 8.8 in [9]. Estimate (4.7) is proved. □

Corollary 2. *If $R_s^{sym} > 0$ for any s, then there exists a $c_4 > 0$ such that*

$$(R_Qu, u)_{L_2(\Omega_T)} = \left(R_Q^{sym}u, u\right)_{L_2(\Omega_T)} = \sum_s \left\|\sqrt{R_{Qs}^{sym}}\,U_sP_su\right\|^2_{L_2^N(Q_{s1}\times(0,T))}$$
$$\geqslant c_4\|u\|^2_{L_2(\Omega_T)} \qquad \forall u \in L_2(\Omega_T). \quad (4.8)$$

Definition 4. A linear operator $\Lambda : L_p(0,T;\mathring{W}_p^1(Q)) \supset \mathscr{D}(\Lambda) \to L_q(0,T;W_q^{-1}(Q))$ is *monotone* if

$$\langle \Lambda u, u\rangle \geqslant 0 \qquad \forall u \in \mathscr{D}(\Lambda).$$

A linear densely defined monotone operator Λ is *maximally monotone* if there is no linear monotone operator that is a strict extension of Λ.

As is known, in reflexive strictly convex spaces together with its conjugate, the maximal monotonicity of the operator is equivalent to the condition:

$$\langle \Lambda u, u\rangle \geqslant 0 \quad \forall u \in \mathscr{D}(\Lambda), \qquad \langle \Lambda^* v, v\rangle \geqslant 0 \quad \forall v \in \mathscr{D}(\Lambda^*), \quad (4.9)$$

see Lemma 1.1 [18, Chapter 3]. It is well known that the operator $\Lambda = \partial_t$ with the domain

$$\mathscr{D}(\Lambda) = \{u \in L_p(0,T;\mathring{W}_p^1(Q)) : \Lambda u \in L_q(0,T;W_q^{-1}(Q)),\ u|_{t=0} = 0\}, \quad (4.10)$$

is maximal monotone, $\partial_t^* = -\partial_t$, and $\mathscr{D}(\Lambda^*) = \{v \in L_p(0,T;\mathring{W}_p^1(Q)) : \partial_t v \in L_q(0,T;W_q^{-1}(Q)),\ v|_{t=T} = 0\}$, see [18, Chapter 3].

Theorem 3. *Let $R_s^{sym} > 0$ for any s. Then the operator $\Lambda = \partial_t R_Q$ with the domain given by (4.10) is maximal monotone.*

Proof. According to the rules of differentiation and by virtue of Lemma 6, we have

$$\partial_t\left(R_Qu(t), u(t)\right)_{L_2(Q)} = \left(\partial_t R_Qu(t), u(t)\right)_{L_2(Q)} + \left(R_Qu(t), \partial_t u(t)\right)_{L_2(Q)}$$
$$= \left(R_Q\partial_t u(t), u(t)\right)_{L_2(Q)} + \left(u(t), R_Q^*\partial_t u(t)\right)_{L_2(Q)} = 2\left(\partial_t R_Q^{sym}u(t), u(t)\right)_{L_2(Q)}$$

for any $u \in W$. Since $\langle \partial_t R_Q^{sk}u, u\rangle = 0$, we obtain

$$\langle \partial_t R_Qu, u\rangle = \langle \partial_t R_Q^{sym}u, u\rangle = \int_0^T \left(\partial_t R_Q^{sym}u(\tau), u(\tau)\right)_{L_2(Q)} d\tau$$
$$= \frac{1}{2}\left(R_Q^{sym}u(T), u(T)\right)_{L_2(Q)} - \frac{1}{2}\left(R_Q^{sym}u(0), u(0)\right)_{L_2(Q)} \qquad \forall u \in W. \quad (4.11)$$

If $u \in \mathscr{D}(\Lambda)$, we can use estimate (4.7):

$$\langle \partial_t R_Qu, u\rangle = \frac{1}{2}\left(R_Q^{sym}u(T), u(T)\right)_{L_2(Q)} \geqslant \frac{c_3}{2}\|u(T)\|^2_{L_2(Q)}.$$

On the other hand, $\Lambda^* = -\partial_t R^*$ has the domain

$$\mathscr{D}(\Lambda^*) = \{v \in L_p(0,T;\mathring{W}_p^1(Q)) : \partial_t R^* v \in L_q(0,T;W_q^{-1}(Q)),\ v|_{t=T} = 0\},$$

and by (4.7) and (4.11), we get

$$\langle (\partial_t R_Q)^* v, v\rangle = -\langle \partial_t R_Q^* v, v\rangle = -\langle \partial_t R_Q^{sym} v, v\rangle \geqslant \frac{c_3}{2}\|v(0)\|^2_{L_2(Q)} \geqslant 0$$

for any $v \in \mathscr{D}(\Lambda^*)$. Condition (4.9) is fulfilled.

5. PROPERTIES OF THE OPERATOR $\Delta_p R_Q$

Let us consider properties of the operator $\Delta_p R_Q : L_p(0,T;\mathring{W}_p^1(Q)) \to L_q(0,T;W_q^{-1}(Q))$ given by

$$\langle \Delta_p R_Q u, v\rangle = \sum_{1\leqslant i\leqslant n} \int_{\Omega_T} |\partial_i R_Q u|^{p-2}\partial_i R_Q u\, \partial_i v\, dx\, dt \qquad \forall u,v \in L_p(0,T;\mathring{W}_p^1(Q)). \tag{5.1}$$

In order to state some definitions, we denote by X a reflexive Banach space; X^* is conjugate to X.

Definition 5. An operator $A : X \to X^*$ is *demicontinuous* if it is continuous from the strong topology of X to the weak topology of X^*.

A nonlinear analog of the notion of the nonnegative definiteness of an operator is accretivity. To define it, it is necessary to introduce the duality operator. Recall that a mapping $J : X \to X^*$ from a Banach space X to the dual space X^* is called *the duality mapping with respect to a function* Φ if

$$\langle Ju,u\rangle_X = \|Ju\|_{X^*}\|u\|_X \qquad \text{and} \qquad \|Ju\|_{X^*} = \Phi(\|u\|_X) \qquad \text{for all}\, u\in X,$$

see [18, Sec. 2.2]. Note that the standard duality operator J for the Lebesgue space $L_p(\Omega_T)$, which is given by $Ju = |u|^{p-2}u$, is the duality mapping with respect to the function $\Phi(r) = r^{p-1}$. By construction, J is bounded. Moreover, J is demicontinuous, see Proposition 2.4 in [18, Ch. 2, §2].

Definition 6. A linear operator $\hat{R}_Q : L_p(\Omega_T) \to L_p(\Omega_T)$ is said to be *accretive* if

$$\langle Ju, \hat{R}_Q u\rangle := \int_{\Omega_T} Ju\, \hat{R}_Q u\, dx\, dt \geqslant 0 \qquad \text{for any } u \in L_p(\Omega_T).$$

$\hat{R}_Q : L_p(\Omega_T) \to L_p(\Omega_T)$ is **strongly accretive** if there exists $c_5 > 0$ such that

$$\langle Ju, \hat{R}_Q u\rangle \geqslant c_5 \|u\|^p_{L_p(\Omega_T)}.$$

Further we assume that $\hat{R}_Q = R_Q^{-1}$. We denote $\hat{R}_s = R_s^{-1}$, $s = 1,\dots,n_1$.
We need the following auxiliary result.

