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On the Ergodic Theory of Equations of Mathematical Physics

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To Academician Viktor Pavlovich Maslov with admiration

Abstract. Linear evolution equations of mathematical physics admitting an invariant in the form of a positive quadratic form are considered. In particular, this includes the string vibration equation, the Liouville kinetic equation, the Maxwell system of equations and the Schrödinger equation. Conditions for the existence of an invariant Gaussian measure are indicated, which makes it possible to apply well-known results of ergodic theory (Poincaré's recurrence theorem, Birkhoff–Khinchin ergodic theorem, etc.). We discuss the Hamiltonian property of such systems and conditions for their complete integrability. The ergodic properties of Kronecker flows on infinite-dimensional tori are studied. A general theorem on the averaging of quadratic forms is established.

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1. FINITE-DIMENSIONAL LINEAR SYSTEMS

We consider the linear differential equation

$$\dot{x} = Ax, \qquad x \in V, \tag{1.1}$$

in a finite-dimensional Euclidean space V (with the inner product $(\,,\,)$), which admits a first integral (an invariant)

$$f = \frac{1}{2}\,(Bx, x) \tag{1.2}$$

in the form of a nondegenerate quadratic form. It turns out that, in this case, the phase flow of the linear system preserves the standard Lebesgue measure μ in V: $d\mu = d^n x$ ($n = \dim V$). In other words,

$$\operatorname{div} Ax = \operatorname{tr} A = 0. \tag{1.3}$$

Indeed, the condition for the invariance of f has the form

$$BA + A^* B = 0. \tag{1.4}$$

The symbol $*$ stands for the conjugation of an operator with respect to the above inner product. However, then $A^* = -B^{-1}AB$, and hence,

$$\operatorname{tr} A = \operatorname{tr} A^* = -\operatorname{tr}(B^{-1}AB) = -\operatorname{tr}(ABB^{-1}) = -\operatorname{tr} A.$$

This implies (1.3).

However, one cannot immediately use the Poincaré recurrence theorem for system (1.1) because $\mu(V) = \infty$. Suppose now that the quadratic form (1.2) is positive definite. Then one can define a finite measure in V which is invariant with respect to the phase flow of system (1.1):

$$d\gamma = e^{-(Bx,x)}\, d^n x. \tag{1.5}$$

Measures of this kind are said to be Gaussian. In statistical mechanics, they are customarily called Gibbs measures.

Since $\gamma(V) < \infty$, it follows that (by the Poincaré recurrence theorem) almost all points in every γ-measurable domain $D \subset V$ return to D infinitely many times. Even more information is provided by the Birkhoff–Khinchin ergodic theorem, which contains Poincaré's theorem as a special case.

Another way to apply ergodic theory is to consider the restriction of the linear system (1.1) to the level manifold of the quadratic integral

$$\{x \in V : f(x) = c > 0\}.$$

The dynamical system arising on this invariant $(n-1)$-dimensional manifold has a finite invariant measure.

For the typical case in which not only the self-adjoint operator B, but also the operator A, are nondegenerate, $\dim V$ is even, and system (1.1) admits $(\dim V)/2$ independent quadratic integrals. In general, under these assumptions, the linear system (1.1) is a linear Hamiltonian system with respect to the symplectic structure

$$\omega(x', x'') = (BA^{-1}x', x''),$$

and the quadratic integral (1.2) serves as the Hamiltonian (see [1, 2], and the discussions therein). An involutive family of first integrals is given by the quadratic forms

$$\begin{aligned} f_1 &= \frac{1}{2}(Bx, x), \quad f_2 = \frac{1}{2}(A^*BAx, x), \quad \dots, \\ f_k &= \frac{1}{2}\big((A^*)^{k-1}BA^{k-1}x, x\big); \qquad k = n/2. \end{aligned} \tag{1.6}$$

These forms are independent if the spectrum of the operator A contains no multiple eigenvalues [3]. In the case of multiple spectrum, a complete set of involutive integrals has a different form (see [**?**]).

For almost all $c_1, \dots, c_k$, the joint levels of quadratic integrals

$$\{x : f_1(x) = c_1,\ \dots,\ f_k(x) = c_k\} \tag{1.7}$$

are k-dimensional tori, which carry trajectories of quasiperiodic motions. In appropriate angular variables $\varphi_1, \dots, \varphi_k \mod 2\pi$, equations (1.1) acquire the following form on the invariant tori (1.7):

$$\dot\varphi_1 = \omega_1, \quad \dots, \quad \dot\varphi_k = \omega_k.$$

The numbers $\omega_1, \dots, \omega_k$ do not depend on the torus. The pairs of purely imaginary numbers $\pm i\omega_1, \dots, \pm i\omega_k$ form the spectrum of the operator A. The ergodic theory of the linear system (1.1) on the integral invariant manifolds (1.4) is reduced to the classical Weyl's averaging theorem.

