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Asymptotics of the Spectrum of an Integro-Differential Equation Arising in the Study of the Flutter of a Viscoelastic Plate

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Abstract. In the paper, the asymptotics for the spectrum of the symbol of the oscillation equation of a viscoelastic plate in a liquid or gas flow is studied using operator analysis methods. This equation is the Gurtin–Pipkin equation with a relatively compact perturbation. Using an operator analog of Rouche's theorem, we explicitly define an asymptotic representation of nonreal points of the spectrum for the symbol of the equation.

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1. INTRODUCTION

In the present paper, we study the equation of motion of a viscoelastic plate in a liquid or gas flow within the piston model; the equation is written in the form (see, e.g., [1]):

$$D_0\left(\Delta^2 u(x,y,t)-\varepsilon_0\int_0^t \Gamma_0(t-\tau)\Delta^2 u(x,y,\tau)d\tau\right)+\rho h\ddot{u}(x,y,t)$$
$$+\frac{\gamma p_0}{a_0}\left(\dot{u}(x,y,t)+v(\overline{n}_0,\nabla u(x,y,t))\right)=f(x,y,t). \tag{1}$$

The plate is two-dimensional; it is a rectangle with the sides L_0 in the x direction and l_0 in the y direction; $u(x,y,t)$ describes the deviation of the plate at a point with the coordinates (x,y) at the moment t; Δ^2 is the operator

$$\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4};$$

$\overline{n}_0$ is the direction of the velocity for the motion of gas or liquid, and v is the modulus of $\overline{n}_0$. The plate is secured in an articulated manner on both sides along the axis Oy :

$$\frac{\partial^2}{\partial x^2}u(x,y,t)=u(x,y,t)=0 \tag{2}$$

at $x=0$ and $x=L_0$. In this paper, we consider a one-dimensional analog of the equation (1):

$$D_0\left(\frac{\partial^4}{\partial x^4}u(x,t)-\varepsilon_0\int_0^t \Gamma_0(t-\tau)\frac{\partial^4}{\partial x^4}u(x,\tau)d\tau\right)+\rho h\ddot{u}(x,t)$$
$$+\frac{\gamma p_0}{a_0}\left(\dot{u}(x,t)+v\;\overline{n}_{0\,x}\frac{\partial}{\partial x}u(x,t)\right)=f(x,t). \tag{3}$$

Then, instead of condition (2), we take the boundary condition

$$\frac{\partial^2}{\partial x^2}u(x,t)=u(x,t)=0$$

at $x=0$ and $x=L_0$. Moreover, below, to simplify mathematical calculations, we take $L_0=\pi$ and also $n_{0\,x}=1$ (i.e., the liquid or gas moves along the direction x).

In [2], a lower estimate for the critical flow speed was given at which the vibrational motion of the plate ceases to be stable. That is, an interval of speeds v was obtained for which the spectrum of the operator function, which is the symbol of the integro-differential equation (1), is absent in the half-plane $\{\mathrm{Re}z \geqslant 0\}$, and the solution to this problem is asymptotically stable. The purpose of this paper is to find the asymptotics for the spectrum of the symbol of the integro-differential equation (1). This asymptotics can be used in the study of high-frequency vibrations of the plate, when determining the speed of the wave propagation, as well as when compiling the Riesz basis for the given problem.

2. STATEMENT OF THE PROBLEM

Let us represent equation (3) in an operator form more convenient for subsequent investigation:

$$\ddot{u}(t) + 2M_1\dot{u}(t) + M_2^2\left(A^2u(t) - \int_0^t \Gamma(t-\tau)A^2u(\tau)d\tau\right) + M_3Tu(t) = f(t), \tag{4}$$

where

$$2M_1 = \frac{\gamma p_0}{a_0\rho h}, \quad M_2^2 = \frac{D_0}{\rho h}, \quad M_3 = \frac{v\gamma p_0}{a_0\rho h}, \quad \Gamma(t) = \varepsilon_0\Gamma_0(t),$$

and $u(t)$ and $f(t)$, for a fixed t, are already vector functions with the values in the Hilbert space $H = L_2([0,\pi])$ with respect to x. The unbounded operator

$$Av(x) = -\frac{\partial^2}{\partial x^2}v(x)$$

acts on the space H and has the domain

$$D(A) = \{v(x) \in H \,:\, v''(x) \in H,\ v(0) = v(\pi) = 0\}.$$

The operator $A^2v(x) = \dfrac{\partial^4}{\partial x^4}v(x)$ also acts on the space H and has the domain

$$D(A^2) = \{v(x) \in H \,:\, v^{(IV)}(x) \in H,\ v(0) = v(\pi) = v''(0) = v''(\pi) = 0\}.$$