Lemma 11. *Let $\lambda, a, b \in \mathbb{R}_+$, $p > 2$. Then*

$$a^p \pm \lambda ab\left(a^{p-2} - b^{p-2}\right) + b^p \geqslant 0 \qquad \forall a,b\in\mathbb{R}_+ \tag{5.2}$$

if $\lambda \in \mathbb{R}_+$ is such that

$$\frac{\lambda^p}{\lambda+1} \leqslant \frac{p^p}{(p-1)^{p-1}}. \tag{5.3}$$

In particular, estimate (5.2) *is true if*

$$\lambda \leqslant \frac{p}{p-1}\sqrt[p-1]{p}. \tag{5.4}$$

Proof. By virtue of the symmetry in formula (5.2), without loss of generality we can assume that $a \geqslant b$. Then we consider the function

$$a^p \pm \lambda ab\left(a^{p-2} - b^{p-2}\right) + b^p = a^p\left(1 \pm \lambda\frac{b}{a} \mp \lambda\left(\frac{b}{a}\right)^{p-1} + \left(\frac{b}{a}\right)^p\right).$$

Since $p > 2$ and $b/a \leqslant 1$, we obtain $b/a \geqslant (b/a)^{p-1}$ and

$$1 + \lambda\frac{b}{a} - \lambda\left(\frac{b}{a}\right)^{p-1} + \left(\frac{b}{a}\right)^p \geqslant 1 + \left(\frac{b}{a}\right)^p > 0.$$

Now let us consider the left part of (5.2) with the opposite signs before λ. Since $(b/a)^{p-1} \geqslant (b/a)^p$, we have

$$1 - \lambda\frac{b}{a} + \lambda\left(\frac{b}{a}\right)^{p-1} + \left(\frac{b}{a}\right)^p \geqslant 1 - \lambda\frac{b}{a} + (\lambda+1)\left(\frac{b}{a}\right)^p.$$

Using the well-known formula $ab \leqslant a^p/p + b^q/q$, we obtain

$$1 - \lambda\frac{b}{a} + (\lambda+1)\left(\frac{b}{a}\right)^p \geqslant 1 - \frac{\lambda^q}{q\varepsilon^q} - \frac{\varepsilon^p}{p}\left(\frac{b}{a}\right)^p + (\lambda+1)\left(\frac{b}{a}\right)^p \geqslant 0$$

if $1 - \lambda^q/q\varepsilon^q \geqslant 0$ and $(\lambda+1) \geqslant \varepsilon^p/p$. We put $\varepsilon = \lambda/q^{1/q}$. Then $(\lambda+1) \geqslant \varepsilon^p/p = \lambda^p p^{-1} q^{-p/q}$. Hence

$$\frac{\lambda^p}{\lambda+1} \leqslant pq^{p/q} = \frac{p^p}{(p-1)^{p-1}}.$$

Therefore, (5.3) implies that (5.2) holds.

Obviously, if $\lambda \leqslant \frac{p}{p-1}\sqrt[p-1]{p}$, then $\lambda^{p-1} \leqslant p\left(\frac{p}{p-1}\right)^{p-1}$, i.e. $\frac{\lambda^p}{\lambda+1} \leqslant \frac{p^p}{(p-1)^{p-1}}$.

Lemma 12. *Let the operator $\hat{R}_Q : L_p(\Omega_T) \to L_p(\Omega_T)$ be such that corresponding matrices $\hat{R}_s = \{\hat{r}^s_{ml}\}$ satisfy the following conditions: for any $s = 1, \dots, n_1$ and for arbitrary $m = 1, 2, \dots, N(s)$,*

$$2\hat{r}^s_{mm} > \lambda^{-1}\sum_{l\neq m}|\hat{r}^s_{ml} - \hat{r}^s_{lm}| + \sum_{l\neq m}|\hat{r}^s_{ml} + \hat{r}^s_{lm}| \tag{5.5}$$

where λ satisfies estimate (5.3) *or estimate* (5.4)*. Then $\hat{R}_Q$ is strongly accretive.*

Proof. By virtue of (4.5), we have the representation

$$\begin{aligned}\langle Ju, \hat{R}_Q u\rangle &= \sum_s \int_{\Omega_s} |P_s u|^{p-2}(P_s u)\left(U_s^{-1}\hat{R}_s U_s P_s u\right) dx\,dt \\ &= \sum_s \int_{\Omega_{s1}} \left(U_s\left(|P_s u|^{p-2}(P_s u)\right), \hat{R}_s U_s P_s u\right) dx\,dt \\ &= \sum_s \sum_{1\leqslant m,l\leqslant N(s)} \int_{\Omega_{s1}} \hat{r}^s_{ml}|u(x+h_{sm},t)|^{p-2}u(x+h_{sm},t)u(x+h_{sl},t)\,dx\,dt. \end{aligned}\tag{5.6}$$

Let us evaluate the integrand in (5.6). Let $\xi \in \mathbb{R}^{N(s)}$ be an arbitrary vector. We denote by $\hat{r}^{s,sym}_{ml} = \frac{1}{2}\big(\hat{r}^s_{ml} + \hat{r}^s_{lm}\big)$ and $\hat{r}^{s,sk}_{ml} = \frac{1}{2}\big(\hat{r}^s_{ml} - \hat{r}^s_{lm}\big)$ the elements of the symmetric and skew-symmetric parts of $\hat{R}_s$. Then

$$\begin{aligned}\sum_{1\leqslant m,l\leqslant N(s)} \hat{r}^s_{ml}|\xi_m|^{p-2}\xi_m\xi_l &= \sum_{1\leqslant m,l\leqslant N(s)} \hat{r}^{s,sk}_{ml}|\xi_m|^{p-2}\xi_m\xi_l + \sum_{1\leqslant m,l\leqslant N(s)} \hat{r}^{s,sym}_{ml}|\xi_m|^{p-2}\xi_m\xi_l \\ &\geqslant \lambda^{-1}\sum_{1\leqslant m<l\leqslant N(s)} \left|\hat{r}^{s,sk}_{ml}\right|\big(|\xi_m|^p - \lambda|\xi_m\xi_l|\,\big||\xi_m|^{p-2} - |\xi_l|^{p-2}\big| + |\xi_l|^p\big) \\ &+ \sum_{1\leqslant m<l\leqslant N(s)} \left|\hat{r}^{s,sym}_{ml}\right|\big(|\xi_m|^p + \operatorname{sign}(\hat{r}^{s,sym}_{ml})|\xi_m|^{p-2}\xi_m\xi_l + \operatorname{sign}(\hat{r}^{s,sym}_{ml})|\xi_l|^{p-2}\xi_m\xi_l + |\xi_l|^p\big) \\ &+ \sum_{1\leqslant m\leqslant N(s)} \hat{r}^s_{mm}|\xi_m|^p - \lambda^{-1}\sum_{1\leqslant m<l\leqslant N(s)} \left|\hat{r}^{s,sk}_{ml}\right|\big(|\xi_m|^p + |\xi_l|^p\big) \\ &- \sum_{1\leqslant m<l\leqslant N(s)} \left|\hat{r}^{s,sym}_{ml}\right|\big(|\xi_m|^p + |\xi_l|^p\big). \end{aligned}\tag{5.7}$$

The first summand in the right part of (5.7) is nonnegative by virtue of estimate (5.2) in Lemma 11. The second summand in the right-hand side of (5.7) is nonnegative because

$$|\xi_m|^p \pm |\xi_m|^{p-2}\xi_m\xi_l \pm |\xi_l|^{p-2}\xi_m\xi_l + |\xi_l|^p \geqslant (|\xi_m|^{p-1} - |\xi_l|^{p-1})(|\xi_m| - |\xi_l|) \geqslant 0.$$

Thus, we obtain

$$\sum_{1\leqslant m,l\leqslant N(s)} \hat{r}^s_{ml}|\xi_m|^{p-2}\xi_m\xi_l \geqslant \sum_{1\leqslant m\leqslant N(s)} \left(\hat{r}^s_{mm} - \lambda^{-1}\sum_{l\neq m}\left|\hat{r}^{s,sk}_{ml}\right| - \sum_{l\neq m}\left|\hat{r}^{s,sym}_{ml}\right|\right)|\xi_m|^p. \tag{5.8}$$

Substituting estimate (5.8) into (5.6), we obtain

$$\langle Ju, \hat{R}_Q u\rangle = \sum_s \int\limits_{\Omega_s} |P_s u|^{p-2} (P_s u) \left(U_s^{-1}\hat{R}_s U_s P_s u\right) dx\,dt$$

$$\geqslant \sum_s \int\limits_{\Omega_{s1}} \sum_{1\leqslant m\leqslant N(s)} \left(\hat{r}^s_{mm} - \lambda^{-1}\sum_{l\neq m}\left|\hat{r}^{s,sk}_{ml}\right| - \sum_{l\neq m}|\hat{r}^{s,sym}_{ml}|\right) |u(x+h_{sm},t)|^p\,dx\,dt$$

$$= \sum_s \sum_{1\leqslant m\leqslant N(s)} \left(\hat{r}^s_{mm} - \lambda^{-1}\sum_{l\neq m}\left|\hat{r}^{s,sk}_{ml}\right| - \sum_{l\neq m}|\hat{r}^{s,sym}_{ml}|\right) \|u\|^p_{L_p(\Omega_{sm})}, \tag{5.9}$$

where $\Omega_{sm} = Q_{sm}\times(0,T)$, $Q_{sm} = Q_{s1}+h_{sm}$. Obviously, if condition (5.5) holds, the operator $\hat{R}_Q$ is strongly accretive with constant

$$c_5 = \min_m \left\{\hat{r}^s_{mm} - \lambda^{-1}\sum_{l\neq m}|\hat{r}^{s,sk}_{ml}| - \sum_{l\neq m}|\hat{r}^{s,sym}_{ml}|\right\}. \tag{5.10}$$

Remark 2. Let $\lambda_{max}(p)$ be such that $\dfrac{\lambda_{max}(p)^p}{\lambda_{max}(p)+1} = \dfrac{p^p}{(p-1)^{p-1}}$. By virtue of inequality (5.3), the function $\lambda_{max}(p)$ monotonously decreases if p increases, $\lambda_{max}(p)\in(1, 2+2\sqrt{2}]$ for $p\geqslant 2$, and $\lambda_{max}(p)\to 1$ as $p\to\infty$.

