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# On a Time-Dependent Nonholonomic Oscillator

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**Abstract.** In this note, we compare the first integrals and exact solutions of equations of motion for scleronomic and rheonomic, and holonomic and nonholonomic oscillators.

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## 1. INTRODUCTION

Let us consider a general rheonomic Lagrangian system with nonintegrable constraints characterized by a Lagrange function $L(q,\dot q,t)$ and ideal independent constraints that are linear at velocities,

$$f_k(q,\dot q,t)=0\,,\qquad k=1,\dots,m\,.$$

Here $q=(q_1,\dots,q_n)$ are the independent Lagrangian coordinates, $t$ is time, and $\dot q_i = dq_i/dt$ are the generalized velocities. The well-known equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q_i}\right)-\frac{\partial L}{\partial q_i}=\sum_{k=0}^{m}\lambda_k\frac{\partial f_k}{\partial q_i}\,,\qquad \text{and}\qquad f_k=0\,,\quad k=1,\dots,m\,, \tag{1.1}$$

are equivalent to a variational equation expressing the form of the Lagrange-d'Alembert principle or the Hamilton principle in the Hölder form; for details, see [3, 22, 29].

The greatest advantage of Lagrangian formulation is that it brings out the connection between conservation laws and important symmetry properties of dynamical systems. The knowledge of conservation laws is of great importance for the analysis of dynamical systems since they enable one to find exact solutions of dynamical systems. Explicit solutions enable us to test analytical and numerical schemes applied to the given mathematical model and to choose a reasonable approximation to solutions of the model.

If time $t$ does not explicitly enter equations (1.1), then these systems are the so-called scleronomic systems. In other cases, for example, when involving a dependence on time of the constraints $f_k$, time $t$ does explicitly enter, and such systems are called rheonomic Lagrangian systems. Rheonomic nonholonomic systems are divided into two families

1. systems with time-independent Lagrangian
$$\frac{\partial L(q,\dot q,t)}{\partial t}=0;$$
2. systems with time-dependent original Lagrangian
$$\frac{\partial L(q,\dot q,t)}{\partial t}\neq 0\,.$$

In the first case, the infinitesimal work of the constraint forces vanishes for any admissible infinitesimal virtual displacement according to Chetaev's rule [12]. As a result, the Jacobi integral

$$H=\sum_{i=1}^{n}\dot q_i\frac{\partial L}{\partial q_i}-L$$

remains the first integral of equations (1.1) which can be used for the explicit solution of equations of motion (1.1).

In the other case, we have

$$\frac{dH}{dt}=-\frac{\partial L}{\partial t}\,,$$

and, instead of the Jacobi integral, we can use the Ermakov–Lewis type invariants to solve the equations of motion (1.1). For instance, consider a Lagrangian describing the time-dependent harmonic oscillator

$$L(q,\dot q,t)=\frac{1}{2m(t)}\Bigl(\dot q^2-\omega^2(t)q^2\Bigr).$$

Introduce a new dynamical variable $Q$ and a new time $\tau$ using the transformation

$$Q=qf(t)\,,\qquad dt=m(t)\,f^2(t)\,d\tau,$$

where the function $f(t)$ satisfies the Ermakov equation [16]

$$\frac{d}{dt}m\dot f+m\omega^2 f=\frac{\Omega^2}{f^3}\,,\qquad \Omega\in\mathbb{R}\,,$$

depending on a real constant $\Omega$. A simple calculation shows that the Lagrangian becomes

$$L(Q,Q',\tau)=\frac12\Bigl(Q'^2+\Omega^2Q^2\Bigr)+\frac{d}{d\tau}\left(\frac{f'}{2f}Q^2\right)$$

where the prime denotes the derivative with respect to $\tau$, and $\Omega$ is a new constant frequency. Since the total time derivative in the Lagrangian does not affect the equations of motion, we can drop the second term and obtain a standard equation for the oscillator,

$$Q''+\Omega^2Q=0\,.$$

For this scleronomic conservative system, the energy is constant,

$$H=\frac12\left(Q'^2+\Omega^2Q^2\right)=\frac12\left(m^2\bigl(\dot qf-q\dot f\bigr)^2+\Omega^2q^2f^2\right)$$

This is precisely the Ermakov–Lewis invariant extensively discussed in the literature [13, 24, 25, 26].

