

Index of Elliptic Boundary Value Problems Associated with Isometric Group Actions

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Abstract. Given a compact manifold with boundary, endowed with an isometric action of a discrete group of polynomial growth, we state an index theorem for elliptic elements in the algebra of nonlocal operators generated by the Boutet de Monvel algebra of pseudodifferential boundary value problems on the manifold and the shift operators associated with the group action.

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1. INTRODUCTION

Let Γ be a discrete group of diffeomorphisms of a smooth manifold M . We consider the class of operators generated by pseudodifferential operators on M and shift operators $T_\gamma u(x) = u(\gamma^{-1}(x))$ for all $\gamma \in \Gamma$ and $u \in C^\infty(M)$. The Fredholm property for operators in this class is known in a quite general situation (see [1, 2]). However, the index problem was studied in the case of manifolds without boundary only (see [3, 4, 5, 6, 7] and the references cited there).

In this paper, we consider a compact smooth manifold with boundary, endowed with an isometric action of a discrete group of polynomial growth in the sense of Gromov [8]. In this geometric situation, we state an index theorem for elliptic elements in the algebra generated by the Boutet de Monvel pseudodifferential boundary value problems [9] on the manifold and the shift operators associated with the group action. Our index formula gives, as special cases, the index formula for elliptic elements in the Boutet de Monvel algebra [10] (see also [11]) and the index formula for elliptic operators associated with isometric group actions on closed manifolds [4].

2. BOUTET DE MONVEL ALGEBRA

Let M be a compact smooth manifold with boundary denoted by X . Suppose that M is endowed with a Riemannian metric and consider the induced Riemannian metric on X . The local coordinates on M and X are denoted by x and x' , respectively. In addition, in a neighborhood of the boundary, we use coordinates $x = (x', x_n)$, $x_n \geq 0$, such that the boundary has the equation $x_n = 0$, while x_n is equal to the distance to the boundary. The dual coordinates in T^*M are denoted by $\xi = (\xi', \xi_n)$.

We consider the Boutet de Monvel operators of order and type equal to zero. We refer the reader to [9, 12, 13, 14] for a complete exposition of the Boutet de Monvel algebra and recall here only several facts about this algebra, which are used below.

The Boutet de Monvel operators of order and type equal to zero define continuous mappings of the form

$$\mathcal{D} = \begin{pmatrix} A + G & C \\ B & A_X \end{pmatrix} : \begin{matrix} L^2(M) \\ \oplus \\ L^2(X) \end{matrix} \longrightarrow \begin{matrix} L^2(M) \\ \oplus \\ L^2(X) \end{matrix} \quad (2.1)$$

Here A is a classical pseudodifferential operator of order zero on M , and the complete symbol of A satisfies the so-called transmission property; A_X is a classical pseudodifferential operator of order zero on X ; B, C , and G are the boundary (or trace), coboundary (or potential), and Green operators, respectively.

The *symbol* of the operator (2.1) is a pair $\sigma(\mathcal{D}) = (\sigma_M(\mathcal{D}), \sigma_X(\mathcal{D}))$. Here the first component is called the *interior symbol* and is a function

$$\sigma_M(\mathcal{D}) = \sigma(A) \in C^\infty(T_0^*M)$$

homogeneous on the cotangent bundle T_0^*M with zero section deleted and equal to the principal symbol of A . The second component is called the *boundary symbol* and is an operator function

$$\sigma_X(\mathcal{D}) \in C^\infty(T_0^*X, \mathcal{B}(\overline{H}_+ \oplus \mathbb{C})) \simeq C^\infty(T_0^*X, \mathcal{B}(L^2(\mathbb{R}_+) \oplus \mathbb{C})).$$

Here $\overline{H}_+ \subset L^2(\mathbb{R}_{\xi_n})$ is the Fourier image of the subspace $L^2(\mathbb{R}_+) \subset L^2(\mathbb{R}_{x_n})$ of functions vanishing for $x_n \leq 0$. The boundary symbol is twisted homogeneous in the following sense

$$\sigma_X(\mathcal{D})(x', \lambda\xi') = \varkappa_\lambda^{-1} \sigma_X(\mathcal{D})(x', \xi') \varkappa_\lambda \quad \text{for all } (x', \xi') \in T_0^*X, \lambda > 0 \tag{2.2}$$

with respect to the unitary representation of \mathbb{R}_+ on $\overline{H}_+ \oplus \mathbb{C}$: $\varkappa_\lambda(u(\xi_n), v) = (\lambda^{1/2}u(\lambda\xi_n), v)$.