All these simple observations can be carried over (with certain precautions) to the infinite-dimensional case. The Hamiltonian property of a linear system with a quadratic invariant in a Hilbert space was discussed in [8] and [2]. This note should be regarded as a complement to the paper [2].

In the conclusion of this section, we make several remarks on invariant measures of general nonlinear systems

$$\dot x = v(x), \qquad x \in V,$$

admitting a quadratic invariant. Under the assumption of positive definiteness, this invariant is reduced to the sum of squares,

$$f = \frac{1}{2}(x_1^2 + \dots + x_n^2).$$

Since f is invariant, it follows that

$$\dot f = \sum x_j v_j = 0.$$

This implies the equalities

$$\begin{gathered} v_1 + x_1\frac{\partial v_1}{\partial x_1} + \dots + x_n\frac{\partial v_n}{\partial x_1} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ v_n + x_1\frac{\partial v_1}{\partial x_n} + \dots + x_n\frac{\partial v_n}{\partial x_n} = 0. \end{gathered}$$

Hence,

$$\operatorname{div} v = \sum \frac{\partial v_j}{\partial x_j} = -\frac{1}{2}(x_1\Delta v_1 + \dots + x_n\Delta v_n),$$

where Δ is the Laplace operator. In particular, for a linear system with a quadratic invariant, we obtain the presence of a standard invariant measure. In the general case, this can be asserted only under the condition that the components of the vector field v generating the dynamical system are harmonic functions.

In particular, for a system with quadratic right-hand sides

$$v_j = (A_j x, x),$$

the condition for the invariance of the Lebesgue measure is reduced to the equalities

$$\operatorname{tr} A_1 = \cdots = \operatorname{tr} A_n = 0.$$

Certainly, here we must add the condition for the existence of a quadratic invariant. Here there is a simple example of equations with quadratic right-hand sides that have a positive-definite quadratic invariant, but admit no invariant measure with continuous positive density at all:

$$\dot{x}_1 = x_2^2, \qquad \dot{x}_2 = -x_1 x_2.$$

On the other hand, as was established by Volterra [9], if a system with quadratic right-hand sides admits two quadratic integrals

$$f_1 = \frac{1}{2}(Bx, x) \qquad \text{and} \qquad f_2 = \frac{1}{2}(Cx, x),$$

where $B > 0$ and the characteristic equation $|C - \lambda B| = 0$ has no multiple roots, then the phase flow of this system preserves the standard Lebesgue measure in V. Conditions for the Hamiltonian property of these systems of differential equations and for their complete integrability were obtained in [10].

2. LINEAR SYSTEMS IN A HILBERT SPACE

Let $\mathbb{H}$ be a real separable Hilbert space with inner product $(\,,)$ and let A be a bounded linear operator on $\mathbb{H}$ with dense domain $D(A)$. As is well known, the operator A has a unique extension $\overline{A}\colon \mathbb{H} \to \mathbb{H}$ with domain $D(\overline{A}) = \mathbb{H}$ such that $\|\overline{A}\| = \|A\|$. Therefore, we assume from the very beginning that the domain of A coincides with $\mathbb{H}$. In this case, there is a unique linear operator A^* (adjoint to A) such that

- $D(A^*) = \mathbb{H}$,
- $(A^* x, y) = (x, Ay)$ for all $x, y \in \mathbb{H}$,
- $\|A^*\| = \|A\|$.

To the operator A, we can assign the linear differential equation

$$\dot{x} = Ax, \qquad x \in \mathbb{H}. \tag{2.1}$$

Since A is bounded, it follows that, for every $x_0 \in \mathbb{H}$, this system has a unique solution $t \mapsto x(t)$ defined on the entire axis $\mathbb{R} = \{t\}$ with the initial value $x(0) = x_0$ (see, e.g., [11]). In other words, the phase flow of system (2.1) is well defined on the entire Hilbert space.

In Sections 3 and 5, we shall also consider linear differential equations in a Hilbert space with an *unbounded* operator A. For equations of mathematical physics, this case is more natural. However, equation (2.1) will no longer have a solution for *arbitrary* initial data.

In accordance with the general definition, to the linear system (2.1) we can assign the adjoint system of differential equations

$$\dot{y} = -A^* y, \qquad y \in \mathbb{H}. \tag{2.2}$$

Since $\|A^*\| = \|A\| < \infty$, it follows that the adjoint linear system (2.2) is also uniformly solvable. According to Lagrange,

$$\big(x(t), y(t)\big) = \text{const}$$

for any solutions of the linear systems (2.1) and (2.2). Note a simple assertion concerning the properties of adjoint systems that is related to the Lagrange general theorem.