Example 4. Let $Ru(x,t) = u(x_1,x_2,t) - u(x_1+1,x_2,t) + u(x_1+2,x_2,t)$, $Q=(0,3)\times(0,1)$. Here the matrix

$$R_1 = \begin{pmatrix} 1 & -1 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}$$

corresponds to the operator R_Q. The inverse matrix is given by

$$R_1^{-1} = \begin{pmatrix} 1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}.$$

For any $p\in(2,\infty)$, there exists a $\varepsilon>0$ such that $\lambda = 1+\varepsilon$ satisfies inequality (5.3). Then

$$2\cdot 1 > (1+\varepsilon)^{-1}|1-0| + |1+0|, \qquad 2\cdot 1 > (1+\varepsilon)^{-1}|0-1| + |0+1|.$$

Condition (5.5) holds, i.e., the operator R_Q^{-1} is strongly accretive for any $p\in(2,\infty)$.

Below we use the following properties of the duality operator and of accretive operators.

Lemma 13. *Let $u_m \rightharpoonup u$ weakly in $L_p(\Omega_T)$. Then $J(u_m)\rightharpoonup J(u)$ weakly in $L_q(\Omega_T)$ and*

$$\varliminf_{m\to\infty}\langle Ju_m, u_m\rangle \geqslant \langle Ju, u\rangle. \tag{5.11}$$

Moreover, for any strongly accretive operator $\hat{R}_Q$,

$$\varliminf_{m\to\infty}\langle Ju_m, \hat{R}_Q u_m\rangle \geqslant \langle Ju, \hat{R}_Q u\rangle. \tag{5.12}$$

The proof coincides with the proof of the Lemmas 5–7 in [22], since the conditions imposed in the present paper on the Ω_T coincide with the conditions imposed on the Q in [22].

Definition 7. An operator $A: X\to X^*$ is *coercive* if $\langle Au, u-\tilde{u}\rangle \geqslant c(\|u-\tilde{u}\|_X)\|u-\tilde{u}\|_X$ for some fixed $\tilde{u}\in X$, where $c:\mathbb{R}_+\to\mathbb{R}$ and $c(s)\to\infty$ as $s\to\infty$.

Definition 8. $A: X\to X^*$ is *pseudomonotone* if for any $u_m\rightharpoonup u$ weakly in X such that $\varlimsup\limits_{m\to\infty}\langle Au_m, u_m - u\rangle\leqslant 0$, we have

$$\varliminf_{m\to\infty}\langle Au_m, u_m-\xi\rangle \geqslant \langle Au, u-\xi\rangle \qquad \forall \xi\in X. \tag{5.13}$$

Lemma 14. *The operator $\Delta_p R_Q : L_p(0,T; R_Q(\mathring{W}_1^p(Q)) \to L_q(0,T; W_q^{-1}(Q))$ is bounded and demicontinuous.*

Proof. The linear operator $R_Q : L_p(0,T; R_Q(\mathring{W}_1^p(Q)) \to L_p\left(0,T; W_{p,\gamma}^1(Q)\right)$ is bounded, see Lemmas 4 and 8; $\Delta_p : L_p\left(0,T; R_Q(W_{p,\gamma}^1(Q)\right) \to L_q(0,T; W_q^{-1}(Q))$ is bounded and demicontinuous, see, for example, [18, Ch. 2, §1.1–1.2]. Thus, their composition is a bounded, demicontinuous operator.

Theorem 4. *Let $R_Q : L_p(\Omega_T) \to L_p(\Omega_T)$ be nonsingular, and let $R_Q^{-1} : L_p(\Omega_T) \to L_p(\Omega_T)$ be strongly accretive. Then $\Delta_p R_Q$ is pseudomonotone and coercive.*

Proof. Since the difference operator $R_Q : L_p(\Omega_T) \to L_p(\Omega_T)$ is nonsingular, it follows that there exists an inverse operator $R_Q^{-1} : L_p(\Omega_T) \to L_p(\Omega_T)$. Moreover, by virtue of (5.1), for all $u, \xi \in L_p(0,T; \mathring{W}_p^1(Q))$ and $w, \zeta \in L_p(0,T; W_{p,\gamma}^1(Q))$ such that $w = R_Q u$, $\zeta = R_Q \xi$, we have

$$\langle \Delta_p R_Q u, \xi \rangle = \sum_{1 \leqslant i \leqslant n} \langle J \partial_i R_Q u, \partial_i \xi \rangle = \sum_{1 \leqslant i \leqslant n} \langle J \partial_i w, R_Q^{-1} \partial_i \zeta \rangle. \tag{5.14}$$

Here we have taken into account the commutativity of the difference operator R_Q with constant coefficients a_h and the differential operator ∂_i (see Lemma 10): $R_Q^{-1}\partial_i \zeta = R_Q^{-1}\partial_i R_Q \xi = R_Q^{-1} R_Q \partial_i \xi = \partial_i \xi$.

Suppose that $u_m \rightharpoonup u$ weakly in $L_p(0,T; \mathring{W}_p^1(Q))$ and

$$\varlimsup_{m\to\infty} \langle \Delta_p R_Q u_m, u_m - u \rangle \leqslant 0.$$

Consider the sequence $\{w_m = R_Q u_m\}$. By virtue of equalities (4.2),

$$\begin{aligned} \lim_{m\to\infty} \langle \partial_i w_m, \xi \rangle = \lim_{m\to\infty} \langle \partial_i R_Q u_m, \xi \rangle = \lim_{m\to\infty} \langle R_Q \partial_i u_m, \xi \rangle = \lim_{m\to\infty} \langle \partial_i u_m, R_Q^* \xi \rangle \\ = \langle \partial_i u, R_Q^* \xi \rangle = \langle \partial_i R_Q u, \xi \rangle = \langle \partial_i w, \xi \rangle \qquad \forall \xi \in L_q(\Omega_T), \end{aligned} \tag{5.15}$$

i.e., $\partial_i w_m \rightharpoonup \partial_i w = R_Q \partial_i u$ weakly in $L_p(\Omega_T)$ for any $i = 1, \ldots, n$. By virtue of (5.14), we also have

$$\varlimsup_{m\to\infty} \sum_{1 \leqslant i \leqslant n} \langle J \partial_i w_m, R_Q^{-1} \partial_i (w_m - w) \rangle = \varlimsup_{m\to\infty} \langle \Delta_p R_Q u_m, u_m - u \rangle \leqslant 0. \tag{5.16}$$

Thus, there exists at least one index $i = i_1$ such that

$$\varlimsup_{m\to\infty} \langle J \partial_i w_m, R_Q^{-1} \partial_i (w_m - w) \rangle \leqslant 0. \tag{5.17}$$

i.e., using the weak convergence $J\partial_i w_m \rightharpoonup J \partial_i w$ in $L_q(\Omega)$, see Lemma 13, we have

$$\varlimsup_{m\to\infty} \langle J \partial_i w_m, R_Q^{-1} \partial_i w_m \rangle \leqslant \lim_{m\to\infty} \langle J \partial_i w_m, R_Q^{-1} \partial_i w \rangle = \langle J \partial_i w, R_Q^{-1} \partial_i w \rangle.$$

On the other hand, by virtue of strong accretivity of R_Q^{-1}, we get

$$\varliminf_{m\to\infty} \langle J \partial_i w_m, R_Q^{-1} \partial_i w_m \rangle \geqslant \langle J \partial_i w, R_Q^{-1} \partial_i w \rangle, \qquad i = i_1,$$

see estimate (5.12) in Lemma 13. Thus, for $i = i_1$,

$$\lim_{m\to\infty} \langle J \partial_i w_m, R_Q^{-1} \partial_i w_m \rangle = \langle J \partial_i w, R_Q^{-1} \partial_i w \rangle. \tag{5.18}$$