Denote the algebra of boundary symbols by Σ_X . Let us describe the elements of this algebra. To this end, let $\theta_\pm(x_n)$ be the functions on \mathbb{R} equal to 1 for $x_n \in \mathbb{R}_\pm$ and to zero otherwise. Consider the Fréchet spaces $H_\pm = \mathcal{F}_{x_n \rightarrow \xi_n}(\theta_\pm(x_n)S(\mathbb{R}))$, where $S(\mathbb{R})$ is the Schwartz space on \mathbb{R} . We also define the projection $\Pi_\pm : H_+ \oplus H_- \rightarrow H_\pm$ and the continuous functional

$$\begin{aligned} \Pi' : H_+ \oplus H_- &\longrightarrow \mathbb{C} \\ u(\xi_n) &\longmapsto \lim_{x_n \rightarrow 0^+} \mathcal{F}_{\xi_n \rightarrow x_n}^{-1}(u(\xi_n)). \end{aligned}$$

Let us now describe the elements of Σ_X . Consider the smooth functions

- $b(x', \xi', \xi_n) \in C^\infty(T_0^*X, H_-)$; $c(x', \xi', \xi_n) \in C^\infty(T_0^*X, H_+)$;
- $g(x', \xi', \xi_n, \eta_n) \in C^\infty(T_0^*X, H_+ \otimes H_-)$ (here we consider the topological tensor product of locally convex linear topological spaces H_\pm);
- $q(x', \xi') \in C^\infty(T_0^*X)$.

We use these functions to define the family of operators

$$a_X(x', \xi') = \begin{pmatrix} \Pi_+ a(x', 0, \xi', \xi_n) + \Pi'_{\eta_n} g(x', \xi', \xi_n, \eta_n) & c(x', \xi', \xi_n) \\ \Pi'_{\xi_n} b(x', \xi', \xi_n) & q(x', \xi') \end{pmatrix} : \begin{matrix} \overline{H}_+ \\ \oplus \\ \mathbb{C} \end{matrix} \longrightarrow \begin{matrix} \overline{H}_+ \\ \oplus \\ \mathbb{C} \end{matrix} \tag{2.3}$$

parametrized by $(x', \xi') \in T_0^*X$. Here $a(x', 0, \xi', \xi_n)$ is the restriction of a zero-order symbol on M with the transmission property to the boundary; it is called the *principal symbol* of a_X . Suppose that the functions b, c, g, q in (2.3) are chosen in such a way that a_X is twisted-homogeneous (see (2.2)). Then $a_X \in \Sigma_X$, and all elements in Σ_X can be written as in (2.3).

3. Γ -BOUTET DE MONVEL OPERATORS. THE FREDHOLM PROPERTY

Let Γ be a discrete group of isometries of M . Suppose that the boundary is Γ -invariant. Given a $\gamma \in \Gamma$, we define the shift operator

$$T_\gamma : L^2(M) \oplus L^2(X) \longrightarrow L^2(M) \oplus L^2(X), \quad (u(x), v(x')) \longmapsto (u(\gamma^{-1}(x)), v(\gamma^{-1}(x'))).$$

The mapping $\gamma \mapsto T_\gamma$ defines a representation of Γ on $L^2(M) \oplus L^2(X)$.

A Γ -Boutet de Monvel operator is an operator equal to the sum

$$\mathcal{D} = \sum_{\gamma \in \Gamma} \mathcal{D}_\gamma T_\gamma : L^2(M) \oplus L^2(X) \longrightarrow L^2(M) \oplus L^2(X), \tag{3.4}$$

where $\{\mathcal{D}_\gamma\}_{\gamma \in \Gamma}$ are Boutet de Monvel operators. We suppose that the sum in (3.4) is finite, i.e., only finitely many \mathcal{D}_γ 's are nonzero.