Theorem 1. *Let $B\colon \mathbb{H} \to \mathbb{H}$ be a bounded self-adjoint operator. In this case,*

(1) *if $f = (Bx, x)/2$ is an invariant of the linear system* (2.1), *then, for every solution $t \mapsto x(t)$ of the system, the function $t \mapsto Bx(t)$ is a solution of the adjoint system* (2.2);

(2) *if, moreover, the operator B has bounded inverse, then the quadratic form $g = (B^{-1} y, y)/2$ is an invariant of the adjoint system.*

Indeed, the invariance condition of a quadratic form f with respect to the flow of the linear system (2.1) means precisely that the operator BA is skew-symmetric:

$$(BA)^* = A^*B = -BA. \tag{2.3}$$

Hence, if $x(\,\cdot\,)$ is a solution to (2.1), then

$$(Bx)^{\cdot} = BAx = -A^*(Bx).$$

Further, condition (2.3) is equivalent to the following one:

$$(B^{-1}A^*)^* = -B^{-1}A^*.$$

This proves Theorem 1.

The correspondence

$$A, B \mapsto -A^*, B^{-1}$$

is involutive. It defines a remarkable duality between a linear system with a quadratic invariant and its adjoint system.

As is known, in an infinite-dimensional Hilbert space, there is no analog of the Lebesgue measure (a countably additive measure invariant under translations). For a discussion of this circle of questions, see, e.g., [12]. A more natural way is the study of invariance conditions of an infinite-dimensional analog of the Gaussian measure (1.5). For a detailed account of the theory of Gaussian measures, see [13].

Let γ be a finite measure on $\mathbb{H}$ and let

$$\widetilde{\gamma}(y) = \int_{\mathbb{H}} e^{i(y,x)}\,\gamma(dx) \tag{2.4}$$

be its characteristic functional. This is a complex-valued function on $\mathbb{H}$.

Theorem 2. *If $\widetilde{\gamma}$ is a first integral of the adjoint system* (2.2), *then γ is an invariant measure for the linear system* (2.1).

Indeed, by the assumption of the theorem, the substitution

$$y \mapsto e^{-A^*t}y, \qquad y \in \mathbb{H}, \tag{2.5}$$

does not change the value of the integral (2.4). Further, the linear system (2.1) defined a family of transformations of the Hilbert space

$$x \mapsto g^t(x) = e^{tA}x. \tag{2.6}$$

By the Lagrange theorem, the substitutions (2.5) and (2.6) do not change the value of the inner product (y, x). This also follows immediately from the equation

$$(e^{At})^* = e^{A^*t}.$$

Consider the transformed measure

$$\gamma_t = \gamma \circ g^t.$$

By the well-known theorem on the image of the measure under a mapping, the integral (2.4) becomes finally equal to

$$\int_{\mathbb{H}} e^{i(y,x)}\,\gamma_t(dx). \tag{2.7}$$

Since $\widetilde{\gamma}$ is an invariant of the adjoint linear system, it follows that the integral (2.7) does not depend on time. Taking into account the fact that every measure is uniquely defined by its characteristic functional, we come to the conclusion that the measure γ is invariant with respect to the phase flow (2.6) of the linear system (2.1).

From Theorem 2, we can derive a condition for the existence of a Gaussian countably additive invariant measure for the original linear equation.

Theorem 3. *Let K be a symmetric nonnegative trace-class operator such that the quadratic form $(Ky, y)/2$, $y \in \mathbb{H}$, is the first integral of the adjoint system* (2.2). *Then the linear system* (2.1) *has an invariant centered Gaussian measure with covariance operator K.*

Proof. Since the operator K is symmetric, nonnegative, and trace class, it follows that there is a centered Gaussian measure with the covariance operator K. Further, as is known, the characteristic functional of this measure is equal to

$$\exp\left[-\frac{1}{2}(Ky,y)\right].$$

By assumption, this function is an invariant of the adjoint system (2.2). Hence, by Theorem 2, this Gaussian countably additive measure is invariant with respect to the phase flow of the linear system (2.1). ☐

Corollary 1. *Let the conditions of Theorem* 3 *hold and let D be a γ-measurable subdomain of $\mathbb{H}$ of positive Gaussian measure γ. Then, for almost all $x_0 \in D$, the trajectory of the solution*

$$t \mapsto e^{At}x_0$$

of the linear system (2.1) *meets D infinitely many times as $|t| \to \infty$.*

This is a strong version of the Poincaré recurrence theorem. The weak version is related to the return of domains of positive measure. This does not require the countable additivity of the invariant measure.

The existence condition for a finitely additive *invariant* Gaussian measure with a symmetric nonnegative covariance operator K also reduces to the invariance of the quadratic form $(Ky,y)/2$, $y \in \mathbb{H}$, with respect to the phase flow of the adjoint system (2.2). A classical example of finitely additive but not countably additive Gaussian measure is given by the case in which $K = I$ is the identity operator [13].