According (5.16) and (5.18), we have

$$\varlimsup_{m\to\infty} \sum_{i \neq i_1} \langle J \partial_i w_m, R_Q^{-1} (\partial_i w_m - \partial_i w) \rangle \leqslant 0. \tag{5.19}$$

Repeating the arguments used in the proof of (5.18), we see that (5.19) implies

$$\lim_{m\to\infty} \langle J \partial_i w_m, R_Q^{-1} \partial_i w_m \rangle = \langle J \partial_i w, R_Q^{-1} \partial_i w \rangle \qquad \text{for some } i = i_2 \neq i_1.$$

Thus, in a finite number of steps, we conclude that

$$\lim_{m\to\infty}\langle J\partial_i w_m, R_Q^{-1}\partial_i w_m\rangle = \langle J\partial_i w, R_Q^{-1}\partial_i w\rangle \qquad \forall i=1,\dots,n.$$

Therefore,

$$\lim_{m\to\infty}\langle \Delta_p R_Q u_m, u_m\rangle = \lim_{m\to\infty}\sum_{1\leqslant i\leqslant n}\langle J\partial_i w_m, R_Q^{-1}\partial_i w_m\rangle = \sum_{1\leqslant i\leqslant n}\langle J\partial_i w, R_Q^{-1}\partial_i w\rangle = \langle \Delta_p R_Q u, u\rangle. \tag{5.20}$$

Pseudomonotonicity of the operator $\Delta_p R_Q$ is proved.

Let us show that the operator $\Delta_p R_Q$ is coercive. Since the operator R_Q commutes with the ∂_i and the operator R_Q^{-1} is strongly accretive, we have

$$\langle \Delta_p R_Q u, u\rangle = \sum_{1\leqslant i\leqslant n}\langle J\partial_i R_Q u, R_Q^{-1}\partial_i R_Q u\rangle = \sum_{1\leqslant i\leqslant n}\langle J\partial_i w, R_Q^{-1}\partial_i w\rangle \geqslant c_5\sum_{1\leqslant i\leqslant n}\|\partial_i w\|^p_{L_p(\Omega_T)},$$

where c_5 is the coefficient corresponding to the strongly accretive operator R_Q^{-1}. By virtue of (4.4) and (4.6),

$$\|\partial_i w\|^p_{L_p(\Omega_T)} = \|\partial_i R_Q u\|^p_{L_p(\Omega_T)} \geqslant c_6\|\partial_i u\|^p_{L_p(\Omega_T)},$$

i.e.,

$$\langle \Delta_p R_Q u, u\rangle \geqslant c_5 c_6\sum_{1\leqslant i\leqslant n}\|\partial_i u\|^p_{L_p(\Omega_T)} = c_7\|u\|^p_{L_p(0,T;\mathring{W}_p^1(Q))}, \tag{5.21}$$

for some $c_7>0$. The operator $\Delta_p R_Q$ is coercive with respect to $\tilde u = 0$. □

Remark 3. If the conditions of Theorem 4 hold, then the operator

$$\Delta_p R_Q(\cdot+\hat u): L_p(0,T;\mathring{W}_p^1(Q)) \to L_q(0,T;W_q^{-1}(Q))$$

is demicontinuous, pseudomonotone, and coercive.

Proof. By construction, $\Delta_p R_Q(\cdot+\hat u)u := \Delta_p R_Q(u+\hat u)$ for arbitrary $u\in L_p(0,T;\mathring{W}_p^1(Q))$. Thus, using (5.21), we have

$$\langle \Delta_p R_Q(u+\hat u), u+\hat u\rangle \geqslant c_7\|u+\hat u\|^p_{L_p(0,T;\mathring{W}_p^1(Q))},$$

for some $c_7>0$. $\Delta_p R_Q(\cdot+\hat u)$ is coercive with respect to $\tilde u = -\hat u$.

Let $u_m \rightharpoonup u$ weakly in $L_p(0,T;\mathring{W}_p^1(Q))$ and $\overline{\lim\limits_{m\to\infty}}\langle \Delta_p R_Q(u_m+\hat u), u_m - u\rangle \leqslant 0$. Obviously, $\overline{\lim\limits_{m\to\infty}}\langle \Delta_p R_Q(u_m+\hat u), (u_m+\hat u)-(u+\hat u)\rangle \leqslant 0$ and $u_m+\hat u \rightharpoonup u+\hat u$ weakly in $L_p(0,T;\mathring{W}_p^1(Q))$ too. By virtue of (5.20),

$$\begin{aligned}\overline{\lim_{m\to\infty}}\langle \Delta_p R_Q(u_m+\hat u), u_m-\xi\rangle &= \overline{\lim_{m\to\infty}}\langle \Delta_p R_Q(u_m+\hat u), u_m+\hat u-\xi-\hat u\rangle\\ &= \langle \Delta_p R_Q(u+\hat u), u+\hat u-\xi-\hat u\rangle = \langle \Delta_p R_Q(u+\hat u), u-\xi\rangle\end{aligned} \tag{5.22}$$

for any $\xi\in L_p(0,T;\mathring{W}_p^1(Q))$. The operator $\Delta_p R_Q(\cdot+\hat u)$ is pseudomonotone. □

6. EXISTENCE OF SOLUTION

Theorem 5. *Let the operator R_Q^{sym} be positive definite, and let the inverse operator R_Q^{-1} be strongly accretive. Then, for any $f\in L_q(0,T;W_q^{-1}(Q))$ and $\varphi\in L_2(Q)$, there exists at least one solution of problem (3.5), (3.6). Moreover, the set of such solutions is weakly compact in $L_p(0,T;\mathring{W}_p^1(Q))$, and the solutions satisfy the following estimates:*

$$\|u(T)\|^2_{L_2(Q)} \leqslant c_8\|f\|^q_{L_q(0,T;W_q^{-1}(Q))} + c_9\|\varphi\|^2_{L_2(Q)}, \tag{6.1}$$

$$\|u\|^p_{L_p(0,T;\mathring{W}_p^1(Q))} \leqslant c_{10}\|f\|^q_{L_q(0,T;W_q^{-1}(Q))} + c_{11}\|\varphi\|^2_{L_2(Q)}, \tag{6.2}$$

where $c_8, c_9, c_{10}, c_{11} > 0$ do not depend on u, f, and φ.

Proof. First we consider the case $\varphi = 0$. Then by virtue of Theorem 3, the operator $\partial_t R_Q$ is maximal monotone, and by virtue of Lemma 14 and Theorem 4 the operator $\Delta_p R_Q$ is demicontinuous, pseudomonotone, and coercive. Thus, the conditions of Theorem 1.1 [18, Ch.III, §1] hold. Therefore problem (3.5), (3.6) has at least one solution.

If $\varphi \neq 0$, we consider an auxiliary fixed element $\hat{u} \in W \subset C(0,T;L_2(Q))$. Since $C^1(0,T;\mathring{W}_p^1(Q)) \cap W$ densely imbedded into W, see Lemma 1.12 in [21, Ch. IV], and W continuously imbedded into $C(0,T;L_2(Q))$, see Theorem 1.17 in [21, Ch. IV], for any $\varphi \in L_2(Q)$, there exists $\hat{u} \in W \subset C(0,T;L_2(Q))$ such that $\hat{u}|_{t=0} = \varphi$. Substituting $u(x,t) = v(x,t) + \hat{u}(x,t)$, we obtain the equivalent equation

$$\partial_t R_Q v + \Delta_p R_Q (v + \hat{u}) = f - \partial_t R_Q \hat{u} := \widehat{f}, \qquad 0 < t < T, \tag{6.3}$$

$$v(0) = 0. \tag{6.4}$$

By virtue of Remark 3, the operator $\Delta_p R_Q(\cdot + \hat{u})$ is demicontinuous, pseudomonotone, and coercive. Thus, since the conditions of Theorem 1.1 in [18, Ch.III, §1] hold, then problem (6.3), (6.4) has at least one solution. Consequently, problem (3.5), (3.6) has at least one solution too.

