ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 27, No. 1, 2020, pp. 48–60. © Pleiades Publishing, Ltd., 2020.

On Operators with Closed Range and Semi-Fredholm Operators Over W^*-Algebras

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Received August 3, 2019; August 24, 2019; August 30, 2019

Abstract. In this paper, we consider $\mathcal{A}$-Fredholm and semi-$\mathcal{A}$-Fredholm operators on Hilbert C^*-modules over a W^*-algebra $\mathcal{A}$ defined in [3] and [9]. Using the assumption that $\mathcal{A}$ is a W^*-algebra (rather than an arbitrary C^*-algebra), we obtain a generalization of Schechter–Lebow characterization of semi-Fredholm operators and a generalization of the "punctured neighborhood" theorem, as well as some other results generalizing their classical counterparts. We consider both adjointable and nonadjointable semi-Fredholm operators over W^*-algebras. Moreover, we also work with general bounded adjointable operators with closed ranges over C^*-algebras and prove a generalization of a Bouldin result for Hilbert spaces to Hilbert C^*-modules.

DOI 10.1134/S1061920820010057

1. INTRODUCTION

Fredholm theory on Hilbert C*-modules, as a generalization of Fredholm theory on Hilbert spaces, was initiated by Mishchenko and Fomenko in [9]. They have elaborated the notion of Fredholm operator on the standard module $H_{\mathcal{A}}$ and proved a generalization of the Atkinson theorem. In [3], we went further in this direction and defined semi-Fredholm operators on Hilbert C*-modules. We then proved several properties of these generalized semi-Fredholm operators on Hilbert C*-modules as an analog or generalization of the well-known properties of classical semi-Fredholm operators on Hilbert and Banach spaces. Several special properties of $\mathcal{A}$-Fredholm operators in the case of W^*-algebra were described in [10, Sec. 3.6]. The idea in the present paper is to go further in this direction and establish more special properties of $\mathcal{A}$-Fredholm operators defined in [9] and of semi-$\mathcal{A}$-Fredholm operators defined in [3] for the case in which $\mathcal{A}$ is a W^*-algebra; these properties are more closely related to those of the classical semi-Fredholm operators on Hilbert spaces than those in the general case in which $\mathcal{A}$ is an arbitrary C^*-algebra. Moreover, we consider here both adjointable and nonadjointable semi-Fredholm operators over W^*-algebras.

Let us list our main results. Proposition 3.19 and Lemma 3.20 generalize a part of the index theorem which states that if F and D are Fredholm operators on a Hilbert space H, then $\dim\ker FD \leqslant \dim\ker F + \dim\ker D$ and $\dim \operatorname{Im} FD^{\perp} \leqslant \dim \operatorname{Im} F^{\perp} + \dim \operatorname{Im} D^{\perp}$.

Corollary 3.3, Lemma 3.4, and Proposition 3.10 form a generalization of [16, Th. 1.5.7], originally given in [14]. Theorem 3.5, Corollary 3.6, Lemma 3.8, and Proposition 3.10 form an analog of Schechter–Lebow's characterization of semi-Fredholm operators [16, Th. 1.4.4] and [16, Th. 1.4.5], originally given in [8] and [13]. Theorem 3.26 is a generalization of the classical "punctured neighborhood theorem" [16, Th. 1.7.7], originally given in [6]. As compared with the classical version for Hilbert spaces, our generalization (Th. 3.26) needs an additional assumption on the operator $F \in \mathcal{M}\Phi(M)$, denoted by (*). It turns out that, in the case of ordinary Hilbert spaces, (*) is automatically satisfied for any Fredholm operator, and thus, in the case of ordinary Hilbert spaces, Theorem 3.14 reduces to the classical "punctured neighborhood" theorem. However, in Example 3.25, we give an example of a Hilbert C^*-module over a W^*-algebra $\mathcal{A}$ which is not a Hilbert space and where the condition (*) is satisfied for all $\mathcal{A}$-Fredholm operators as long as they have closed image.