The operator T acts by the rule $Tv(x) = \dfrac{d}{dx}v(x)$ and has the domain

$$\{v(x) \in AC([0,\pi]) :\ v'(x) \in H,\ v(0) = v(\pi) = 0\}.$$

This is the Gurtin–Pipkin equation with a small modification in the form of the terms $2M_1\dfrac{d}{dt}u(t)$ and $M_3Tu(t)$.

We also assume that the function $u(t)$ takes values in the Sobolev space $W^1_{2,\gamma}([0,+\infty), A)$ with the following inner product (here and below, all inner products are sesquilinear):

$$(u(t), v(t))_{W^1_{2,\gamma}([0,+\infty),A)} = \int_0^\infty \left((u'(t), v'(t))_H + (Au(t), Av(t))_H e^{-2\gamma t}dt\right).$$

Adding the initial conditions

$$u(0) = \varphi_0(x), \quad u'(0) = \varphi_1(x), \tag{5}$$

to the equation (4), we obtain a Cauchy problem.

In the present paper, we consider the kernel of the form

$$\Gamma(t) = \begin{cases} 0, & x < 0, \\ \Gamma_1(t), & x \geqslant 0, \end{cases} \quad \text{where} \quad \Gamma_1(t) = \sum_{k=1}^\infty c_k e^{-\gamma_k t} \tag{6}$$

$$c_j, \gamma_j > 0, \quad \gamma_{j+1} > \gamma_j, \quad \sum_{j=1}^\infty \frac{c_j}{\gamma_j} < 1. \tag{7}$$

It is necessary to note further that the operator A^2 introduced above is a self-adjoint positive operator and the operator A^{-2} (inverse to A^2) is compact. Then, by the Hilbert–Schmidt theorem, there is a basis of eigenfunctions of the operator A^2 in H; $e_n = \sin(nx)$, $n \in \mathbb{N}$.

Moreover, the operator T^* acts by the rule $T^*v(x) = \dfrac{d}{dx}v(x)$ with the domain

$$D(T^*) = \{v(x) \in AC([0,\pi]) : \ v'(x) \in H\}$$

(see, e.g., [3]). Since the operator T is closed, we can represent t in the form of a product of a partial isometry and a self-adjoint operator using the polar decomposition:

$$T = U\sqrt{T^*T} = UA^{\frac{1}{2}}. \tag{8}$$

It is also appropriate here to cite the result of the paper [2] in which an estimate for the constant M_3 for which the spectrum of the symbol (4) lies in the half-plane $\{z : \operatorname{Re} z < 0\}$ is given.

Theorem 1. *Suppose that $\Gamma(t)e^{\gamma_1 t}$ is a monotone decreasing positive integrable function for some $\gamma_1 > 0$ and*

$$2M_1M_2^2(1 - ||\Gamma||_{L_1})(\sqrt{a_1})^3 - M_3M_2\sqrt{a_1} - 2M_1M_3 > 0,$$

where a_1 is the least eigenvalue of A. Then, for some $\gamma > 0$, the spectrum of the operator function $L(z)$, which is the symbol of the equation (4), *is absent in the right half-plane* $\{\operatorname{Re}\, z \geqslant -\gamma\}$.

3. GENERAL CASE OF THE GURTIN–PIPKIN EQUATION WITH PERTURBATION

To begin with, we consider the case of the standard Gurtin–Pipkin equation with an A^2-compact perturbation in the Kato sense (see [4]):

$$u''(t) + A^2u(t) - \Gamma(t) * A^2u(t) + RA^\theta u(t) = f(t), \tag{9}$$

$$u(0) = \varphi_0, \tag{10}$$

$$u'(0) = \varphi_1, \tag{11}$$

where $\theta \in [0, 2)$. Here u and f are some functions from $\mathbb{R}^+$ to the Hilbert space H; $*$ stands for the convolution operation, A is a self-adjoint positive operator on H having compact inverse, R is a bounded operator with the norm $\|R\| = B$. The function Γ is defined by (6), (7). Below we shall see that problem (4), (5) can be reduced to (9)–(11) by a change of variables.