For a solution of problem (3.5), (3.6), we have

$$\langle \partial_t R_Q u, u\rangle + \langle \Delta_p R_Q u, u\rangle = \langle f, u\rangle. \tag{6.5}$$

Repeating the arguments from (4.11), we obtain

$$\langle \partial_t R_Q u, u\rangle = \frac{1}{2}(R_Q^{sym} u(T), u(T))_{L_2(Q)} - \frac{1}{2}(R_Q^{sym} u(0), u(0))_{L_2(Q)} = \frac{1}{2}\left\|\sqrt{R_Q^{sym}}u(T)\right\|_{L_2(Q)}^2 - \frac{1}{2}\left\|\sqrt{R_Q^{sym}}\varphi\right\|_{L_2(Q)}^2.$$

The second summand in (6.5) was evaluated in (5.21). That is

$$\frac{1}{2}\left\|\sqrt{R_Q^{sym}}u(T)\right\|_{L_2(Q)}^2 - \frac{1}{2}\left\|\sqrt{R_Q^{sym}}\varphi\right\|_{L_2(Q)}^2 + c_7\|u\|_{L_p(0,T;\mathring{W}_p^1(Q))}^p \leqslant \langle \partial_t R_Q u, u\rangle + \langle \Delta_p R_Q u, u\rangle = \langle f, u\rangle. \tag{6.6}$$

Let's estimate the right part using Hölder's inequality and the well-known formula $ab \leqslant a^p/p + b^q/q$. We obtain

$$\langle f, u\rangle \leqslant \|f\|_{L_q(0,T;W_q^{-1}(Q))}\|u\|_{L_p(0,T;\mathring{W}_p^1(Q))} \leqslant \frac{1}{q\varepsilon^q}\|f\|_{L_q(0,T;W_q^{-1}(Q))}^q + \frac{\varepsilon^p}{p}\|u\|_{L_p(0,T;\mathring{W}_p^1(Q))}^p.$$

Let $\varepsilon > 0$ be such that $\varepsilon^p/p = c_7/2$. The first term of the left part of inequality (6.6) is nonnegative, thus

$$\frac{c_7}{2}\|u\|_{L_p(0,T;\mathring{W}_p^1(Q))}^p \leqslant \frac{1}{q\varepsilon^q}\|f\|_{L_q(0,T;W_q^{-1}(Q))}^q + \frac{1}{2}\left\|\sqrt{R_Q^{sym}}\,\varphi\right\|_{L_2(Q)}^2,$$

i.e.

$$\|u\|_{L_p(0,T;\mathring{W}_p^1(Q))}^p \leqslant \frac{1}{q}\left(\frac{2}{pc_7}\right)^{q-1}\|f\|_{L_q(0,T;W_q^{-1}(Q))}^q + \frac{1}{c_7}\left\|\sqrt{R_Q^{sym}}\,\varphi\right\|_{L_2(Q)}^2. \tag{6.7}$$

By virtue of boundedness of the linear operator R_Q^{sym}, we conclude that $\left\|\sqrt{R_Q^{sym}}\,\varphi\right\|_{L_2(Q)}^2 \leqslant c_{12}\|\varphi\|_{L_2(Q)}^2$ for some $c_{12} > 0$. This estimate and inequality (6.7) prove that inequality (6.2) is true.

On the other hand, the third term of the left part of (6.6) is also nonnegative. Using estimates (4.7) and (6.2), we obtain

$$\begin{aligned}\frac{c_3}{2}\|u(T)\|_{L_2(Q)}^2 &\leqslant \frac{1}{2}\left\|\sqrt{R_Q^{sym}}\,u(T)\right\|_{L_2(Q)}^2 \leqslant \langle f, u\rangle + \frac{1}{2}\left\|\sqrt{R_Q^{sym}}\,\varphi\right\|_{L_2(Q)}^2 \\ &\leqslant \frac{1}{q}\|f\|_{L_q(0,T;W_q^{-1}(Q))}^q + \frac{1}{p}\|u\|_{L_p(0,T;\mathring{W}_p^1(Q))}^p + \frac{c_{12}}{2}\|\varphi\|_{L_2(Q)}^2 \\ &\leqslant \left(\frac{1}{q} + \frac{c_{11}}{p}\right)\|f\|_{L_q(0,T;W_q^{-1}(Q))}^q + \left(\frac{c_{10}}{p} + \frac{c_{12}}{2}\right)\|\varphi\|_{L_2(Q)}^2.\end{aligned} \tag{6.8}$$

This proves estimate (6.1).

Now let us prove the weak compactness of the set of solutions. Let $\{u_m\}$ belong to the set of solutions of problem (3.5), (3.6) such that $u_m \rightharpoonup u$ in $L_p(0,T;\mathring{W}_p^1(Q))$. Since, $\{\partial_t R_Q u_m = f - \Delta_p R_Q u_m\}$ is a bounded set in $L_q(0,T;W_q^{-1}(Q))$, without loss of generality we can assume that $u_m \rightharpoonup u$ in W (up to the subsequences), here $u \in W$. Obviously, $u|_{t=0} = u_m|_{t=0} = \varphi$. By virtue of (4.11) and (4.7), we have

$$\langle \partial_t R_Q u_m - \partial_t R_Q u, u_m - u\rangle = \frac{1}{2}\left\|\sqrt{R_Q^{sym}}\,(u_m(T) - u(T))\right\|^2_{L_2(Q)} \geqslant 0 \quad \forall m.$$

Moreover, by virtue of weak convergence of the sequence $\{u_m\}$, we obtain

$$\lim_{m\to\infty}\langle f, u_m - u\rangle = 0 \qquad \text{and} \qquad \lim_{m\to\infty}\langle \partial_t R_Q u, u_m - u\rangle = 0.$$

Thus,

$$\overline{\lim_{m\to\infty}}\langle \Delta_p R_Q u_m, u_m - u\rangle = \overline{\lim_{m\to\infty}}\langle f - \partial_t R_Q u_m, u_m - u\rangle \leqslant 0.$$

Using the pseudomonotonocity of $\Delta_p R_Q$, we get

$$\langle \Delta_p R_Q u, u - \xi\rangle \leqslant \underline{\lim}_{m\to\infty}\langle \Delta_p R_Q u_m, u_m - \xi\rangle = \underline{\lim}_{m\to\infty}\langle f - \partial_t R_Q u_m, u_m - \xi\rangle$$
$$\leqslant \langle f - \partial_t R_Q u, u - \xi\rangle \qquad \forall \xi \in L_p(0,T;\mathring{W}_p^1(Q)). \quad (6.9)$$

By virtue of (6.9), u is a solution of operator equation (3.5). The set of solutions of problem (3.5), (3.6) is weakly compact in $L_p(0,T;\mathring{W}_p^1(Q))$.

Theorem 6. *Let Conditions* 1–3 *hold,* $R_s^{sym} > 0$*, and let* R_Q^{-1} *be strongly accretive operator. Then, for any* $f \in L_q(0,T;W_q^{-1}(Q))$ *and* $\psi \in L_2(Q)$*, there exists at least one generalized solution of problem* (1.1)–(1.3)*. Moreover, the set of such solutions is weakly compact in* $L_p(0,T;W_p^1(Q))$*, solutions satisfy the following estimates:*

$$\|w(T)\|^2_{L_2(Q)} \leqslant c_{13}\|f\|^q_{L_q(0,T;W_q^{-1}(Q))} + c_{14}\|\psi\|^2_{L_2(Q)}, \tag{6.10}$$

$$\|w\|^p_{L_p(0,T;W_p^1(Q))} \leqslant c_{15}\|f\|^q_{L_q(0,T;W_q^{-1}(Q))} + c_{16}\|\psi\|^2_{L_2(Q)}. \tag{6.11}$$

where $c_{13}, c_{14}, c_{15}, c_{16} > 0$ *do not depend on* w, f*, and* ψ*.*

Proof. From Conditions 1–3 and Theorem 1 it follows that there exists a set $\gamma = \{\gamma_{lj}^r\}$ such that the operator R_Q is an isomorphism of $L_p(0,T;\mathring{W}_p^1(Q))$ onto $L_p(0,T;W_{p,\gamma}^1(Q))$. Therefore, by virtue of Theorem 5 and Theorem 2, there exists a generalized solution of nonlocal boundary value problem (1.1)–(1.3) with the above mentioned set γ. It remains to prove estimates (6.10) and (6.11). Let us prove estimate (6.10). From inequality (6.1) and boundedness of operators $R_Q : L_2(Q) \to L_2(Q)$ and $R_Q^{-1} : L_2(Q) \to L_2(Q)$ it follows that

$$\|w(T)\|^2_{L_2(Q)} = \|R_Q u\|^2_{C(0,T;L_2(Q))} \leqslant c_{17}\|u(T)\|^2_{L_2(Q)}$$
$$\leqslant c_{17}\left(c_8\|f\|^q_{L_q(0,T;W_q^{-1}(Q))} + c_9\|\varphi\|^2_{L_2(Q)}\right) = c_{17}\left(c_8\|f\|^q_{L_q(0,T;W_q^{-1}(Q))} + c_9\|R_Q^{-1}\psi\|^2_{L_2(Q)}\right)$$
$$\leqslant c_{17}\left(c_8\|f\|^q_{L_q(0,T;W_q^{-1}(Q))} + c_9 c_{18}\|\psi\|^2_{L_2(Q)}\right).$$