Below, Γ -Boutet de Monvel operators are referred to as Γ -operators for short. One can show that, given a Boutet de Monvel operator \mathcal{D} and a $\gamma \in \Gamma$, the composition $T_\gamma \mathcal{D} T_\gamma^{-1}$ is also a Boutet de Monvel operator. This implies that the operators (3.4) form an algebra. Moreover, the interior and boundary symbols of $T_\gamma \mathcal{D} T_\gamma^{-1}$ are equal to

$$\sigma_M(T_\gamma \mathcal{D} T_\gamma^{-1})(x, \xi) = \sigma_M(\mathcal{D})(\partial\gamma^{-1}(x, \xi)), \quad \sigma_X(T_\gamma \mathcal{D} T_\gamma^{-1})(x', \xi') = \sigma_X(\mathcal{D})(\partial\gamma^{-1}(x', \xi')).$$

Here the action of Γ on M and X is lifted to the bundles T^*M and T^*X by the codifferentials $\partial\gamma = (d\gamma^t)^{-1}$ of the corresponding diffeomorphisms.

Consider smooth crossed products $C^\infty(T_0^*M) \rtimes \Gamma$ and $\Sigma_X \rtimes \Gamma$, in the sense of [15], of algebras of interior and boundary symbols with Γ acting on these algebras by automorphisms. Recall that the smooth crossed product $\mathcal{A} \rtimes \Gamma$ of a Fréchet algebra \mathcal{A} with the seminorms $\|\cdot\|_m, m > 0$, and a group Γ of polynomial growth

acting on \mathcal{A} by automorphisms $a \mapsto \gamma(a)$ for all $a \in \mathcal{A}$ and $\gamma \in \Gamma$ is equal to the vector space of functions $f : \Gamma \rightarrow \mathcal{A}$ which decay rapidly at infinity, which means that the following estimates hold:

$$\|f(\gamma)\|_m \leq C_N(1 + |\gamma|)^{-N}$$

for all $N, m > 0$ and $\gamma \in \Gamma$, where the constant C_N does not depend on γ . Here $|\gamma|$ is the length of γ in the word metric on Γ . Finally, the action of Γ on \mathcal{A} is required to be tempered: for any m there exists k and a real polynomial $P(z)$ such that $\|\gamma(a)\|_m \leq P(|\gamma|)\|a\|_k$ for all a and γ . The product in $\mathcal{A} \rtimes \Gamma$ is defined by the formula

$$\{f_1(\gamma)\} \cdot \{f_2(\gamma)\} = \left\{ \sum_{\gamma_1 \gamma_2 = \gamma} f_1(\gamma_1) \gamma_1(f_2(\gamma_2)) \right\}.$$

The symbol of a Γ -operator (3.4) is the pair $\sigma(\mathcal{D}) = (\sigma_M(\mathcal{D}), \sigma_X(\mathcal{D}))$ consisting of the interior and the boundary symbols

$$\sigma_M(\mathcal{D}) = \{\sigma(A_\gamma)\}_{\gamma \in \Gamma} \in C^\infty(T_0^*M) \rtimes \Gamma, \quad \sigma_X(\mathcal{D}) = \{\sigma_X(\mathcal{D}_\gamma)\}_{\gamma \in \Gamma} \in \Sigma_X \rtimes \Gamma. \tag{3.5}$$

A Γ -operator is *elliptic* if its interior and boundary symbols are invertible in the corresponding crossed products (3.5).

Theorem 1. *Any elliptic G -operator has the Fredholm property.*

The proof is standard (e.g., see [1]). More precisely, if \mathcal{D} is elliptic, then its symbol is invertible. Denote the inverse symbol by

$$(\{b_{\gamma,M}\}_{\gamma \in \Gamma}, \{b_{\gamma,X}\}_{\gamma \in \Gamma}) \in (C^\infty(T_0^*M) \oplus \Sigma_X) \rtimes \Gamma.$$

Then, given $\gamma \in \Gamma$, we choose a Boutet de Monvel operator B_γ with the symbol $(b_{\gamma,M}, b_{\gamma,X})$. Finally, we define the operator (one can choose B_γ such that the series is norm convergent)

$$\mathcal{B} = \sum_{\gamma \in \Gamma} B_\gamma T_\gamma.$$

A direct computation shows that \mathcal{B} is a regularizer for \mathcal{D} modulo compact operators.