In the present note, we consider the time-dependent nonholonomic oscillator which is chosen as a sample toy-model following [1, 9, 20, 28, 27, 30, 19]. Our calculations are quite elementary and do not involve sophisticated algebro-geometric technique of [3, 17, 22, 23].

The paper is structured as follows: in Sec. 2, we discuss the first integrals for scleronomic nonholonomic systems including the oscillator. In Sec. 3, we study the specific case of rheonomic nonholonomic systems describing a rigid body sliding along the stationary blade on a plane. After this, we obtain the equations of motion for the rheonomic nonholonomic oscillator which can be reduced to the well-studied equations of motion for the time-dependent harmonic oscillator. The corresponding first integrals and solutions of equations of motion are discussed.

## 2. SCLERONOMIC NONHOLONOMIC OSCILLATOR

Consider a scleronomic mechanical system in the configuration manifold $Q=\mathbb{R}^3$ defined by the Lagrangian

$$L\left(q_1,q_2,q_3,\dot q_1,\dot q_2,\dot q_3\right)=\frac12\Bigl(\dot q_1^2+\dot q_2^2+\dot q_3^2\Bigr)\,,\tag{2.2}$$

and subjected to the ideal nonholonomic constraint

$$f=\dot q_3-q_2\dot q_1=0\,.\tag{2.3}$$

The equations of motion (1.1) are

$$\begin{cases}\ddot q_1=-\lambda q_2\\ \ddot q_2=0\\ \ddot q_3=\lambda\\ \dot q_3-q_2\dot q_1=0\end{cases}\qquad\Rightarrow\qquad\begin{cases}\ddot q_1=-\lambda q_2\frac{\dot q_1\dot q_2}{1+q_2^2}\\ \ddot q_2=0\\ \ddot q_3=\frac{\dot q_1\dot q_2}{1+q_2^2}\\ \dot q_3-q_2\dot q_1=0\end{cases}\,,\tag{2.4}$$

where the value of the Lagrange multiplier

$$\lambda=\frac{\dot q_1\dot q_2}{1+q_2^2}$$

is computed with the help of the following equation for the constraint:

$$\frac{d}{dt}f = \frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\frac{\partial f}{\partial q_i}\,\dot q_i + \frac{\partial f}{\partial \dot q_i}\,\ddot q_i\right).$$

The equations of motion (2.4) with the initial condition

$$q_1(t=0) = q_{1,0}\,, \qquad q_2(t=0) = q_{2,0}\,, \qquad q_3(t=0) = q_{3,0}$$

and

$$\dot q_1(t=0) = v_1\,, \qquad \dot q_2(t=0) = v_2 \neq 0\,, \qquad \dot q_3(t=0) = v_3$$

have an explicit solution given by

$$\begin{aligned} q_1(t) &= \frac{v_1}{v_2}\sqrt{q_{2,0}^2+1}\,\big(\operatorname{arcsinh}(v_2\,t+q_{2,0}) - \operatorname{arcsinh}(q_{2,0})\big) + q_{1,0}\,,\\ q_2(t) &= v_2\,t + q_{2,0}\,,\\ q_3(t) &= \frac{v_1}{v_2}\sqrt{q_{2,0}^2+1}\,\Big(\sqrt{(v_2\,t+q_{2,0})^2+1} - \sqrt{q_{2,0}+1}\Big) + q_{3,0}\,. \end{aligned}$$

Let us add a potential $V(q_1,q_2,q_3)$ to the kinetic energy (2.2); the Lagrangian now reads as

$$L\,(q_1,q_2,q_3,\dot q_1,\dot q_2,\dot q_3) = \frac{1}{2}\Big(\dot q_1^2+\dot q_2^2+\dot q_3^2\Big) - V(q_1,q_2,q_3)\,. \tag{2.5}$$

In this case, the equations of motion (1.1) are equal to

$$\begin{cases} \ddot q_1 = -\dfrac{\partial V(q_1,q_2,q_3)}{\partial q_1} - \lambda q_2\\ \ddot q_2 = -\dfrac{\partial V(q_1,q_2,q_3)}{\partial q_2}\\ \ddot q_3 = -\dfrac{\partial V(q_1,q_2,q_3)}{\partial q_3} + \lambda\\ \dot q_3 - q_2\dot q_1 = 0 \end{cases} \tag{2.6}$$

where

$$\lambda = \frac{\dot q_1\dot q_2 + \dfrac{\partial V}{\partial q_3} - q_2\dfrac{\partial V}{\partial q_1}}{1+q_2^2}\,.$$