By the Hilbert–Schmidt theorem, there is a basis of the eigenvectors e_n of A in H; the eigenvalues a_n correspond to these eigenvectors. For the problem (4), (5), we have $e_n = \sin(nx),\ a_n = n^2,\ n \in \mathbb{N}$. Moreover, our operator obviously satisfies the relation

$$a_n^{\theta-1}(a_n - a_{n-1})^{-1} \to 0, \quad n \to \infty. \tag{12}$$

The application of the Laplace transform to equation (4) leads to the operator function

$$L(z) = z^2I + (1 - K(z))A^2 + RA^\theta, \tag{13}$$

which is the symbol of the original equation. Here $K(z)$ stands for the Laplace transform of $\Gamma(t)$:

$$K(z) = \sum_{k=1}^{\infty} \frac{c_k}{z + \gamma_k}. \tag{14}$$

Definition 1. By the *resolvent* set $R(L)$ of $L(z)$ we mean the set of all values $z \in \mathbb{C}$ for which the operator function $L^{-1}(z)$ exists and is bounded. The complement of the set $R(L)$ to the complex plane, i.e., $\sigma(L) = \{\mathbb{C} \ \setminus R(L)\}$, is called the *spectrum* of $L(z)$. We are to find the asymptotics of the *spectrum* $\sigma(L)$ of the symbol of the given equation.

Write

$$l_n(z) = (L(z)e_n, e_n) = z^2 + (1 - K(z))a_n^2. \tag{15}$$

This function of a complex variable has only one root in the upper half-plane $\{\operatorname{Im} z > 0\}$. This is proved, e.g., in [5]. Denote the root by μ_n^+. When c_j and γ_j have polynomial behavior, the asymptotics of μ_n^+ is studied in detail in [6]. Further, write $D_{n,C} = \{z : |z - \mu_n^+| < Ca_n^{\theta-1}\}$.

Theorem 2. *Suppose that the eigenvalues of the operator A satisfy relation* (12)*. Then there are positive constants* y_0 *and* C *such that the spectrum* $\sigma(L)$ *of* $L(z)$ *lying in the upper half-plane* $\{z : \Im z > y_0\}$ *can be represented in the form of a family of points* $\{\widetilde{\mu}_n^+,\ n > n_0\}$ *in such a way that* $\widetilde{\mu}_n^+ \in D_{n,C}$*. The number* n_0 *is here the least positive integer such that, for every* $n > n_0$*,*

$$D_{n,C} \subset \{z : \Im z > y_0\}. \tag{16}$$

Remark 1. Using the definition of $D_{n,C}$, we can describe the asymptotics of the spectrum of $L(z)$:

$$\widetilde{\mu}_n^+ = \mu_n^+ + O(a_n^{\theta-1}).$$

Proof. We need an operator analog of Rouche's theorem (see [7]). To begin with, we introduce some definitions.

Definition 2. Let a point z_0 be a pole of the operator function $L(z)$. In some neighborhood of the point z_0, we have an expansion

$$L(z) = \sum_{j=-n}^{\infty} (z - z_0)^j L_j.$$

If the operators $L_j,\ j = -n,\ -n+1, \ldots, -1$ in this decomposition are finite-dimensional, then the operator function $L(z)$ is said to be *finite meromorphic* at the point z_0. If $L(z)$ is finite meromorphic at any point of some domain G, then we say that $L(z)$ is finite meromorphic in G. A bounded linear operator L acting on a Banach space $\mathbb{L}$ is called an *F-operator* if it is normally solvable (i.e., the range $\operatorname{Im} L$ is closed), its kernel $\operatorname{Ker} L$ is finite-dimensional, and $\dim(\mathbb{L} \setminus \operatorname{Im} L) < \infty$. The operator function $L(z)$ is said to be *Fredholm* at a point k if the operator L_0 in the expansion (15) is am F-operator. If $L(z)$ is Fredholm at any point of some domain G, then we call it *Fredholm* in G. A point z_0 is called a *normal* point of the operator function $L(z)$ if $L(z)$ is finite meromorphic and Fredholm at the point z_0 and all points of some punctured disk $0 < |z - z_0| < \rho$ are regular for $L(z)$. Let Γ be a simple closed rectifiable contour bounding the domain G. An operator function $L(z)$ which is finite meromorphic and Fredholm in G and continuous up to Γ, is said to be *normal with respect to the contour* Γ if the operator $L(z)$ is invertible for all $z \in G \cup \Gamma$ except for finitely many points of the domain G that are normal points $L(z)$.