In the same way, from inequality (6.2) and boundedness of operators $R_Q : L_p(0,T;\mathring{W}_p^1(Q)) \to L_q(0,T;W_q^{-1}(Q))$ and $R_Q^{-1} : L_2(Q) \to L_2(Q)$ it follows that

$$\|w\|^p_{L_p(0,T;W_p^1(Q))} = \|R_Q u\|^p_{L_p(0,T;W_p^1(Q))} \leqslant c_{19}\|u\|^p_{L_p(0,T;W_p^1(Q))}$$
$$\leqslant c_{19}\left(c_{10}\|f\|^q_{L_q(0,T;W_q^{-1}(Q))} + c_{11}\|\varphi\|^2_{L_2(Q)}\right)$$
$$\leqslant c_{19}\left(c_{10}\|f\|^q_{L_q(0,T;W_q^{-1}(Q))} + c_{11}c_{18}\|\psi\|^2_{L_2(Q)}\right).$$

Now we prove the weak compactness of the set of solutions. Let $\{w_m\}$ belong to the set of generalized solutions of problem (1.1)–(1.3) such that $w_m \rightharpoonup w$ in $L_p(0,T;W_p^1(Q))$. Obviously, $w|_{t=0} = w_m|_{t=0} = \psi$. Then there exists a sequence $\{u_m\}$ such that $w_m = R_Q u_m$ and u_m is a solution of (3.5), (3.6). As it was proved in (5.15), $u_m \rightharpoonup u$ in W if and only if $w_m \rightharpoonup w$ in W_γ. But in the previous theorem it is proved that in this case u is a solution of (3.5), (3.6). Therefore, $w = R_Q u$ is a generalized solution of problem (1.1)–(1.3).

7. EXAMPLES

Example 5. We continue to consider problem (1.1), (1.2) in the rectangular parallelepiped $\Omega_T = (0,2) \times (0,1) \times (0,T)$ with Bitsadze–Samarskii nonlocal boundary conditions (1.5). As was proved in Example 2,

$$R_1 = \begin{pmatrix} 1 & \gamma_1 \\ \gamma_2 & 1 \end{pmatrix},$$

i.e., $R_s^{sym} > 0$ if

$$|\gamma_1 + \gamma_2| < 2. \tag{7.1}$$

At the same time,

$$R_1^{-1} = \frac{1}{1 - \gamma_1\gamma_2} \begin{pmatrix} 1 & -\gamma_1 \\ -\gamma_2 & 1 \end{pmatrix}.$$

Thus, R_Q^{-1} is strongly accretive if

$$2 > |\gamma_1 + \gamma_2| + \lambda^{-1}|\gamma_1 - \gamma_2|, \tag{7.2}$$

where λ satisfies estimate (5.3) or estimate (5.4), see Lemma 12. Obviously, if condition (7.2) holds, then condition (7.1) is also satisfied, i.e., by virtue of Theorem 6, problem (1.1), (1.2), (1.5) has at least one generalized solution.

Example 6. We consider problem (1.1), (1.2) in the rectangular domain $\Omega_T = (0,3) \times (0,1) \times (0,T)$ with boundary conditions

$$\left.\begin{array}{ll} w(x_1,0,t) = w(x_1,1,t) = 0 & (0 \leqslant x_1 \leqslant 2; 0 < t < T), \\ w(0,x_2,t) = \gamma_1 w(1,x_2,t) + \gamma_2 w(2,x_2,t) & (0 < x_2 < 1; 0 < t < T), \\ w(3,x_2,t) = \gamma_2 w(1,x_2,t) + \gamma_1 w(2,x_2,t) & (0 < x_2 < 1; 0 < t < T) \end{array}\right\} \tag{7.3}$$

for $\gamma_1 = 39/28$ and $\gamma_2 = -6/7$.

In this case, the difference operator is given by $Ru(x) = \sum\limits_{-2 \leqslant k \leqslant 2} a_k u(x_1 + k, x_2, t)$. We shall find a nonsingular martix

$$R_1 = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_0 & a_1 \\ a_2 & a_1 & a_0 \end{pmatrix}$$

such that the relations

$$\begin{cases} a_1 = \gamma_1 a_0 + \gamma_2 a_1, \\ a_2 = \gamma_2 a_0 + \gamma_1 a_1 \end{cases} \tag{7.4}$$

hold. This means that Condition 3 is fulfilled if $a_0 = 1$, $a_1 = 3/4$, and $a_2 = 3/16$. It is easy to show that the symmetric matrices R_1 and $R_{10} = \begin{pmatrix} 1 & 3/4 \\ 3/4 & 1 \end{pmatrix}$ are positive definite. However, the inverse matrix

$$R_1^{-1} = \begin{pmatrix} 8.615 & -12 & 7.385 \\ -12 & 19 & -12 \\ 7.385 & -12 & 8.615 \end{pmatrix}$$

does not satisfy the conditions of Lemma 12: $8.615 < 12 + 7.385$. We cannot prove that R_Q^{-1} is strongly accretive, and we cannot guarantee that problem (1.1), (1.2), (7.3) has at least one generalized solution.

Note that if $p = 2$, then problem (1.1), (1.2), (7.3) is linear and has a unique generalized solution, see [14].

Example 7. We consider problem (1.1), (1.2), (7.3) with $\gamma_1 = 1/2$ and $\gamma_2 = 0$.

As in Example 6, the difference operator is given by $Ru(x) = \sum\limits_{-2 \leqslant k \leqslant 2} a_k u(x_1 + k, x_2, t)$. We shall find a nonsingular martix R_1 such that the relations (7.4) hold. This means that Condition 3 is fulfilled if

$$R_1 = \begin{pmatrix} 1 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 1 \end{pmatrix} \quad \text{and} \quad R_{10} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

It is easy to show that the symmetric matrices R_1 and R_{10} are positive definite. The inverse matrix

$$R_1^{-1} = \begin{pmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 5/3 & -2/3 \\ 0 & -2/3 & 4/3 \end{pmatrix}$$

satisfies condition (7.1), i.e., the operator R_Q^{-1} is strongly accretive. Therefore, by virtue of Theorem 6, problem (1.1), (1.2), (7.3) with $\gamma_1 = 1/2$ and $\gamma_2 = 0$ has at least one generalized solution for any $p \in (2, \infty)$.

8. SOLVABILITY OF QUASILINEAR NONLOCAL PARABOLIC PROBLEMS

The above results can be applied not only to the equation with the p–Laplacian. In this section, we will consider nonlocal quasilinear parabolic problems. We will use the properties of quasilinear differential–difference operator that was studied in [23].

In the cylinder $\Omega_T = Q \times (0, T)$, we consider the differential equation

$$\partial_t w(x,t) - \sum_{1\leqslant i\leqslant n} \partial_i A_i(x,t,w,\nabla w) + A_0(x,t,w,\nabla w) = f(x,t) \quad ((x,t) \in \Omega_T) \tag{8.1}$$

with initial condition

$$w(x,0) = \psi(x) \qquad (x \in Q) \tag{8.2}$$

and with nonlocal boundary conditions

$$\left.\begin{aligned} w|_{\Gamma_{rl}^T} &= \sum_{j=1}^{J_0} \gamma_{lj}^r w|_{\Gamma_{rj}^T} && (r \in B,\ l = J_0+1, \dots, J), \\ w|_{\Gamma_{rl}^T} &= 0 && (r \notin B,\ l = 1, \dots, J), \end{aligned}\right\} \tag{8.3}$$

see Section 1. Assume that $f \in L_q(0,T; W_q^{-1}(Q))$ and $\psi \in L_2(Q)$. Introduce the nonlinear operator $A : L_p(0,T; W_{p,\gamma}^1(Q)) \to L_q(0,T; W_q^{-1}(Q))$ given by the formula

$$\langle Aw, v\rangle = \sum_{0\leqslant i\leqslant n} \int_{\Omega_T} A_i(x,t,w,\nabla w)\, \partial_i v \, dx\, dt \qquad \forall v \in L_p(0,T; \mathring{W}_p^1(Q)). \tag{8.4}$$

Here and below we write $\partial_0 w := w$. Also we assume that for some $c_{20} > 0$ and $g_0 \in L_Q(\Omega_T)$,

$$|A_i(x,t,\xi)| \leqslant g_0(x,t) + c_{20} \sum_{0\leqslant i\leqslant n} |\xi_i|^{p-1} \qquad (i = 0, 1, \dots, n). \tag{8.5}$$

Definition 9. A function $w \in W_\gamma$ is called a *generalized solution of problem* (8.1)–(8.3) if it satisfies the operator equation

$$\partial_t w + Aw = f, \qquad w \in W_\gamma \tag{8.6}$$

and initial condition (8.2).