Substituting $\dot q_3 = q_2\dot q_1$ (2.3) into this Lagrangian, one obtains the reduced Lagrangian

$$L_r = \frac{1}{2}\Big((1+q_2^2)\dot q_1^2 + \dot q_2^2\Big) - V(q_1,q_2,q_3)$$

and the Jacobi integral

$$H = \sum_{i=1}^{2}\dot q_i\frac{\partial L_r}{\partial \dot q_i} - L_r = \frac{1}{2}\left(\frac{p_1^2}{1+q_2^2} + p_2^2\right) + V(q_1,q_2,q_3)\,, \tag{2.7}$$

which is a first integral of the equations of motion (2.6)

$$\begin{aligned} \dot q_1 &= \frac{p_1}{1+q_2^2}\,, \qquad \dot q_2 = p_2\,, \qquad \dot q_3 = \frac{q_2p_1}{1+q_2^2}\,,\\ \dot p_1 &= \frac{q_2p_1p_2}{1+q_2^2} - \frac{\partial V(q_1,q_2,q_3)}{\partial q_1} - q_2\frac{\partial V(q_1,q_2,q_3)}{\partial q_3}\,, \qquad \dot p_2 = -\frac{\partial V(q_1,q_2,q_3)}{\partial q_2} \end{aligned} \tag{2.8}$$

on the five-dimensional reduced phase space with the coordinates $(q_1,q_2,q_3,p_1,p_2)$. Here $p_{1,2}$ are the momenta obtained via the Legendre transformation of the reduced Lagrangian

$$p_1 = \frac{\partial L_r}{\partial \dot q_1} = (1+q_2)\dot q_1\,, \qquad p_2 = \frac{\partial L_r}{\partial \dot q_2} = \dot q_2\,.$$

The last Jacobi multiplier for these equations is equal to

$$\mu=\frac{1}{\sqrt{1+q_2^2}}$$

see [20, 27] for details. The corresponding vector field

$$X=(\dot q_1,\dot q_2,\dot q_3,\dot p_1,\dot p_2)$$

is an almost Hamiltonian vector field

$$X=\tilde P dH$$

with respect to the bivector

$$P=\begin{pmatrix} 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & q_2 & 0\\ -1 & 0 & -q_2 & 0 & \dfrac{q_2p_1}{1+q_2^2}\\ 0 & -1 & 0 & -\dfrac{q_2p_1}{1+q_2^2} & 0\end{pmatrix}.$$

We can divide this bivector into two parts and present the vector field $X$ as a linear combination of two Hamiltonian vector fields

$$X(q,p)=\mu P_1 dH+\mu P_2 dH\,, \tag{2.9}$$

where $P_{1,2}$ are two incompatible Poisson bivectors,

$$P_1=\mu^{-1}\begin{pmatrix} 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & \dfrac{q_2p_1}{1+q_2^2}\\ 0 & -1 & 0 & -\dfrac{q_2p_1}{1+q_2^2} & 0\end{pmatrix},\qquad P_2=\mu^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & q_2 & 0\\ 0 & 0 & -q_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

The Schouten brackets for these Poisson bivectors are equal to

$$[\![P_1,P_1]\!]=[\![P_2,P_2]\!]=0\,,\quad\text{and}\qquad [\![P_1,P_2]\!]\neq 0\,.$$

Equation (2.9) guarantees that $H$ is the first integral of the vector field $X$ [4]. Unfortunately, changing the form of equations from (2.6) to (2.8–2.9) does not bring us closer to the explicit solution of these equations.

### *2.1. Polynomial Integrals of Motion*

There is a fundamental theorem due to Noether which shows that, indeed, for every spatial continuous symmetry of a system which can be described by a Lagrangian, some physical quantity is conserved, and the theorem also enables us to find that quantity. The situation is different in the nonholonomic context, but there have been several recent extensions of the Noether theorem to the nonholonomic setting [10, 21].