We need the following simplified version of the operator analog of Rouche's theorem.

Theorem 3. *Let* $L_1(z)$ *be an operator function normal with respect to some simple closed rectifiable contour* Γ *bounding a domain* G*. If an operator function* $L_2(z)$*, which is finite meromorphic in* G *and continuous up to* Γ*, satisfies the condition*

$$||L_2(z)L_1^{-1}(z)|| < 1, \quad z \in \Gamma, \tag{17}$$

then the operator function $L_3(z) = L_1(z) + L_2(z)$ *is also normal with respect to* Γ*. If the operator functions* $L_1(z)$ *and* $L_2(z)$ *have no poles in the domain* G *and, moreover, the spectrum of the operator function* $L_1(z)$ *consists of a single point of the point spectrum of multiplicity one, then the spectrum of the operator function* $L_3(z)$ *in the domain* G *consists of a single point of the point spectrum of multiplicity one.*

If we set $L_1(z) = z^2 I + (1 - K(z))A^2$ and $L_2(z) = RA^\theta$ and then prove inequality (17) on the set

$$\mathbb{D} = \{z : \operatorname{Im} z > y_0\} \setminus \bigcup_{n=1}^{\infty} D_{n,C}$$

for some constants n and C, then we will be able to prove Theorem 1. Indeed, it can readily be seen that the operator function $L_1(z)$ is normal with respect to every simple closed rectifiable contour contained in $\{z : \operatorname{Im} z > y_0\}$ or $y_0 > 0$, and also has here only isolated points of the point spectrum; moreover, $L_2(z)$ is by definition finite meromorphic in every domain contained in $\{z : \operatorname{Im} z > y_0\}$ and is continuous on the closure of the domain.

By condition (12), $a_n^{\theta-1} \ll a_n - a_{n-1}$ for large n. Then $a_n^{\theta-1} \ll |\mu_n^+ - \mu_{n-1}^+|$, since $|\mu_n^+| \sim a_n$ (see, e.g., [8]). Therefore, we have $D_{n,C} \cap D_{m,C} = \varnothing$ for sufficiently large n and m. Then, if we shall prove relation (17) in the domain $\mathbb{D}$, then we shall see that, inside $D_{n,C}$, for $n > n_0$ (n_0 is defined from the condition of Theorem 1), by Theorem 2, there is only one point of the spectrum $L_3(z) = L(z)$, because the very center of the domain $D_{n,C}$, the point μ_n^+, is a unique spectral point of the operator function $L_1(z)$ inside the domain $D_{n,C}$ by the spectral theorem. In this case, the contour Γ is $\partial D_{n,C}$. Indeed, by definition, μ_n^+ is a simple zero of the function $l_n(z) = (L(z)e_n, e_n)$, which is the projection of the operator function $L(z)$ to the eigensubspace $\langle e_n \rangle$.

Outside the domains $D_{n,C}$, all points of the half-plane $\{z : \operatorname{Im} z > y_0\}$ are regular for $L(z)$ under condition (17) (see, e.g., Theorem 1.16 of [4]).

Thus, it suffices to prove condition (17) on $\mathbb{D}$ for sufficiently large y_0 and C. By the spectral theorem,

$$||L_2(z)L_1^{-1}(z)|| = ||RA^\theta L_1^{-1}(z)|| \leqslant ||R||\,||A^\theta L_1^{-1}(z)|| = B \sup_n |a_n^\theta l_n^{-1}(z)| = B \left(\inf_n |a_n^{-\theta} l_n(z)|\right)^{-1}, \tag{18}$$

since the norm of the operator R is equal to B.

We also need some known facts concerning the structure of zeros of $l_n(z)$ and the behavior of the function $K(z)$ as $\operatorname{Im} z \to +\infty$:

1^0 : $\mu_n^+ = ia_n(1 + \overline{o}(1))$ as $a_n \to +\infty$.