4. CHERN CHARACTER FOR ELLIPTIC Γ -OPERATORS

The aim of this paper is to obtain an index formula for elliptic Γ -operators. To state our cohomological index formula, we introduce two cohomology groups for T^*M , which are important for our index formula. To this end, note that T^*M is a manifold with boundary $\partial T^*M \simeq T^*X \times \mathbb{R}$. Denote the embedding $\partial T^*M \subset T^*M$ by i and the projection $\partial T^*M \rightarrow T^*X$ by π . Consider the complex

$$\Omega^j(T^*M, \pi) = \Omega^j(T^*M) \oplus \Omega^{j-2}(T^*X), \quad \partial = \begin{pmatrix} (-1)^j d & 0 \\ \pi_* i^* & (-1)^{j+1} d \end{pmatrix} \tag{4.6}$$

of compactly supported differential forms, where $\pi_* : \Omega^*(\partial T^*M) \rightarrow \Omega^{*-1}(T^*X)$ stands for the integration along the fibers of π . The cohomology of the complex $(\Omega^j(T^*M, \pi), \partial)$ is denoted by $H^*(T^*M, \pi)$.

Consider also the complex

$$\tilde{\Omega}^j(T^*M, \pi) = \{(\omega, \omega_X) \in \Omega^j(T^*M) \oplus \Omega^j(T^*X) \mid i^* \omega = \pi^* \omega_X\}, \quad \tilde{\partial} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}. \tag{4.7}$$

Denote its cohomology by $\tilde{H}^*(T^*M, \pi)$.

The componentwise products of differential forms give us the product

$$\wedge : \Omega^j(T^*M, \pi) \times \tilde{\Omega}^k(T^*M, \pi) \longrightarrow \Omega^{j+k}(T^*M, \pi).$$

The following Leibniz rule: $\partial(a \wedge b) = \partial a \wedge b + (-1)^j a \wedge \tilde{\partial} b$, $a \in \Omega^j(M, \pi), b \in \tilde{\Omega}^k(M, \pi)$ implies that the product \wedge defines a product in the cohomology

$$\wedge : H^j(T^*M, \pi) \times \tilde{H}^k(T^*M, \pi) \longrightarrow H^{j+k}(T^*M, \pi).$$

Finally, since T^*M and T^*X are oriented, we define the integration mapping

$$\begin{aligned} \langle \cdot, [T^*M, \pi] \rangle : H^*(T^*M, \pi) &\longrightarrow \mathbb{C} \\ (\omega, \omega_X) &\longrightarrow \int_{T^*M} \omega + \int_{T^*X} \omega_X. \end{aligned}$$

Denote by $\tilde{\Sigma}_X$ the algebra of boundary symbols, which are defined on T^*X and twisted homogeneous for large $|\xi'|$. Consider the actions of Γ on the algebras $\Omega(T^*M), \tilde{\Sigma}_X \otimes_{C^\infty(X)} \Omega(T^*X)$ of compactly supported differential forms and the corresponding smooth crossed products,

$$\Omega(T^*M) \rtimes \Gamma, \quad \left(\tilde{\Sigma}_X \otimes_{C^\infty(X)} \Omega(T^*X) \right) \rtimes \Gamma.$$

These crossed products are differential graded algebras.

Given $\gamma \in \Gamma$, let us define mappings (cf. [4])

$$\tau^\gamma : \Omega(T^*M) \rtimes \Gamma \longrightarrow \Omega(T^*M^\gamma), \tag{4.8}$$

$$\tau_X^\gamma : \left(\tilde{\Sigma}_X \otimes_{C^\infty(X)} \Omega(T^*X) \right) \rtimes \Gamma \longrightarrow \Omega(T^*X^\gamma). \tag{4.9}$$

To define these mappings, we introduce some notation. Denote by $\bar{\Gamma}$ the closure of Γ in the compact Lie group of all isometries of M . This closure is a compact Lie group. Let $C^\gamma \subset \bar{\Gamma}$ be the centralizer of γ . The centralizer is a closed Lie subgroup of $\bar{\Gamma}$. The elements of the centralizer are denoted by h , while the induced Haar measure on the centralizer is denoted by dh . Below, given $\gamma' \in \langle \gamma \rangle$ (here $\langle \gamma \rangle \subset \Gamma$ stands for the conjugacy class of γ), we choose an arbitrary element $z = z(\gamma, \gamma')$ which conjugates γ and $\gamma' = z\gamma z^{-1}$. Any such element defines a diffeomorphism $\partial z : T^*M^\gamma \rightarrow T^*M^{\gamma'}$ of the corresponding fixed point sets.