Let the original linear system (2.1) admit a first integral in the form of a nonnegative quadratic form $(Bx,x)/2$. If the operator B has the inverse B^{-1}, then (in accordance with Theorem 1) the quadratic form $(B^{-1}y,y)/2$ must be an invariant of the adjoint linear system. If, moreover, the inverse operator is of trace class, then the Poincaré recurrence theorem can be applied.

Let $\{\lambda_k\}$ be the eigenvalues of the operator B. As a rule, $\lambda_k \to \infty$ as $k \to \infty$ for equations of mathematical physics. Hence, the eigenvalues $\{\lambda_k^{-1}\}$ of the inverse operator B^{-1} tend to zero. If, in addition, the series $\sum \lambda_k^{-1}$ converges, then B^{-1} is a trace-class operator.

As an illustration, consider the Schrödinger equation on the one-dimensional torus

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2}, \qquad x \mod 2\pi, \tag{2.8}$$

which describes the dynamics of a quantum rotator. The wave functions $\psi(x,t)$ are 2π-periodic with respect to x and satisfy the normalization condition

$$\int_0^{2\pi} \psi\overline{\psi}\,dx = 1. \tag{2.9}$$

Actually, the integral on the left-hand side defines a Hermitian inner product in the complex Hilbert space of square-summable functions. It can readily be seen that, after a realification, equation (2.8) is *self-adjoint* (in the sense of the definition (2.2)).

Setting

$$\psi = \sum \psi_n(t)e^{inx},$$

we obtain from (2.8) a chain of equations for the coefficients:

$$i\dot{\psi}_n = n^2\psi_n.$$

Hence, equation (2.8) has an entire family of invariants

$$f_n = \psi_n\overline{\psi}_n.$$

In general, the quadratic forms

$$f = \sum \mu_n f_n \tag{2.10}$$

with bounded positive coefficients μ_n are continuous invariants of equation (2.8). If $\sum \mu_n < \infty$, then (2.10) can be taken for the covariance operator of an invariant countably additive Gaussian measure.

In particular, to the energy integral

$$-\int_0^{2\pi} \frac{\partial^2\psi}{\partial x^2}\overline{\psi}\,dx$$

there corresponds the case in which $\mu_n = 2\pi n^2$. The quadratic form (2.10) with the coefficients μ_n^{-1} $(n \neq 0)$, where $\mu_0 > 0$ is arbitrary, is also a first integral of equation (2.8), and the self-adjoint operator B generating this form is of trace class.

To the integral (2.9), there corresponds a quadratic form for which $B = 2\pi I$. This form generates an invariant Gaussian measure, which is only finitely additive.

Thus, equation (2.8) admits a whole family of different invariant Gaussian measures. The application of the ergodic theory rests on a meaningful problem of domains in a Hilbert space with positive Gaussian measure. In the next section, these observations are clarified.

3. COMPLETE INTEGRABILITY, PRODUCT MEASURES AND RETURNABILITY

Linear equations with a quadratic invariant in a Hilbert space also admit a series of quadratic invariants of the form (1.6):

$$f_k = \frac{1}{2}\big((A^*)^{k-1} B A^{k-1} x, x\big), \qquad k \geqslant 1. \tag{3.1}$$

These invariants can be used to construct Gaussian invariant measures for system (2.1), as well as the original invariant $f = f_0$.

Moreover, as in the finite-dimensional case, the linear equation (2.1) with a quadratic invariant f_0 can be represented in Hamiltonian form, and the quadratic invariants (3.1) will be pairwise in involution [2]. The Poisson bracket itself is defined naturally on the space of all continuous quadratic forms given in a real Hilbert space. However, it is still impossible to conclude from this that the infinite-dimensional Hamiltonian system is completely integrable. To prove the complete integrability, another set of independent quadratic invariants was used in [2].

Assume that a bounded symmetric operator $B\colon \mathbb{H} \to \mathbb{H}$ is positive definite:

$$(Bx, x) \geqslant c\|x\|^2$$

for all $x \in \mathbb{H}$ with some constant c. Regarding a densely defined operator A, we do not assume its boundedness. However, we assume that its eigenvectors enter the domain of the operator A in the complexified Hilbert space. The theory of such linear systems of differential equations was developed in [2] (Sec. 4).