Proof. A short proof is given here; for details, see [6], Theorem 3.2.5, and [8]. We make the change of variables $z = ia_n v$ in the equation $l_n(z) = z^2 + (1 - K(z))a_n^2 = 0$. We then have $-a_n^2 v^2 + (1 - K(ia_n v))a_n^2 = 0$. Hence,

$$v^2 - 1 + K(ia_n v) = 0. \tag{19}$$

By property 2^0, we have $K(ia_n v) \to 0$ as $n \to \infty$ uniformly with respect to v for $v \approx 1$. Thus, by Rouche's theorem, we see that the root of the equation (19) is $\widetilde{v}_n = 1 + \overline{o}(1)$. Thus, the root of the equation $l_n(z) = 0$ is equal to $\mu_n^+ = ia_n(1 + \overline{o}(1))$.

2^0 : $|K(z)| \to 0$ as $\operatorname{Im} z \to +\infty$,

3^0 : $|zK'(z)| \to 0$ as $\operatorname{Im} z \to +\infty$.

The proofs of 2^0 and 3^0 are presented in [8].

Let us pass to the proof of condition (17) on $\mathbb{D}$. For sufficiently large values of y, consider the complex points $z = x + iy,\ x, y \in \mathbb{R},\ y > y_0 > 0$. Choose a point z and consider three cases, exhausting all numbers $n \in \mathbb{N}$.

I. Consider n such that $a_n < y/2$. Then

$$y - a_n > y/2. \tag{20}$$

We have $|z - ia_n| \geqslant |\operatorname{Im}(z - ia_n)| = |y - ia_n|$. Similarly, $|z + ia_n| \geqslant |y + a_n|$. Then

$$\begin{aligned} |l_n(z)| = |(z - ia_n)(z + ia_n) - a_n^2 K(z)| &\geqslant |z - ia_n||z + ia_n| - a_n^2|K(z)| \\ &\geqslant |y - a_n||y + a_n| - a_n^2|K(z)|. \end{aligned} \tag{21}$$

Since $y > 0$, we have $|y + a_n| = y + a_n > y$. Taking also into account that $a_n < y/2$, by the equation (20), we obtain $|y - a_n||y + a_n| > y/2 \cdot y$. Moreover, $a_n^2 < y^2/4$. Therefore,

$$|l_n(z)| > \frac{y}{2} \cdot y - \frac{y^2}{4}|K(z)|.$$

If y_0 is so large that $|K(z)| < 1$ by property 2^0, then $|l_n(z)| > y^2/4$. Since $a_n < y/2$, we have

$$|a_n^{-\theta} l_n(z)| > a_n^{-\theta}\frac{y^2}{4} > \left(\frac{y}{2}\right)^{-\theta}\frac{y^2}{4} = \left(\frac{y}{2}\right)^{2-\theta}.$$

II. Consider the values n for which $a_n > 2y$. Then $a_n - y > a_n/2$.

By the chain of inequalities (21), we see that $|l_n(z)| \geqslant |y - a_n||y + a_n| - a_n^2|K(z)|$. Further,

$$|y - a_n| = a_n - y > a_n/2 \quad \text{and} \quad |y + a_n| = a_n + y > a_n.$$

However,

$$|l_n(z)| > \frac{a_n}{2} \cdot a_n - a_n^2|K(z)|.$$

If we take y_0 so large that $|K(z)| < 1/4$ by property 2^0, then $|l_n(z)| > a_n^2/4$.

Hence,

$$|a_n^{-\theta} l_n(z)| > \frac{a_n^2}{4a_n^\theta} > \frac{a_n^{2-\theta}}{4} > \frac{(2y)^{2-\theta}}{4} = \frac{y^{2-\theta}}{2^\theta}.$$

III. Consider the values n for which $a_n \in [y/2, 2y]$.

Take an arbitrary $\varepsilon > 0$. We have $|z| > |\operatorname{Im} z| = y > a_n/2$. Hence, $a_n < 2|z|$ and

$$|a_n^2 K'(z)| = a_n^2 |K'(z)| < 2a_n |zK'(z)| < \varepsilon a_n \tag{22}$$

for sufficiently large y_0 by property 3^0. Since μ_n^+ is the upper root of $l_n(z) = 0$, it follows that the modulus of $l_n(z)$ can be represented in the form

$$|l_n(z)| = \left| \int_{\mu_n^+}^{z} l_n'(\lambda)d\lambda + l_n(\mu_n^+) \right| = \left| \int_{\mu_n^+}^{z} l_n'(\lambda)d\lambda \right| = \left| \int_{\mu_n^+}^{z} (2\lambda - a_n^2 K'(\lambda))d\lambda \right|$$

$$\geqslant \left| \int_{\mu_n^+}^{z} 2\lambda d\lambda \right| - \left| \int_{\mu_n^+}^{z} a_n^2 K'(\lambda)d\lambda \right|. \tag{23}$$