Nevertheless, for low dimensional systems, the brute force method remains the most productive way of finding the constant of motion by using modern computer technologies [18, 19]. Indeed, substituting the standard polynomial anzats

$$Z=\sum_{k=0}^{n}\sum_{j=0}^{N-k} f_{jk}(q_1,q_2,q_3)p_1^j p_2^{N-k-j} \tag{2.10}$$

into the equation

$$\frac{dZ}{dt}=\sum_{i=1}^{3}\frac{\partial Z}{\partial q_i}\,\dot q_i+\sum_{j=1}^{2}\frac{\partial Z}{\partial p_j}\,\dot p_i=0 \tag{2.11}$$

one obtains a system of equations for the functions $f_{jk}$ and the potential $V$, which are the coefficients of the polynomial (2.11) for $p_1$ and $p_2$ [2, 14]. From the form of (2.8) we see that the even and odd terms in $Z$ are independent. The terms of the order $N+1$ in the momenta in (2.11) form the so-called Killing equation, which is independent of the potential $V$, and therefore it can be solved without any knowledge concerning the potential [15].

Below we set $N = 2$ in (2.10) and consider the following second-order polynomials in the momenta:

$$Z = K_{11}(q_1, q_2, q_3)p_1^2 + 2K_{12}(q_1, q_2, q_3)p_1p_2 + K_{2,2}(q_1, q_2, q_3)p_2^2 + U(q_1, q_2, q_3)\,,$$

where $K_{ij}$ are the entries of the symmetric Killing tensor of valency two. In this case, the Killing equation is

$$\partial_2 K_{22} = 0\,, \qquad 2\Big((1+q_2^2)\partial_2 + q_2\Big)K_{12} + (q_2\partial_3 + \partial_1)K_{22} = 0\,,$$

$$(\partial_1 + \partial_3)K_{11} = 0\,, \qquad \partial_2(1+q_2^2)K_{11} + 2(q_2\partial_3 + \partial_1)K_{12} = 0\,,$$

where $\partial_i = \partial/\partial q_i$. Since our metric in (2.7) is independent of $q_3$, the generic solution $K$ of these equations with the entries

$$K_{11} = F_{11}(q_2, q_3 - q_1q_2)\,, \quad K_{22} = F_{22}(q_1, q_3)$$

and

$$K_{12} = \frac{F_{12}(q_1q_3)}{\sqrt{1+q_2^2}} - \frac{1}{2}\left(\frac{\operatorname{arcsinh} q_2}{\sqrt{1+q_2^2}}\partial_1 - \partial_3\right)F_{22}(q_1, q_3),$$

is labelled by the functions $F_{11}, F_{12}$, and $F_{22}$ in such a way that

$$\partial_2(1+q_2^2)K_{11} + 2(q_2\partial_3 + \partial_1)K_{12} = 0\,.$$

The remaining equations for the potentials have the following form:

$$\phi_1 = (\partial_1 + q_2\partial_3)U(q_1, q_2q_3) - 2(1+q_2^2)\Big(K_{11}(\partial_1 + q_2\partial_3) + K_{12}\partial_2\Big)V(q_1, q_2q_3) = 0$$

and

$$\phi_2 = \partial_2 U(q_1, q_2q_3) - 2\Big(K_{12}(\partial_1 + q_2\partial_3) + K_{22}\partial_2\Big)V(q_1, q_2q_3) = 0\,.$$

If $U$ is independent of the variable $q_3$, then we can readily obtain an equation for the potential $V$,

$$\partial_2\phi_1 - \partial_1\phi_1 = dK(dV) = 0,$$

which is similar to that for the standard Bertrand–Darboux theory for Hamiltonian systems [2, 14, 18, 19]. In order to obtain integrals of motion, we must solve this system of equations with respect to $V$ and $U$ for a given tensor $K$.

### *2.2. First Partial Solution*

If the potential $V(q_1, q_2, q_3)$ does not depend on $q_3$, then the equation

$$\dot{q_3} = \frac{q_2p_1}{1+q_2^2}$$

decouples from the rest of system (2.8),

$$\dot{q_1} = \frac{p_1}{1+q_2^2}\,, \quad \dot{q_2} = p_2\,, \quad \dot{p_1} = \frac{q_2p_1p_2}{1+q_2^2} - \frac{\partial V}{\partial q_1}\,, \quad \dot{p_2} = -\frac{\partial V}{\partial q_2}\,, \tag{2.12}$$

which determines a conformally Hamiltonian system with two degrees of freedom on the plane.