We define the functional (4.8) by

$$\tau^\gamma(\omega) = \sum_{\gamma' \in \langle \gamma \rangle} \int_{C^\gamma} h^*(z^*\omega(\gamma')) \Big|_{T^*M^\gamma} dh, \quad \text{where } \omega \in \Omega(T^*M) \rtimes \Gamma, \tag{4.10}$$

while the functional (4.9) by

$$\tau_X^\gamma(\omega_X) = \sum_{\gamma' \in \langle \gamma \rangle} \int_{C^\gamma} \text{tr}_X h^*(z^*\omega_X(\gamma')) \Big|_{T^*X^\gamma} dh, \quad \text{where } \omega_X \in \left(\tilde{\Sigma}_X \otimes_{C^\infty(X)} \Omega(T^*X) \right) \rtimes \Gamma. \tag{4.11}$$

Here

$$\text{tr}_X \left(\sum_I \omega_I(z) dz^I \right) = \sum_I \text{tr}'(\omega_I(z)) dz^I,$$

where the regularized trace $\text{tr}' : \tilde{\Sigma}_X \rightarrow C^\infty(T^*X)$ is that of Fedosov [10]

$$\text{tr}' \begin{pmatrix} \Pi_+ a + \Pi' g & c \\ \Pi' b & q \end{pmatrix} = \Pi'_{\xi_n} g(x', \xi', \xi_n, \xi_n) + q(x', \xi').$$

One can show that the expressions (4.10) and (4.11) do not depend on the choice of z .

Let now \mathcal{D} be an $N \times N$ matrix elliptic Γ -operator. Then its interior and boundary symbols are invertible elements in the corresponding crossed products, and we denote the inverse symbols by

$$\sigma_M(\mathcal{D})^{-1} \in C^\infty(T_0^*M, \text{Mat}_N) \rtimes \Gamma, \quad \sigma_X(\mathcal{D})^{-1} \in (\Sigma_X \otimes \text{Mat}_N) \rtimes \Gamma.$$

Extend $\sigma_M(\mathcal{D})^{\pm 1}$ to T^*M up to smooth symbols satisfying the transmission property and homogeneous at infinity, and extend $\sigma_X(\mathcal{D})^{\pm 1}$ to T^*X as smooth symbols that are twisted homogeneous at infinity. Denote these extensions by

$$a, r \in C^\infty(T^*M, \text{Mat}_N) \rtimes \Gamma, \quad a_X, r_X \in \left(\tilde{\Sigma}_X \otimes \text{Mat}_N \otimes_{C^\infty(X)} \Omega(T^*X) \right) \rtimes \Gamma.$$

Suppose that these extensions are compatible, i.e., the symbol of the boundary symbol is equal to the restriction of the interior symbol to the boundary.

Define noncommutative connections

$$\nabla_M = d + rda\wedge, \quad \nabla_X = d + r_X da_X \wedge$$

in the trivial bundles over T^*M and T^*X . Their curvature forms are equal to

$$\Omega_M = \nabla_M^2 = dr \wedge da + (rda)^2, \quad \Omega_X = \nabla_X^2 = dr_X \wedge da_X + (r_X da_X)^2.$$

Define compactly supported differential forms

$$\text{ch}_{T^*M}^\gamma \sigma(\mathcal{D}) \in \Omega^{ev}(T^*M^\gamma), \quad \text{ch}_{T^*X}^\gamma \sigma_X(\mathcal{D}) \in \Omega^{ev}(T^*X^\gamma)$$

(here X^γ is the boundary of M^γ) on the cotangent bundles of the fixed point submanifolds by

$$\begin{aligned} \text{ch}_{T^*M}^\gamma \sigma(\mathcal{D}) &= \tau^\gamma \left(\exp \left(-\frac{\Omega_M}{2\pi i} \right) (1_N - ra) \right) - \tau^\gamma \left(1_N - a \exp \left(-\frac{\Omega_M}{2\pi i} \right) r \right) \\ \text{ch}_{T^*X}^\gamma \sigma(\mathcal{D}) &= \tau_X^\gamma \left(\exp \left(-\frac{\Omega_X}{2\pi i} \right) (1_N - r_X a_X) \right) - \tau_X^\gamma \left(1_N - a_X \exp \left(-\frac{\Omega_X}{2\pi i} \right) r_X \right). \end{aligned}$$

The boundary $\partial(T^*M^\gamma) \simeq T^*X^\gamma \times \mathbb{R}$ is fibered over T^*X^γ with fiber \mathbb{R} . Denote the corresponding projection by $\pi^\gamma : \partial(T^*M^\gamma) \rightarrow T^*X^\gamma$ and the embedding $\partial(T^*M^\gamma) \subset T^*M^\gamma$ by i_γ .