We have

$$\left| \int_{\mu_n^+}^{z} 2\lambda d\lambda \right| = \left| \lambda^2 \Big|_{\mu_n^+}^{z} \right| = \left| z^2 - (\mu_n^+)^2 \right| = |z - \mu_n^+||z + \mu_n^+|. \tag{24}$$

Moreover, we can estimate according to the bound (22),

$$\left| \int_{\mu_n^+}^{z} a_n^2 K'(\lambda)d\lambda \right| \leqslant |z - \mu_n^+| \sup_{[\mu_n^+, z]} |a_n^2 K'(\lambda)| \leqslant |z - \mu_n^+| \varepsilon a_n. \tag{25}$$

By property 1^0, we can write

$$|z + \mu_n^+| \geqslant |\operatorname{Im}(z + \mu_n^+)| = y + \operatorname{Im} \mu_n^+ > \operatorname{Im} \mu_n^+ > (1 - \varepsilon)a_n \tag{26}$$

for every sufficiently large y_0.

Considering also that z belongs to the set

$$\mathbb{D} = \{z : \operatorname{Im} z > y_0\} \setminus \bigcup_{n=1}^{\infty} D_{n,C}$$

and, on this set, by the definition of $D_{n,C}$, the inequality $|z - \mu_n^+| \geqslant C a_n^{\theta - 1}$ holds, we can combine the inequalities (23)–(26):

$$|l_n(z)| > |z - \mu_n^+|(|z + \mu_n^+| - \varepsilon a_n) > C a_n^{\theta-1}(a_n(1 - \varepsilon) - \varepsilon a_n) = C a_n^{\theta}(1 - 2\varepsilon).$$

Finally, we obtain

$$|a_n^{-\theta} l_n(z)| > C(1 - 2\varepsilon).$$

Thus, taking into account cases I–III, we note that

$$|a_n^{-\theta} l_n(z)| > \max\left(C(1 - 2\varepsilon),\ \left(\frac{y}{2}\right)^{2-\theta},\ \frac{y^{2-\theta}}{2^{\theta}} \right).$$

It can readily be seen that, taking $C > B(1 - 2\varepsilon)^{-2}$, for a sufficiently large value y_0, we have the inequality $|a_n^{-\theta} l_n(z)| > B(1 - 2\varepsilon)^{-1}$. Thus, we also have

$$\left(\inf_n |a_n^{-\theta} l_n(z)| \right) \geqslant B(1 - 2\varepsilon)^{-1} > B.$$

Then

$$\left(\inf_n |a_n^{-\theta} l_n(z)| \right)^{-1} < B^{-1},$$

which, taking into account (18), gives the validity of (17), as was to be proved.

Remark 2. For $\theta < 1$, we see that the radii $D_{n,C}$ tend to zero, and hence, $\widetilde{\mu}_n^+$ comes infinitely close to μ_n^+, which gives us a *localization of the spectrum* of the perturbed operator function (13).

Remark 3. As mentioned earlier for problem (4), (5), condition (12) holds. Moreover, obviously, for $a_n = n^\alpha,\ \alpha > 0$, for this condition to hold, it is necessary that

$$a_n^{\theta-1}(a_n - a_{n-1})^{-1} = n^{\alpha(\theta-1)}(n^\alpha - (n-1)^\alpha)^{-1} \sim n^{\alpha(\theta-1)}n^{1-\alpha} \to 0, \quad n \to \infty,$$

since $(n^\alpha - (n-1)^\alpha) \sim n^{\alpha-1}$. This condition holds for $\alpha > 1/(2-\theta)$.

Corollary 1. *If* $\sum\limits_{k=1}^{\infty} c_k < \infty$, *we can write out the asymptotics of the spectrum of* $L(z)$ *explicitly*:

$$\widetilde{\mu}_n^+ = ia_n - \frac{\sum\limits_{k=1}^{\infty} c_k}{2} + O(a_n^\kappa), \quad \textit{where} \quad \kappa = \max(-1, \theta - 1). \tag{27}$$

Proof. By [8], we have

$$(\mu_n^+)_1 = ia_n - \frac{\sum\limits_{k=1}^{\infty} c_k}{2} + O(a_n^{-1}).$$

Thus, by Theorem 1,

$$\widetilde{\mu}_n^+ = ia_n - \frac{\sum\limits_{k=1}^{\infty} c_k}{2} + O(a_n^{-1}) + O(a_n^{\theta-1}).$$

This implies (27), as was to be proved.