It turns out that, under these assumptions, all eigenvalues of the operator A are purely imaginary:

$$\pm i\omega_1, \quad \pm i\omega_2, \quad \ldots, \quad \pm i\omega_k, \quad \ldots \tag{3.2}$$

Assume that all of them are distinct (and, in particular, nonzero). Let

$$A(\xi_k \pm i\eta_k) = \pm i\omega_k(\xi_k \pm i\eta_k),$$

where ξ_k and η_k are vectors in $\mathbb{H}$ (they belong to $D(A)$). Then

$$A\xi_k = -\omega_k \eta_k, \qquad A\eta_k = \omega_k \xi_k.$$

It can be proved that here we have

$$(B\xi_k, \xi_k) = (B\eta_k, \eta_k) \qquad \text{and} \qquad (B\xi_k, \eta_k) = 0. \tag{3.3}$$

Let π_k be an invariant plane of A spanned by the linearly independent vectors ξ_k and η_k, and let P_k be the orthogonal projection to the two-dimensional subspace π_k. All projections $P_1, P_2, \ldots$ are bounded self-adjoint operators on $\mathbb{H}$. Since $\omega_k \neq \pm\omega_l$ for $k \neq l$, it follows that the two-dimensional planes π_k and π_l are orthogonal with respect to the inner product $(x, y)' = (Bx, y)$.

Write

$$f_k = \frac{1}{2}\,(BP_k x, P_k x), \qquad k = 1, 2, \ldots \tag{3.4}$$

These quadratic forms are first integrals of the linear system (2.1); they are independent and are pairwise in involution. In [2], a natural condition is indicated under which the integrals (3.3) form a *complete* involutive family: the vector system

$$\xi_1, \quad \eta_1, \quad \ldots, \quad \xi_k, \quad \eta_k, \quad \ldots$$

orthonormal with respect to the scalar product $(\,,)'$ is closed.

Under these assumptions, the invariant algebraic manifold

$$J_c = \{x \in \mathbb{H} : f_1(x) = c_1,\ f_2(x) = c_2,\ \dots\},$$

under the positive values of $c_1, c_2, \dots$ such that $\sum c_k < \infty$, is an infinite-dimensional torus

$$\mathbb{T}^\infty = \mathop{\mathrm{X}}_{k=1}^{\infty} \mathbb{T}^1_k;$$

on each of these tori, one can choose angular coordinates $\varphi_1 \mod 2\pi, \varphi_2 \mod 2\pi, \dots$, which are uniformly changes in the course of time:

$$\dot\varphi_1 = \omega_1, \quad \dot\varphi_2 = \omega_2, \quad \dots \tag{3.5}$$

The frequencies $\omega_1, \omega_2, \dots$ are numbers in the spectrum (3.2) of the linear operator A. Our task is to discuss ergodic properties of system (3.5) on the infinite-dimensional torus $\mathbb{T}^\infty$. Its phase flow is often called the *Kronecker flow* (actually related to studied by Kronecker quasiperiodic motions on finite-dimensional tori).

Every one-dimensional torus $\mathbb{T}^1_k$ is a topological group (the group of complex numbers with unit modulus with respect to multiplication) on which there is a probabilistic measure μ_k invariant under the action of this group. Since all tori $\mathbb{T}^1_k$ are compact, it follows that (by Tikhonov's theorem) the product $\mathbb{T}^\infty$ is also compact. Therefore, $\mathbb{T}^\infty$, as the direct product of countably many one-dimensional tori and can also be equipped with the structure of an infinite-dimensional commutative compact group with invariant measure μ, which is defined as the *product measure*

$$\mu = \mathop{\mathrm{X}}_{k=1}^{\infty} \mu_k.$$

If $X_1, X_2, \dots$ are measurable subsets of $\mathbb{T}^1_1, \mathbb{T}^1_2, \dots$, it follows that

$$\mu(X_1 \times X_2 \times \cdots) = \mu_1(X_1)\mu_2(X_2)\cdots.$$

It is clear that $\mu(\mathbb{T}^\infty) = 1$. For a measure of a set $X_1 \times X_2 \times \cdots$ to be positive, it is necessary that $\mu_k(X_k) \to 1$ as $k \to \infty$. These problems are discussed in detail in [14].

The product measure can be identified with a unique probability measure on a compact topological group that is invariant simultaneously with respect to all left and right shifts on the group. As is known, this measure is called the *Haar measure*.

The flow of system (3.5) on $\mathbb{T}^\infty$ preserves the product measure μ. This assertion is obvious: a transformation in the flow of system (3.5) performs the rotation of all points of every torus $\mathbb{T}^1_k$ by the same angle. The measure μ_k does not change under these rotations, which ultimately leads to the invariance of the measure μ. This readily implies the following theorem.

Theorem 4. *Let X_k be measurable subsets of $\mathbb{T}^1_k$ and let*

$$\prod_{k=1}^{\infty} \mu_k(X_k) > 0.$$

Then for almost all (with respect to the measure μ) points $\varphi^0 = (\varphi^0_1, \varphi^0_2, \dots)$, $\varphi^0_k \in X_k$, the trajectory of the motion

$$t \mapsto \varphi(t) = \omega t + \varphi^0$$

meets the set

$$X_1 \times X_2 \times \cdots \subset \mathbb{T}^\infty$$

infinitely many times.