Proposition 1. *Given $\gamma \in \Gamma$, the pair $(\text{ch}_{T^*M}^\gamma \sigma(\mathcal{D}), \text{ch}_{T^*X}^\gamma \sigma_X(\mathcal{D}))$ enjoys the properties*

$$d(\text{ch}_{T^*M}^\gamma \sigma(\mathcal{D})) = 0, \quad d(\text{ch}_{T^*X}^\gamma \sigma(\mathcal{D})) = \pi_*^\gamma i_\gamma^* (\text{ch}_{T^*M}^\gamma \sigma(\mathcal{D})),$$

i.e., it is closed in the complex $(\Omega^(T^*M^\gamma, \pi^\gamma), \partial)$, see (4.6), and its cohomology class, denoted by*

$$\text{ch}^\gamma \sigma(\mathcal{D}) \in H^{ev}(T^*M^\gamma, \pi^\gamma),$$

does not depend on the choice of a, r, a_X, r_X and does not change under homotopies of elliptic symbols.

5. INDEX FORMULA

To state the index theorem, we define the necessary equivariant characteristic classes. First, we define the Todd form on M^γ :

$$\text{Td}(T^*M^\gamma \otimes \mathbb{C}) = \det \left(\frac{-\Omega^\gamma/2\pi i}{1 - \exp(\Omega^\gamma/2\pi i)} \right),$$

where Ω^γ is the curvature form of the Levi-Civita connection on M^γ . One similarly defines the Todd form $\text{Td}(T^*X^\gamma \otimes \mathbb{C})$ on X^γ . The pair of these forms is closed in the complex $(\tilde{\Omega}^*(M^\gamma, \pi^\gamma), \tilde{\partial})$, see (4.7), and its cohomology class is denoted by

$$\text{Td}^\gamma(T^*M \otimes \mathbb{C}) \in \tilde{H}^{ev}(M^\gamma, \pi^\gamma).$$

Second, let N^γ be the normal bundle of $M^\gamma \subset M$. Then we have a natural action of γ on N^γ , and the following differential form on M^γ is defined:

$$\text{ch}^\gamma \Lambda(N^\gamma \otimes \mathbb{C}) = \text{tr}_{\Lambda^{ev}(N^\gamma)} (\gamma \exp(-\Omega/2\pi i)) - \text{tr}_{\Lambda^{odd}(N^\gamma)} (\gamma \exp(-\Omega/2\pi i)),$$

where Ω stands for the curvature form of the exterior bundle $\Lambda(N^\gamma)$, and γ is regarded as an endomorphism of the subbundles $\Lambda^{ev/odd}(N^\gamma)$ of even/odd forms and $\text{tr}_{\Lambda^{ev/odd}(N^\gamma)}$ is the fiberwise trace functional of endomorphisms of $\Lambda^{ev/odd}(N^\gamma)$. Similarly, let N_X^γ be the normal bundle of $X^\gamma \subset X$; then we define the form $\text{ch}^\gamma \Lambda(N_X^\gamma \otimes \mathbb{C})$ on X^γ . The pair $(\text{ch}^\gamma \Lambda(N^\gamma \otimes \mathbb{C}), \text{ch}^\gamma \Lambda(N_X^\gamma \otimes \mathbb{C}))$ is closed in the complex (4.7). Denote its cohomology class by

$$\text{ch}^\gamma \Lambda(N^\gamma \otimes \mathbb{C}) \in \tilde{H}^{ev}(M^\gamma, \pi^\gamma).$$

The last class is invertible, since its zero degree component is a nonzero complex number (see [16] or [4] for a proof).

Theorem 2. *Given an elliptic Γ -operator \mathcal{D} , its Fredholm index is equal to*

$$\text{ind} \mathcal{D} = \sum_{\langle \gamma \rangle \in \Gamma} \langle \text{ch}^\gamma \sigma(\mathcal{D}) \wedge \text{Td}^\gamma(T^*M \otimes \mathbb{C}) \wedge (\text{ch}^\gamma \Lambda(N^\gamma \otimes \mathbb{C}))^{-1}, [T^*M^\gamma, \pi^\gamma] \rangle,$$

where the summation ranges over all conjugacy classes in Γ .

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