4. SPECTRUM ASYMPTOTICS IN THE FLUTTER PROBLEM

Now consider the operator function which is the symbol of the original equation of oscillation of a viscoelastic plate in a liquid or gas flow (4), (5):

$$L(z) = z^2 I + 2M_1 z I + M_2^2(1 - K(z))A^2 + M_3 U A^\theta.$$

It is clear from relation (8) that $\theta = 1/2$; for greater generality, we consider $\theta \in [0, 1)$.

Let us make the change of the spectral variable: $\rho = (z + M_1)/M_2;\ z = M_2\rho - M_1$. Then

$$z^2 + 2M_1 z = (z + M_1)^2 - M_1^2 = M_2^2\rho^2 - M_1^2.$$

In this case, the symbol of the equation is transformed,

$$\begin{aligned} L(\rho) &= M_2^2\rho^2 I + M_2^2\left(1 - K\left(M_2\rho - M_1\right)\right)A^2 + M_3 U A^\theta - M_1^2 I \\ &= M_2^2\rho^2 I + M_2^2\left(1 - K\left(M_2\rho - M_1\right)\right)A^2 + M_2^2\left(M_2^{-2}M_3 U - M_2^{-2}M_1^2 A^{-\theta}\right)A^\theta. \end{aligned}$$

Since the spectrum of an operator function is preserved under multiplication by a positive constant M_2, we can consider the following operator function:

$$L(\rho) = \rho^2 I + \left(1 - K\left(M_2\rho - M_1\right)\right)A^2 + \left(M_2^{-2}M_3 U - M_2^{-2}M_1^2 A^{-\theta}\right)A^\theta.$$

Define a bounded operator $R = M_2^{-2}M_3 U - M_2^{-2}M_1^2 A^{-\theta}$ and a function of the variable ρ,

$$\widetilde{K}(\rho) = K\left(M_2\rho - M_1\right) = \sum_{k=1}^{\infty} \frac{c_k}{M_2\rho - M_1 + \gamma_k} = \sum_{k=1}^{\infty} \frac{c_k/M_2}{\rho + (\gamma_k - M_1)/M_2}. \tag{28}$$

$\widetilde{K}(\rho)$ is a function of the form (14) with other coefficients.

Then $L(\rho)$ can be represented in the form

$$L(\rho) = \rho^2 I + (1 - \widetilde{K}(\rho))A^2 + RA^\theta. \tag{29}$$

Denote by $(\mu_n^+)_1$ the upper root of the equation $\rho^2 + (1 - \widetilde{K}(\rho))a_n^2 = 0$ and the domain $(D_{n,C})_1$:

$$(D_{n,C})_1 = \{\rho : |\rho - (\mu_n^+)_1| < Ca_n^{\theta-1}\} = \{z : |z - (M_2(\mu_n^+)_1 - M_1)| < CM_2 a_n^{\theta-1}\}.$$

Applying Theorem 1 to the function $L(\rho)$ from equality (14) and making the inverse change of the spectral variable, we obtain the following result.

Theorem 4. *Suppose that the eigenvalues of the operator A satisfy relation* (12)*. Then there are positive constants y_0 and C such that the spectrum $\sigma(L)$ of the operator function $L(z)$ lying in the upper half-plane $\{z : \operatorname{Im} z > y_0\}$ can be represented as a family of points $\{\widetilde{\mu}_n^+,\ n > n_0\}$ in such a way that $\widetilde{\mu}_n^+ \in (D_{n,C})_1$; here n_0 is the least positive integer such that, for every $n > n_0$,*

$$(D_{n,C})_1 \subset \{z : \operatorname{Im} z > y_0\}.$$

Remark. In view of the definition of $D_{n,C}$, we can describe the asymptotics of the spectrum $L(z)$ as

$$\widetilde{\mu}_n^+ = M_2(\mu_n^+)_1 - M_1 + O(a_n^{\theta-1}).$$

Corollary 2. *If $\sum\limits_{k=1}^{\infty} c_k < \infty$, then the asymptotics of the spectrum of $L(z)$ can be written explicitly as*

$$\widetilde{\mu}_n^+ = iM_2 a_n - \frac{\sum\limits_{k=1}^{\infty} c_k}{2} - M_1 + O(a_n^{\theta-1}). \tag{30}$$