This is an immediate consequence of the Poincaré recurrence theorem; it does not depend on the arithmetic properties of the set of frequencies $\omega_1, \omega_2, \dots$ Since $\mu_k(X_k) \to 1$ as $k \to \infty$, it follows that, for large k, the set X_k "almost coincides" with the entire torus $\mathbb{T}^1_k$. However, if we set $X_k = \mathbb{T}^1_k$ for all $k \geqslant k_0$, then Theorem 4 becomes the classical Kronecker–Weyl theorem about quasiperiodic motions on finite-dimensional tori.

4. KRONECKER FLOWS

Let us continue the consideration of system (3.5) on $\mathbb{T}^\infty$. A family of frequencies $\omega_1, \omega_2, \dots$ is said to be *nonresonance* if, for every $n \geqslant 1$, it follows from the relation

$$k_1\omega_1 + k_2\omega_2 + \dots + k_n\omega_n = 0$$

with integer k_j that $k_1 = k_2 = \dots = k_n = 0$. Here is a constructive example of a nonresonance set, going back to H. Bohr:

$$\omega_1 = \ln 2, \quad \omega_2 = \ln 3, \quad \omega_3 = \ln 5, \quad \dots, \quad \omega_n = \ln p_n, \quad \dots .$$

Here p_n stands for the n-th prime.

As is known, if the set of frequencies ω is nonresonance, then the Kronecker flow on $\mathbb{T}^\infty$ is ergodic with respect to the product measure μ of Section 3 (see the discussion in [15, 16] and the references therein). In fact, this result was first formulated by von Neumann in [17], mentioning the classical works of Kronecker and Weyl in the finite-dimensional case. Technical means (like Tikhonov's theorem on infinite products of compacta) needed in the proof arose later. Recently, ergodic properties of the so-called quantum Kronecker flows are discussed (see, e.g., [18]). However, under broad assumptions, the Schrödinger equations are completely integrable infinite-dimensional Hamiltonian systems, and their phase spaces are foliated into invariant ergodic sets, i.e., the tori $\mathbb{T}^\infty$ with ordinary (nonquantum) Kronecker flows [2].

Let us indicate some new properties of temporal and spatial averages for Kronecker flows.

Theorem 5. *If a family of frequencies ω is nonresonance, then, for any continuous function $f\colon \mathbb{T}^\infty \to \mathbb{R}$,*

$$f(\omega t + \varphi_0) \rightrightarrows \int_{\mathbb{T}^\infty} f\,d\mu \qquad (C). \tag{4.1}$$

The symbol $\rightrightarrows$ stands for uniform convergence (with respect to $\varphi_0 \in \mathbb{T}^\infty$) and C stands for the convergence as $t \to \infty$ in the Cesàro sense (the convergence of mean values).

As well as the classical finite-dimensional Weyl theorem on the uniform distribution, Theorem 5 is proved in two stages. The first stage considers the case when f is a trigonometric polynomial in the angular variables $\varphi_1, \varphi_2, \dots$:

$$f = \sum_{k=-N}^{N} f_k e^{i(m_k,\varphi)}, \qquad f_{-k} = \overline{f}_k, \quad m_{-k} = -m_k. \tag{4.2}$$

Here $m_k \in \mathbb{Z}^n$ and $\varphi = (\varphi_1, \dots, \varphi_n)$; the positive integer N does not depend on n. Taking into account the condition of nonresonance for ω of such functions, the property (4.1) is quite elementary. Here $m_k \in \mathbb{Z}^n$ and $\varphi = (\varphi_1, \dots, \varphi_n)$; a positive integer N does not depend on n. Taking into account the nonresonance property ω, we see that, for these functions, the property (4.1) is quite elementary.

At the second step, an approximation of continuous functions on the compact set $\mathbb{T}^\infty$ by suitable trigonometric polynomials (4.2) is used. This possibility is provided by the well-known Stone–Weierstrass theorem.

According to the old Oxtoby theorem [19], relation (4.1) implies the *strict ergodicity* of the dynamical system (3.4) on $\mathbb{T}^\infty$. This means that, in the nonresonance case, the Kronecker flow admits only one invariant probability Borel measure. In particular, this flow is ergodic (with respect to the Borel product measure μ): for almost all $\varphi^0 \in \mathbb{T}^\infty$, the temporal mean of any μ-integrable function coincides with its spatial mean.