Moreover, we suppose that Conditions 1–3 hold. If $R_s^{sym} > 0$, we can consider equation

$$\partial_t R_Q u + A R_Q u = f, \qquad u \in W, \tag{8.7}$$

with initial condition

$$u(x,0) = R_Q^{-1}\psi(x) = \varphi(x) \qquad (x \in Q) \tag{8.8}$$

instead of (8.6), (8.2).

Theorem 7. *Let Conditions* 1–3 *hold, and let* $R_s^{sym} > 0$ $(s = s(r), r \in B)$*. Suppose that the operator* $A : L_p(0,T; W_p^1(Q)) \to L_q(0,T; W_q^{-1}(Q))$ *is given by the formula* (8.4)*, where the functions* $A_i(x,t,\xi)$ *are measurable in* $(x,t) \in \Omega_T$ *and differentiable in* $\xi_j \in \mathbb{R}\, (j = 0, 1, \dots, n)$*. Moreover, assume that*

$$\sum_{1\leqslant m,l\leqslant N(s)} \sum_{0\leqslant i,j\leqslant n} r_{ml}^s A_{ij}\left(x + h_{sm}, t, \zeta_{m\cdot}\right) \eta_{lj}\, \eta_{mi} \geqslant c_{21} \sum_{1\leqslant m\leqslant N(s)} \sum_{1\leqslant i\leqslant n} |\zeta_{mi}|^{p-2} \left|\eta_{mi}\right|^2, \tag{8.9}$$

$$|A_{ij}(x,t,\xi)| \leqslant g_1(x,t) + c_{22} \sum_{0\leqslant i\leqslant n} |\xi_i|^{p-2} \qquad (i,j = 0, 1, \dots, n), \tag{8.10}$$

where $A_{ij}(x,t,\xi) = \dfrac{\partial A_i(x,t,\xi)}{\partial \xi_j}$, $g_1 \in L_{p/(p-2)}(\Omega_T)$, *and* $c_{21}, c_{22} > 0$ *do not depend on* $(x,t) \in \Omega_T$, $\xi \in \mathbb{R}^{n+1}$, $\zeta, \eta \in \mathbb{R}^{N(s)\times(n+1)}$, $\zeta_{m\cdot} = \{\zeta_{m0}, \zeta_{m1}, \dots, \zeta_{mn}\}$, $m = 1, \dots, N(s)$.

Then, for any $f \in L_q(0,T;W_q^{-1}(Q))$ *and* $\psi \in L_2(Q)$, *there exists a unique generalized solution of problem* (8.1)–(8.3). *Moreover, the following estimates hold:*

$$\|w_1(T) - w_2(T)\|_{L_2(Q)} \leqslant c_{23}\|f_1 - f_2\|^{q/2}_{L_q(0,T;W_q^{-1}(Q))} + c_{24}\|\psi_1 - \psi_2\|_{L_2(Q)}, \tag{8.11}$$

$$\|w_1 - w_2\|_{L_p(0,T;W_p^1(Q))} \leqslant c_{25}\|f_1 - f_2\|^{q/p}_{L_q(0,T;W_q^{-1}(Q))} + c_{26}\|\psi_1 - \psi_2\|^{2/p}_{L_2(Q)}, \tag{8.12}$$

here w_1 *and* w_2 *are generalized solutions of problem* (8.1)—(8.3) *with the right parts* f_1 *and* f_2 *and with initial conditions* ψ_1 *and* ψ_2, *respectively; the positive constant* $c_{23}, c_{24}, c_{25}, c_{26}$ *do not depend on* w_k, f_k, *and* ψ_k.

Proof. First we consider the case $\psi = 0$, i.e., $\varphi = 0$. By virtue of (8.9)–(8.10), the operator $AR_Q : L_p(0,T;\mathring{W}_p^1(Q)) \to L_q(0,T;W_q^{-1}(Q))$ is demicontinuous, monotone, and coercive. This follows from Theorem 1 in [23]. Moreover, inequalities (8.9) and (8.10) imply that there exists $c_{27} > 0$ such that

$$\langle AR_Q u - AR_Q y, u - y\rangle \geqslant c_{27}\|u - y\|^p_{L_p(0,T;\mathring{W}_p^1(Q))}, \tag{8.13}$$

see Theorem 1 in [23]. Thus, the conditions of Theorem 1.1 in [18, Ch.III, §1] hold, problem (8.7), (8.8) has at least one solution. Moreover, since AR_Q is monotone, this solution is unique. Since R_Q is an isomorphism, there exists a unique generalized solution of (8.1)–(8.3).

If $\varphi \neq 0$, we consider an auxiliary fixed element $\hat{u} \in W \subset C(0,T;L_2(Q))$. Since $C^1(0,T;\mathring{W}_p^1(Q)) \cap W$ densely imbedded into W, see Lemma 1.12 in [21, Ch. IV], and W continuously imbedded into $C(0,T;L_2(Q))$, see Theorem 1.17 in [21, Ch. IV], for any $\varphi \in L_2(Q)$, there exists $\hat{u} \in W \subset C(0,T;L_2(Q))$ such that $\hat{u}|_{t=0} = \varphi$. Substituting $u(x,t) = v(x,t) + \hat{u}(x,t)$, we obtain the equivalent equation

$$\partial_t R_Q v + AR_Q(v + \hat{u}) = f - \partial_t R_Q \hat{u} := \widehat{f}, \tag{8.14}$$

$$v(0) = 0. \tag{8.15}$$

Clearly, $\hat{f} \in L_q(0,T;W_q^{-1}(Q))$. By virtue of construction of the operator $AR_Q(\cdot + \hat{u})$, we obtain estimates that are similar to estimates (8.5)–(8.10). Therefore, the operator

$$AR_Q(\cdot + \hat{u}) : L_p(0,T;\mathring{W}_p^1(Q)) \to L_q(0,T;W_q^{-1}(Q))$$

is demicontinuous, monotone, and coercive too for any fixed $\hat{u} \in L_p(0,T;\mathring{W}_p^1(Q))$. Thus, problem (8.14), (8.15) has a unique solution, see Theorem 1.1 in [18, Ch.III, §1]. Therefore problem (8.1)–(8.3) has a unique generalized solution.

Let $w_1 \in W_\gamma$ and $w_2 \in W_\gamma$ be solutions of (8.1)–(8.3) with the right parts $f_1 \in L_q(0,T;W_q^{-1}(Q))$ and $f_2 \in L_q(0,T;W_q^{-1}(Q))$ and with initial conditions $\psi_1 \in L_2(Q)$ and $\psi_2 \in L_2(Q)$, respectively. We denote $u_i = R_Q^{-1}w_i$ and $\varphi_i = R_Q^{-1}\psi_i$. Since u_i is a solution of (8.7), (8.8) with the right parts f_i and with initial conditions φ_i, we have

$$\langle \partial_t R_Q u_1 - \partial_t R_Q u_2, u_1 - u_2\rangle + \langle AR_Q u_1 - AR_Q u_2, u_1 - u_2\rangle = \langle f_1 - f_2, u_1 - u_2\rangle.$$

Using equality (4.11), we obtain

$$\begin{aligned} \frac{1}{2}\left\|\sqrt{R_Q^{sym}}\,(u_1(T) - u_2(T))\right\|^2_{L_2(Q)} &+ \langle AR_Q u_1 - AR_Q u_2, u_1 - u_2\rangle \\ &= \langle f_1 - f_2, u_1 - u_2\rangle + \frac{1}{2}\left\|\sqrt{R_Q^{sym}}(\varphi_1 - \varphi_2)\right\|^2_{L_2(Q)}. \end{aligned} \tag{8.16}$$