For the conformally Hamiltonian system (2.12), the first integral $F$ is also independent of $q_3$, and the general solution of the corresponding Killing equation

$$K = \begin{pmatrix} \dfrac{c_1\,\operatorname{arcsinh}^2 q_2 + c_2\,\operatorname{arcsinh} q_2 + c_3}{1+q_2^2} & -\dfrac{(2c_1\,q_1 + c_4)\operatorname{arcsinh} q_2 + c_2\,q_1 - 2c_6}{2\sqrt{1+q_2^2}} \\ -\dfrac{(2c_1\,q_1 + c_4)\operatorname{arcsinh} q_2 + c_2\,q_1 - 2c_6}{2\sqrt{1+q_2^2}} & c_1q_1^2 + c_4\,q_1 + c_5 \end{pmatrix} \tag{2.13}$$

depends on six constants of integration $c_1, \ldots, c_6$, similarly to other holonomic [18] and nonholonomic systems [19]. Substituting this general solution into (2.11), one obtains a Bertrand–Darboux type equation $dK(dV) = 0$ or

$$\Big(A\partial_{11} + B\partial_{22} + C\partial_{12} + a\partial_1 + b\,\partial_2\Big)V(q_1, q_2) = 0\,, \tag{2.14}$$

where $\partial_i = \partial/\partial q_i$, $\partial_{ik} = \partial^2/\partial q_i \partial q_k$ and

$$A = 2c_6 - (2c_1 q_1 - c_4)\operatorname{arcsinh} q_2 - c_2 q_1\,, \quad B = -(1+q_2^2)A\,,$$

$$C = -2\sqrt{1+q_2^2}(c_1 \operatorname{arcsinh}^2 q_2 - c_1 q_1^2 + c_2 \operatorname{arcsinh} q_2 - c_4 q_1 + c_3 - c5)$$

$$a = -3(2c_1 \operatorname{arcsinh} q_2 + c_2)\,, \quad b = -q_2 A + 3(2c_1 q_1 + c_4)\sqrt{1+q_2^2}\,.$$

Solving (2.14) for $V(q_1,q_2)$ amounts to finding admissible potentials of the conformally Hamiltonian system (2.12) whose integrability is provided for the existence of the two first integrals and the invariant measure.

If $c_1 = c_2 = c_4 = 0$, then the general solution of (2.14) is equal to

$$V(q_1,q_2) = F_1(\zeta_+) + F_2(\zeta_-)\,, \qquad \zeta_\pm = q_1 \pm \frac{2c_6 \operatorname{arcsinh} q_2}{\sqrt{c_3^2 - 2c_3c_5 + c_5^2 + 4c_6^2} - c_3 + c_5}\,.$$

If $c_6 = 0$ as well, then the general solution of (2.14) has the following form:

$$V(q_1,q_2) = F_1(q_1) + F_2(q_2)\,.$$

This solution can be associated with a Cartesian characteristic coordinate system. For other characteristic coordinates, we have only an integral representation in contrast to the standard polar ($c_1 \neq 0$), parabolic ($c_2 \neq 0$), and elliptic coordinates on the plane ($c_1 \neq 0, c_4 \neq 0$), which can be expressed in terms of elementary functions of the original variables $q_{1,2}$.

### *2.3. Second Partial Solution*

According to [30], if

$$V(q_1,q_2,q_3) = F(q_2) + G(q_1,q_3),$$

then the two equations

$$\dot q_2 = p_2,, \qquad \dot p_2 = -\frac{dF(q_2)}{dq_2}$$

decouple from the rest of system (2.8):

$$\dot q_1 = \frac{p_1}{1+q_2^2}\,, \quad \dot p_1 = \frac{q_2 p_1 p_2}{1+q_2^2} - \frac{\partial G(q_1,q_3)}{\partial q_1}\,, \quad \dot q_3 = \frac{q_2 p_1}{1+q_2^2}\,, \tag{2.15}$$

Subsequently, the Jacobi integral

$$H = \frac{1}{2}\left(\frac{p_1^2}{1+q_2^2} + p_2^2\right) + V(q_1,q_2,q_3) = E$$

is the sum of two integrals of motion

$$\frac{p_2^2}{2} + F(q_2) = E_1\,, \qquad \frac{p_1^2}{2(1+q_2^2)} + G(q_1,q_3) = E_2\,, \qquad E_1 + E_2 = E\,,$$

which are second-order polynomials in momenta. Using the first integral, we obtain a standard Abel's quadrature for the variable $q_2$:

$$\int^{q_2} \frac{dx}{\sqrt{2(E_1 - F(x))}} = t\,.$$

If we substitute the corresponding solutions $q_2 = \phi(t)$ and $p_2 = \psi(t)$ into (2.15), we obtain a three-dimensional rheonomic flow which explicitly depends on time

$$\dot q_1 = \frac{p_1}{1+\phi^2(t)}\,, \quad \dot p_1 = \frac{\phi(t)\psi(t)}{1+\phi^2(t)}\,p_1 - \frac{\partial G(q_1,q_3)}{\partial q_1}\,, \quad \dot q_3 = \frac{\phi(t)}{1+\phi^2(t)}\,p_1\,,$$

with the first integral

$$E_2 = \frac{p_1^2}{2(1+\phi^2 t)} + G(q_1,q_3)\,.$$

This dynamical system belongs to a first family of rheonomic nonholonomic Lagrangian systems with a time-independent Lagrangian and an ideal rheonomic constraint

$$f = \dot q_3 - \phi(t)\dot q_1\,.$$

Similar rheonomic constraints have been considered in [7].

*2.4. Nonholonomic Oscillator*

If we take

$$V(q_1,q_2,q_3)=aq_1^2+bq_2^2+cq_3^2\,,\qquad a,b,c\in\mathbb{R}\,, \tag{2.16}$$

then the general solutions of the equations

$$\dot q_2=p_2,\,,\qquad \dot p_2=-2bq_2$$

are

$$\begin{aligned} q_2(t) &= -\frac{1}{\sqrt{2b}}\Big(c_1\cos(\sqrt{2b}t)-c_2\sin(\sqrt{2b}t)\Big)\\ p_2(t) &= c_1\sin(\sqrt{2b}t)+c_2\cos(\sqrt{2b}t)\,,\qquad c_{1,2}\in\mathbb{R}\,. \end{aligned}$$

where $c_{1,2}$ are constants of integration depending on the initial conditions.

For brevity, we set $b=1/2$, $c_1=0$, and $c_2=1$, then $E_1=1/2$ and

$$q_2(t)=\sin t\,,\qquad p_2(t)=\cos t\,.$$

Substituting this partial solution into the remaining three equations of (2.8), we obtain a rheonomic system of equations

$$\dot p_1=\frac{\sin t\cos t}{1+\sin^2 t}\,p_1-2aq_1-2cq_3\sin t\,,\quad \dot q_1=\frac{p_1}{1+\sin^2 t}\,,\quad \dot q_3=\frac{\sin t}{1+\sin^2 t}\,p_1\,,$$

associated with the time-dependent constraint

$$f=\dot q_3-\sin t\,\dot q_1=0\,.$$

However, since the original Lagrangian is independent of time, and the nonholonomic constraint is ideal, it follows that there is an integral of motion

$$E_2=\frac{p_1^2}{2(1+\sin^2 t)}+aq_1^2+cq_3^2\,,$$

which can be used to reducte this system of equations to one equation. If $c\neq 0$, then we have

$$\begin{aligned} 4cE_2\sin^2 t &= (1+\sin^2 t)^2\left(\frac{d^2q_1}{dt^2}\right)^2+2\Big((1+\sin^2 t)\cos(t)\frac{dq_1}{dt}+2aq_1\Big)\frac{d^2q_1}{dt^2}\\ &+(2c\sin^2 t+\cos^2 t+2c)\sin^2 t\left(\frac{dq_1}{dt}\right)^2+4a\cos t\sin t q_1\frac{dq_1}{dt}+4a(c\sin^2 t+a)q_1^2\,. \end{aligned}$$

If $c=0$, then we have

$$E_2=\frac{(1+\sin^2 t)}{2}\left(\frac{d^2q_1}{dt^2}\right)^2+aq_1\,,$$

or

$$\frac{dq_1}{\sqrt{2(E_2-aq_1^2)}}=d\tau\quad\text{where}\quad d\tau=\frac{dt}{\sqrt{1+\sin^2 t}}\,.$$

If $a=1/2$ and $E_2=1$, then the general solution of this equation is a linear combination

$$q_1(t)=c_1\sin(\tau)+c_2\cos(\tau)\,,$$

where

$$\tau=-\frac{1}{\sqrt{2}}\,\mathbb{F}\Big(\cos t,1/\sqrt{2}\Big)$$

and $\mathbb{F}$ is an incomplete elliptic integral of the first kind.