Proof. By [8], and also by definition (28) of the function $\widetilde{K}(\rho)$, we have

$$(\mu_n^+)_1 = ia_n - \frac{\sum\limits_{k=1}^{\infty} c_k/M_2}{2} + O(a_n^{-1}).$$

Thus, by Theorem 3,

$$\widetilde{\mu}_n^+ = iM_2 a_n - \frac{\sum\limits_{k=1}^{\infty} c_k}{2} - M_1 + O(a_n^{-1}) + O(a_n^{\theta-1}).$$

This, together with the condition $\theta \in [0, 1)$, implies (30) as was to be proved.

BIBLIOGRAPHIC COMMENTARY

The equation in question was mainly studied for stability and asymptotic stability (flutter phenomenon). One can mention, e.g., [1] and [9], where a numerical study of the dependence of the critical speed $v_{\rm cr}$ (at which the solution (3) becomes unstable) on the parameters $h, D_0, \varepsilon_0, \gamma$, and so on, and also papers of V.V. Vedeneeva [10] and [11], where a numerical investigation of the single-mode flutter is carried out for a nonviscoelastic plate. In addition, the work of A.I. Miloslavsky [12] should be noted, in which the instability of integro-differential equations of the Gurtin–Pipkin type occurring in the study of models of a viscoelastic water supply system was studied by methods of functional analysis.

The Cauchy problem of the form (4), (5) arises in problems viscoelasticity and heat conductivity. If we remove the terms $2M_1\frac{d}{dt}u(t)$ and $M_3Tu(t)$, we obtain the Gurtin–Pipkin equation, originally obtained in [13]. Many works, including foreign ones, are devoted to his study. We note here the works of V.V. Vlasov and N.A. Rautian [6] and [8], Subsection 3.2, in which the correct solvability of the Gurtin–Pipkin equation in weighted Sobolev spaces for $M_2 = 1$ is established, the spectral analysis of the symbol of the equation (4) is carried out, the asymptotics of nonreal points of the spectrum and the localization of real clusters is obtained, which was used to write out a representation of the solution in the form of a series of exponentials. In addition, results on the correct solvability of this problem were obtained. The natural continuation of this research can be seen in [14], where the classical Gurtin–Pipkin equation with fractional-exponential relaxation kernels is studied. In [15] and [16], problems of control of solutions of the Gurtin–Pipkin equation by means of boundary effects were considered. In [17], the dependence of the energy decay rate on the decay rate of the kernel in the Gurtin–Pipkin heat conductivity model is established.

In the monograph [18] and in [19] and [20], an approach to solving the problem (4), (5) from the standpoint of semigroup theory is developed, where, for the case in which we remove the terms $2M_1\frac{d}{dt}u(t)$ and $M_3Tu(t)$ and for a more general form of the kernels $\Gamma(t)$, the form of the generator of the semigroup is established and it is proved that the semigroup is contracting and exponentially stable. The semigroup approach to more general problems in which the integral kernel has compact support was developed in the works of N.D. Kopachevskii and D.A. Zakora [21], [22]. In these papers, the exponential stability of the corresponding contracting semigroups is established. The newest research on the semigroup approach for the investigation of equations of Gurtin=-Pipkin type with two noncommuting operators were published in [23]. There the construction of a fundamentally new semigroup related to these equations is described and used to prove the exponential stability of solutions of these equations, the classical solvability of these equations, and also to construct the energy equality. In addition, in [24], a description of semigroups arising in the study of equations of the Gurtin–Pipkin type with a Kelvin–Voight friction is given. It is also worth noticing the paper [25], in which the generalized solvability of equations of Gurtin–Pipkin type with two noncommuting operators is studied.

We should also mention the paper [5], where it was proved that the spectrum of the symbol of an integro-differential equation of Gurtin–Pipkin type with a nonzero Kelvin–Voigt friction term contains only finitely many nonreal points. In the paper [26], the question concerning the presence and localization of an infinite nonreal spectrum of the symbol of this equation is studied in the case of a kernel representable by an infinite sum of exponentials.

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