Let also $u_3 \in W$ be a generalized solution of the problem (8.7), (8.8) with the initial condition $\varphi = \varphi_1$ and the right-hand sides of equation $f = f_2$. Then,

$$\frac{1}{2}\left\|\sqrt{R_Q^{sym}}\,(u_1(T) - u_3(T))\right\|^2_{L_2(Q)} + \langle AR_Q u_1 - AR_Q u_3, u_1 - u_3\rangle = \langle f_1 - f_2, u_1 - u_3\rangle, \tag{8.17}$$

$$\frac{1}{2}\left\|\sqrt{R_Q^{sym}}\,(u_3(T) - u_2(T))\right\|^2_{L_2(Q)} + \langle AR_Q u_3 - AR_Q u_2, u_3 - u_2\rangle = \frac{1}{2}\left\|\sqrt{R_Q^{sym}}R_Q^{-1}(\psi_1 - \psi_2)\right\|^2_{L_2(Q)}. \tag{8.18}$$

By virtue of estimate (8.13) and equality (8.17), we have

$$\begin{aligned} c_{27}\|u_1-u_3\|^p_{L_p(0,T;\mathring{W}^1_p(Q))} &\leqslant \langle AR_Qu_1-AR_Qu_3,u_1-u_3\rangle \\ &= \langle f_1-f_2,u_1-u_3\rangle-\frac{1}{2}\left\|\sqrt{R_Q^{sym}}\,(u_1(T)-u_3(T))\right\|^2_{L_2(Q)} \\ &\leqslant \langle f_1-f_2,u_1-u_3\rangle \leqslant \|f_1-f_2\|_{L_q(0,T;W_q^{-1}(Q))}\|u_1-u_3\|_{L_p(0,T;\mathring{W}^1_p(Q))} \\ &\leqslant \frac{1}{\varepsilon^q q}\|f_1-f_2\|^q_{L_q(0,T;W_q^{-1}(Q))}+\frac{\varepsilon^p}{p}\|u_1-u_3\|^p_{L_p(0,T;\mathring{W}^1_p(Q))}, \end{aligned} \tag{8.19}$$

i.e., for $\varepsilon^p/p\leqslant c_{27}/2$, we get

$$\|u_1-u_3\|^p_{L_p(0,T;\mathring{W}^1_p(Q))}\leqslant c_{28}\|f_1-f_2\|^q_{L_q(0,T;W_q^{-1}(Q))}, \tag{8.20}$$

$$\langle f_1-f_2,u_1-u_3\rangle\leqslant c_{29}\|f_1-f_2\|^q_{L_q(0,T;W_q^{-1}(Q))}. \tag{8.21}$$

On the other hand, the second summand in the left-hand side of (8.17) is nonnegative, too, because AR_Q is monotone. Thus, substituting (8.21), we obtain

$$\frac{1}{2}\left\|\sqrt{R_Q^{sym}}\,(u_1(T)-u_3(T))\right\|^2_{L_2(Q)}\leqslant\langle f_1-f_2,u_1-u_3\rangle\leqslant c_{29}\|f_1-f_2\|^q_{L_q(0,T;W_q^{-1}(Q))}. \tag{8.22}$$

Substituting estimate (4.7) into (8.22), we obtain

$$c_3\,\|u_1(T)-u_3(T)\|^2_{L_2(Q)}\leqslant\left\|\sqrt{R_Q^{sym}}\,(u_1(T)-u_3(T))\right\|^2_{L_2(Q)}\leqslant 2c_{29}\|f_1-f_2\|^q_{L_q(0,T;W_q^{-1}(Q))}. \tag{8.23}$$

Nonnegativity of the second summand in the left part of (8.18) implies that

$$\left\|\sqrt{R_Q^{sym}}(u_3(T)-u_2(T))\right\|^2_{L_2(Q)}\leqslant\left\|\sqrt{R_Q^{sym}}(\varphi_1-\varphi_2)\right\|^2_{L_2(Q)}. \tag{8.24}$$

Using estimate (4.7) and the boundedness of the operators $\sqrt{R_Q^{sym}}$ and R_Q^{-1}, from (8.24), we obtain

$$\begin{aligned} c_3\,\|(u_3(T)-u_2(T))\|^2_{L_2(Q)} &\leqslant \left\|\sqrt{R_Q^{sym}}(u_3(T)-u_2(T))\right\|^2_{L_2(Q)} \\ &\leqslant \left\|\sqrt{R_Q^{sym}}(\varphi_1-\varphi_2)\right\|^2_{L_2(Q)}=\left\|\sqrt{R_Q^{sym}}R_Q^{-1}(\psi_1-\psi_2)\right\|^2_{L_2(Q)} \\ &\leqslant c_{12}c_{18}\,\|\psi_1-\psi_2\|^2_{L_2(Q)}. \end{aligned} \tag{8.25}$$

Using the triangle inequality and (8.23) and (8.25), we can write

$$\begin{aligned} \|u_1(T)-u_2(T)\|_{L_2(Q)} &\leqslant \|u_1(T)-u_3(T)\|_{L_2(Q)}+\|u_3(T)-u_2(T)\|_{L_2(Q)} \\ &\leqslant \sqrt{\frac{2c_{29}}{c_3}}\|f_1-f_2\|^{q/2}_{L_q(0,T;W_q^{-1}(Q))}+\sqrt{\frac{c_{12}c_{18}}{c_3}}\|\psi_1-\psi_2\|_{L_2(Q)}. \end{aligned}$$

Therefore, by virtue of boundedness of operator $R_Q:L_2(Q)\to L_2(Q)$, we obtain

$$\begin{aligned} \|w_1(T)-w_2(T)\|_{L_2(Q)} &= \|R_Q(u_1(T)-u_2(T))\|_{L_2(Q)}\leqslant c_{17}\|u_1(T)-u_2(T)\|_{L_2(Q)} \\ &\leqslant c_{17}\left(\sqrt{\frac{2c_{29}}{c_3}}\|f_1-f_2\|^{q/2}_{L_q(0,T;W_q^{-1}(Q))}+\sqrt{\frac{c_{12}c_{18}}{c_3}}\|\psi_1-\psi_2\|_{L_2(Q)}\right). \end{aligned}$$

Estimate (8.11) is proved.

On the other hand, (8.13), (8.18), and (8.25) imply

$$\begin{aligned} c_{27}\|u_3-u_2\|^p_{L_p(0,T;\mathring{W}^1_p(Q))} &\leqslant \langle AR_Qu_3-AR_Qu_2,u_3-u_2\rangle \\ &\leqslant \frac{1}{2}\left\|\sqrt{R_Q^{sym}}(u_3(T)-u_2(T))\right\|^2_{L_2(Q)}+\langle AR_Qu_3-AR_Qu_2,u_3-u_2\rangle \\ &= \frac{1}{2}\left\|\sqrt{R_Q^{sym}}R_Q^{-1}(\psi_1-\psi_2)\right\|^2_{L_2(Q)}\leqslant\frac{c_{12}c_{18}}{2}\,\|\psi_1-\psi_2\|^2_{L_2(Q)}, \end{aligned} \tag{8.26}$$

i.e. $\|u_3 - u_2\|^p_{L_p(0,T;\mathring{W}^1_p(Q))} \leqslant \dfrac{c_{12}c_{18}}{2c_{27}}\|\psi_1 - \psi_2\|^2_{L_2(Q)}$. Hence, from (8.20) and (8.26) it follows that

$$\|u_1 - u_2\|_{L_p(0,T;\mathring{W}^1_p(Q))} \leqslant \|u_1 - u_3\|_{L_p(0,T;\mathring{W}^1_p(Q))} + \|u_3 - u_2\|_{L_p(0,T;\mathring{W}^1_p(Q))}$$
$$\leqslant c_{28}^{1/p}\|f_1 - f_2\|^{q/p}_{L_q(0,T;W_q^{-1}(Q))} + \left(\frac{c_{12}c_{18}}{2c_{27}}\right)^{1/p}\|\psi_1 - \psi_2\|^{2/p}_{L_2(Q)}.$$

Thus, by virtue of boundedness of the operator $R_Q : L_p(0,T;\mathring{W}^1_p(Q)) \to L_q(0,T;W_q^{-1}(Q))$, we derive the following estimate

$$\|w_1 - w_2\|_{L_p(0,T;W^1_p(Q))} = \|R_Q(u_1 - u_2)\|_{L_p(0,T;W^1_p(Q))} \leqslant c_{19}\|u_1 - u_2\|_{L_p(0,T;\mathring{W}^1_p(Q))}$$
$$\leqslant c_{19}\left(c_{28}^{1/p}\|f_1 - f_2\|^{q/p}_{L_q(0,T;W_q^{-1}(Q))} + \left(\frac{c_{12}c_{18}}{2c_{27}}\right)^{1/p}\|\psi_1 - \psi_2\|^{2/p}_{L_2(Q)}\right).$$

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