## 3. RHEONOMIC NONHOLONOMIC OSCILLATOR

Following [30], consider a rigid body sliding along a stationary blade on a plane. Since the velocity of the body is to be directed along the blade, we have a nonholonomic constraint similar to (2.3),

$$\dot q_3 = q_2\dot q_1$$

where $q_2$ and $q_3$ are the coordinates of the rigid body center of mass, and $q_1$ is the angle of rotation of the body in a suitable coordinate system. The kinetic energy of the rigid body is

$$T=\frac{1}{2}\left(\rho^2\dot q_1^2+\dot q_2^2+\dot q_2^2\right)\,.$$

Assume that the body has the unit mass $m = 1$, $\rho$ is a central moment of inertia, and there is some potential force acting on the body, i.e., the Lagrangian is

$$L=\frac{1}{2}\left(\rho^2\dot q_1^2+\dot q_2^2+\dot q_2^2\right)-V(q_1,q_2,q_3)$$

If we also suppose that the central moment of inertia is a function of time $\rho = \rho(t)$, then the equations of motion (1.1) are

$$\begin{cases} \ddot q_1 = -\dfrac{\dot q_1\dfrac{d\rho(t)^2}{dt}-\dfrac{\partial V}{\partial q_1}-\lambda q_2}{\rho(t)^2} \\[2ex] \ddot q_2 = -\dfrac{\partial V}{\partial q_2} \\[2ex] \ddot q_3 = -\dfrac{\partial V}{\partial q_3}+\lambda \\[2ex] \dot q_3 - q_2\dot q_1 = 0 \end{cases} \tag{3.17}$$

where

$$\lambda=\frac{\dot q_1\dot q_2\rho^2(t)-q_2\dot q_1\dfrac{\rho(t)^2}{dt}+\rho(t)^2\dfrac{\partial V}{\partial q_3}-q_2\dfrac{\partial V}{\partial q_1}}{\rho(t)^2+q_2^2}\,.$$

Substituting $\dot q_3 = q_2\dot q_1$ (2.3) into this Lagrangian, we determine the reduced Lagrangian

$$L_r=\frac{1}{2}\Big((\rho^2(t)+q_2^2)\dot q_1^2+\dot q_2^2\Big)-V(q_1,q_2,q_3)$$

and the variables

$$p_1=\frac{\partial L_r}{\partial\dot q_1}=\big(\rho^2(t)+q_2^2\big)\dot q_1\,,\qquad p_2=\frac{\partial L_r}{\partial \dot q_2}=\dot q_2\,.$$

Equations of motion in these variables have the following form

$$\begin{aligned} \dot q_1 &= \frac{p_1}{\rho^2(t)+q_2^2}\,,\qquad \dot q_2=p_2\,,\qquad \dot q_3=\frac{q_2p_1}{\rho^2(t)+q_2^2}\,,\\ \dot p_1 &= \frac{q_2p_1p_2}{\rho^2(t)+q_2^2}-\frac{\partial V(q_1,q_2,q_3)}{\partial q_1}-q_2\frac{\partial V(q_1,q_2,q_3)}{\partial q_3}\,,\qquad \dot p_2=-\frac{\partial V(q_1,q_2,q_3)}{\partial q_2} \end{aligned} \tag{3.18}$$

The corresponding time-dependent vector field is a linear combination of time-dependent Hamiltonian vector fields similar to (2.9)

$$X(q,p,t)=\mu(t)P_1(t)dH(t)+\mu(t)P_2dH(t)\,,$$

where

$$P_1(t)=\mu^{-1}(t)\begin{pmatrix} 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & \dfrac{q_2p_1}{\rho^2(t)+q_2^2}\\ 0 & -1 & 0 & -\dfrac{q_2p_1}{\rho^2(t)+q_2^2} & 0 \end{pmatrix},$$

and

$$P_2=\mu^{-1}(t)\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&q_2&0\\0&0&-q_2&0&0\\0&0&0&0&0\end{pmatrix},\qquad \mu(t)=\frac{1}{\sqrt{\rho^2(t)+q_2^2}}\,.$$

In this case, the Poisson bivectors $P_{1,2}(t)$ depend on time similarly to the Jacobi integral

$$H=\sum_{i=1}^{2}\dot q_i\frac{\partial L_r}{\partial\dot q_i}-L_r=\frac12\left(\frac{p_1^2}{\rho^2(t)+q_2^2}+p_2^2\right)+V(q_1,q_2,q_3)\,,$$

and thus

$$\frac{d}{dt}H=-\frac{\rho(t)\dot\rho(t)}{(\rho^2(t)+q_2^2)^2}\,p_1^2\,.$$

As above, changing the form of equations from (3.17) to (3.18) does not bring us closer to an explicit solution of these equations.