Relation (4.1) can be extended to a wider class of functions, which can be called "Riemann integrable" (below we call them $\mathfrak{R}$-integrable functions). We say that a function $f\colon \mathbb{T}^\infty \to \mathbb{R}$ is $\mathfrak{R}$-*integrable* if, for for any $\varepsilon > 0$, there are two continuous functions f_1 and f_2 such that

$$f_1(\varphi) \leqslant f(\varphi) \leqslant f_2(\varphi) \qquad \text{for all} \quad \varphi \in \mathbb{T}^\infty, \tag{4.3}$$

and

$$\int_{\mathbb{T}^\infty} (f_2 - f_1)\,d\mu < \varepsilon. \tag{4.4}$$

Since $\mathbb{T}^\infty$ is a compact set, it follows that all $\mathfrak{R}$-functions are bounded and all continuous functions are $\mathfrak{R}$-integrable.

The integral of an $\mathfrak{R}$-integrable function is defined in the following natural way. Let $\varepsilon_n \to 0$. Consider the corresponding sequences of continuous functions $f_1^{(n)}$ and $f_2^{(n)}$ satisfying the conditions (4.3) and (4.4). It can readily be shown that the limits of the sequences of integrals of $f_1^{(n)}$ and $f_2^{(n)}$ over $\mathbb{T}^\infty$ as $n \to \infty$ exist and coincide. We call this number the $\mathfrak{R}$*-integral of* f *with respect to the Borel measure* μ and denote it by

$$\int_{\mathbb{T}^\infty} f \, d\mu.$$

The $\mathfrak{R}$-integral is well defined: it does not depend on the choice of the sequences ε_n, $f_1^{(n)}$, and $f_2^{(n)}$.

If we replace $\mathbb{T}^\infty$ in this definition by a finite-dimensional torus $\mathbb{T}^m$ with the standard measure, then the class of $\mathfrak{R}$-integrable functions on $\mathbb{T}^m$ coincides with the class of Riemann integrable functions.

Theorem 5 admits a generalization.

Theorem 6. *The Kronecker flow on* $\mathbb{T}^\infty$ *is strictly ergodic if and only if equality* (4.1) *holds for every* $\mathfrak{R}$*-integrable function* $f\colon \mathbb{T}^\infty \to \mathbb{R}$.

A similar statement for strictly ergodic cascades is noted in [20].

A domain $D \subset \mathbb{T}^\infty$ is said to be $\mathfrak{R}$-measurable if its characteristic function 1_D is $\mathfrak{R}$-integrable. Write

$$\operatorname{mes} D = \int_{\mathbb{T}^\infty} 1_D \, d\mu.$$

It is clear that every $\mathfrak{R}$-measurable function is μ-measurable and $\operatorname{mes} D = \mu(D)$.

Example 1. Let I_k be an interval in $\mathbb{T}^1_k$. Then the domain $D = I_1 \times I_2 \times \cdots \subset \mathbb{T}^\infty$ is $\mathfrak{R}$-measurable. Recall that $\operatorname{mes} D > 0$ if the product of the lengths of the intervals is positive.

Corollary 2. *Let* D *be a measurable domain in* $\mathbb{T}^\infty$ *and let* $\tau_D(\varphi^0)$ *be the total time in the interval* $[0, \tau]$ *during which the point* $\varphi(\omega t + \varphi^0)$ *belongs to the domain* D*. If the family of frequencies* ω *is nonresonance, then*

$$\frac{\tau_D(\varphi^0)}{\tau} \rightrightarrows \operatorname{mes} D \tag{4.5}$$

as $\tau \to \infty$.

The difference between this statement and the general ergodic theorem for Kronecker flows is that the limit relation (4.5) is valid for *all* $\varphi^0 \in \mathbb{T}^\infty$.

5. ERGODIC THEOREM FOR QUADRATIC FORMS

Let us begin with a simple remark concerning the Kronecker flow on $\mathbb{T}^n$. Suppose that the family of frequencies $\omega_1, \dots, \omega_n$ is such that

$$\sum_{j=1}^{n} k_j \omega_j \neq 0$$

for $|k| = |k_1| + \dots + |k_n| \leqslant m$. Consider the trigonometric polynomial (4.2)

$$f(\varphi) = \sum f_k e^{i(k,\varphi)}, \qquad f_{-k} = \overline{f}_k,$$

for which $|k| \leqslant m$. Then, for this polynomial, the ergodic theorem (4.1) holds (the temporal mean of f is equal to its spatial mean), although the Kronecker flow need not be ergodic.

Indeed,

$$\int_0^t f(\omega t + \varphi^0) \, dt = f_0 t + F(\omega t + \varphi_0) - F(\varphi_0), \tag{5.1}$$

where f_0 in the mean of f over the torus $\mathbb{T}^n$, and

$$F(\varphi) = \sum_{|k| \leqslant m}{}' \frac{f_k}{i(k,\omega)} e^{i(k,\varphi)}$$

is a well defined continuous function on $\mathbb{T}^n$. After dividing both the sides of (5.1) by t and passing to the limit as $t \to \infty$, we obtain the desired formula (4.1).