The first integrals for this rheonomic system can be obtained by using the extended Noether's theorem, see [11, 31] and the references therein. The same invariants can be also obtained by using the polynomial anzats

$$Z(t)=\sum_{k=0}^{n}\sum_{j=0}^{N-k}f_{jk}(q_1,q_2,q_3,t)p_1^jp_2^{N-k-j}$$

with the coefficients $f_{jk}(q_1,q_2,q_3,t)$ depending on time, and thus

$$\frac{dZ}{dt}=\frac{\partial Z}{\partial t}+\sum_{i=1}^{3}\frac{\partial Z}{\partial q_i}\,\dot q_i+\sum_{j=1}^{2}\frac{\partial Z}{\partial p_j}\,\dot p_i=0$$

instead of (2.11). Solutions of there time-dependent equations were discussed in [13].

In this note, we prefer to reduce the equations of motion (3.17) to well-studied equations of motion for the time-dependent oscillator in order to discuss the form of their solutions.

### *3.1. Nonholonomic Time-Dependent Oscillator*

Write

$$V(q_1,q_2,q_3)=aq_1^2+bq_2^2+cq_3^2\,,\qquad a,b,c\in\mathbb{R}\,,$$

and substitute the partial solutions

$$q_2(t)=\sin t\,,\qquad p_2(t)=\cos t\,,$$

of equations (3.17) for $b=1/2$ into the Lagrangian

$$L_r=\frac{\rho^2(t)+\sin^2 t}{2}\,\dot q_1^2-aq_1^2-cq_3^2\,.$$

and the constraint

$$\dot q_3=\frac{\sin t}{\rho^2(t)+\sin^2 t}\,p_1\,.$$

If $c=0$, then the Lagrangian $L_r$ coincides with the Lagrangian of the time-dependent oscillator

$$L(q_1,\dot q_1,t)=\frac{1}{2m(t)}\Big(\dot q_1^2-\omega^2(t)q_1^2\Big)$$

with

$$m(t)=\frac{1}{\rho^2(t)+\sin^2 t}\quad\text{and}\quad \omega^2(t)=2am(t)\,.$$

This enables us to apply all the known machinery of [13, 24, 25, 26] to the solution of this system. For instance, we can say that the general solution has the following "explicit" form

$$q_1(t)=f^{-1}(t)\Big(c_1\sin\Omega\tau+c_2\cos\Omega\tau)\,,\qquad dt=m(t)\,f^2(t)\,d\tau\,,$$

where the function $f(t)$ satisfies the Ermakov–Milne–Pinney-type equation

$$\frac{d}{dt}\, m\dot{f} + m\omega^2 f = \frac{\Omega^2}{f^3}\,, \qquad \Omega \in \mathbb{R}\,,$$

depending on a real constant $\Omega$. The meaning of such a solution is discussed in detail in the theory of the time-dependent holonomic oscillator.

## 4. CONCLUSION

In this note, we compare explicit solutions of equations of motion for holonomic, nonholonomic, and time-dependent nonholonomic oscillators. In the holonomic case, the general solution is a well-known combination of trigonometric functions

$$q(t) = c_1 \sin \Omega t + c_2 \cos \Omega t\,.$$

For nonholonomic scleronomic oscillator, the solutions have the similar form,

$$q(t) = c_1 \sin \Omega\tau + c_2 \cos \Omega\tau\,,$$

where $\tau$ is an elliptic function of the original time variable $t$.
For nonholonomic rheonomic oscillator, the solutions are

$$q(t) = c_1(t) \sin \Omega\tau + c_2(t) \cos \Omega\tau\,,$$

where the functions $c_1(t)$ and $\tau(t)$ and the constant $\Omega$ are related via Ermakov's equation.

There are some open problems which have popped up in this note. Firstly, it would be useful to study the rheonomic or time-dependent nonholonomic systems with nonideal constraints similar to [8]. Secondly, we have considered the rheonomic nonholonomic system with the Lewis–Ermakov invariant, and hence it would be great to compare such nonholonomic systems with nonholonomic systems under time-dependent control [5, 6].

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