For linear systems of differential equations with a quadratic invariant in a Hilbert space $\mathbb{H}$, quadratic forms, rather than arbitrary functions on $\mathbb{H}$, play the key role. There are several reasons for this. First, these systems are represented in a Hamiltonian form on the space of quadratic forms. Further, a complete set of independent involutive first integrals is also formed by continuous quadratic forms. Finally, for quantum systems whose evolution is governed by the Schrödinger equation, the observables are Hermitian operators, and the quadratic forms generated by them express the expectations (mean values) of these observables at the current state of the quantum system.

In general, a linear system with a quadratic invariant cannot be ergodic, even if it is restricted to the level manifold of this invariant. The reason for the nonergodicity is in the presence of other quadratic invariants (e.g., of the form (3.1)). Nevertheless, one can prove the following version of the ergodic theorem for quadratic forms: if we restrict this quadratic form to an invariant torus $\mathbb{T}^\infty$, then its temporary mean is equal to the spatial one provided that there are no multiple eigenvalues of the operator A. Here, as above, the Kronecker flow on $\mathbb{T}^\infty$ need not be ergodic at all.

The condition

$$\omega_k \neq \pm\omega_l, \qquad k \neq l, \tag{5.2}$$

plays a crucial role in quantum statistical mechanics. Von Neumann called these systems *ergodic*. Their characteristic property is that any Hermitian operator commuting with the Hamiltonian operator is a function of the latter. It is curious that, in the paper [17] and the book [21], where von Neumann discusses the "ergodic theorem," the theorem itself (as the equality of time and spatial averages) is in fact not formulated. The ergodic theorem was treated by von Neumann himself as a meaningful correspondences between temporal and spatial averages for *macroscopic* values (taking into account an additional combinatorial analysis of the ensemble of microscopic subsystems of the quantum system). This tradition continues also in modern manuals on quantum statistical mechanics (see, e.g., [22]).

We assume the validity of the condition on the operators A and B of Section 3 under which the Hilbert space $\mathbb{H}$ is foliated into invariant infinite-dimensional tori

$$\mathbb{T}^\infty_c = \{x \in \mathbb{H} : f_1(x) = c_1,\ f_2(x) = c_2, \dots\}, \qquad \text{where} \qquad f_n(x) = \frac{1}{2}(P_n B P_n x, x).$$

In particular, it is assumed that condition (5.2) holds. Let

$$f = \frac{1}{2}(Cx, x) \tag{5.3}$$

be a continuous quadratic form in a real Hilbert space $\mathbb{H}$ generated by a bounded self-adjoint operator C.

Theorem 7. *The temporal mean of the quadratic form* (5.3)

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau (e^{At}x)\, dt$$

at the points $x \in \mathbb{T}^\infty_c$ *is equal to*

$$\frac{1}{2}\sum_{n=1}^{\infty} \operatorname{tr}(P_n C P_n) c_n \tag{5.4}$$

and coincides with the mean over the torus $\mathbb{T}^\infty_c$ *with respect to the product measure* μ *of the restriction of* f *to* $\mathbb{T}^\infty_c$.

This assertion can readily be derived from Theorem 5, taking into account the continuity of the restriction of f to $\mathbb{T}^\infty_c$ and the considerations outlined at the beginning of this section: the expansion of the restriction of f on $\mathbb{T}^\infty_c$ into a trigonometric series with respect to $\varphi_1, \varphi_2, \dots$ contains harmonics no higher than of the second order. The derivation of the formula (5.4) uses the parameterization of $\mathbb{T}^\infty_c$ by the formula

$$x = \sum_{n=1}^{\infty} (p_n \xi_n + q_n \eta_n),$$

where the coefficients p_n and q_n, as functions of time, satisfy the following linear system:

$$\dot{p}_n = \omega_n q_n, \quad \dot{q}_n = -\omega_n p_n \qquad (n \geqslant 1).$$

Three remarks.

1. Theorem 7 holds also for an unbounded self-adjoint operator C; however, one only needs to assume that the vectors ξ_n and η_n $(n \geqslant 1)$ belong to the domain of C and that the restriction of f to $\mathbb{T}^\infty_c$ is a continuous function.

2. Formula (5.4) holds also for temporal means of expectations for a Hermitian operator in quantum mechanics after the realification of complex Schrödinger equations.

3. For the quantum rotator (see Section 2) and the quantum harmonic oscillator, all ratios ω_k/ω_l are rational numbers which differ from ± 1 for $k \neq l$. In these cases, the corresponding Kronecker flows are certainly not ergodic; however, the conclusion of Theorem 7 holds.

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