In several results in this paper, we consider semi-$\mathcal{A}$-Fredholm operators with closed image. Ordinary semi-Fredholm operators on Hilbert spaces always have closed images; however, in our generalizations to modules, we need sometimes to provide this additional assumption in order to obtain an analog of classical results. This leads us to study, in general, bounded adjointable operators over C^*-algebras with closed image rather than semi-Fredholm operators over W^*-algebra only. We prove in Lemma 3.17 that, if F and D are two bounded adjointable operators on a standard module with closed images, then $\operatorname{Im} DF$ is closed if the Dixmier angle between $\operatorname{Im} F$ and $\ker D \cap (\ker D \cap \operatorname{Im} F)^{\perp}$ is positive, or, equivalently, if the Dixmier angle between $\ker D$ and $\operatorname{Im} F \cap (\ker D \cap \operatorname{Im} F)^{\perp}$ is positive. This is a generalization, for Hilbert C^*-modules, of a well known result in [1] on Hilbert spaces. Moreover, our Lemma 3.20, which generalizes the aforementioned second part of the classical index theorem, works for arbitrary bounded adjointable operators F and D on a standard C^* module provided that $\operatorname{Im} F, \operatorname{Im} D, \operatorname{Im} DF$ are closed. Next, our Lemma 3.13 gives another (simplified) proof of the result in [12]. This result follows from Lemma 3.13 and Corollary 3.14. Finally, in Lemma 3.21, we prove that, if F and G are two $\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$, not necessarily adjointable and such that $\operatorname{Im} G, \operatorname{Im} F$ are closed, then $\operatorname{Im} GF$ is closed if and only if $\operatorname{Im} F + \ker G$ is closed and complementable.

Important tools for proving most of the results in this paper are [10, Corollary 3.6.4], [10, Corollary 3.6.7], and [10, Proposition 3.6.8] originally given in [2, 5, 7], and these results assume that $\mathcal{A}$ is a W^*-algebra. For this reason, we deal here mainly with Hilbert C*-modules over W^*-algebras. However, our Lemma 3.11, Corollary 3.12, Lemma 3.13, Lemma 3.17, Corollary 3.14, Corollary 3.18, and Lemma 3.20 hold also in the case when $\mathcal{A}$ is an arbitrary unital C^*-algebra rather than a W^*-algebra.

2. PRELIMINARIES

Throughout this paper, $\mathcal{A}$ stands for a W^*-algebra, $H_{\mathcal{A}}$ for the standard Hilbert C^*-module over $\mathcal{A}$, and $B^a(H_{\mathcal{A}})$ for the set of all bounded adjointable operators on $H_{\mathcal{A}}$. Similarly, if M is an arbitrary Hilbert C^*-module, then $B^a(M)$ stands for the set of all bounded adjojntable operators on M. Let $B(H_{\mathcal{A}})$ denote the set of all $\mathcal{A}$-linear bounded and not necessarily adjointable operators on $H_{\mathcal{A}}$. According to [10, Definition 1.4.1], we say that a Hilbert C*-module M over $\mathcal{A}$ is finitely generated if there exists a finite set $\{x_i\} \subseteq M$ such that M is equal to the linear span (over C and $\mathcal{A}$) of this set.

The notation $\tilde{\oplus}$ means a direct sum of modules without orthogonality, as in [10].

Definition 2.1. [3, Definition 2.1] Let $\mathrm{F} \in B^a(H_{\mathcal{A}})$. We say that F is an upper semi-$\mathcal{A}$-Fredholm operator if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathrm{F}} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix

$$\begin{bmatrix} \mathrm{F}_1 & 0 \\ 0 & \mathrm{F}_4 \end{bmatrix},$$

where F_1 is an isomorphism of M_1 and M_2, N_1 and N_2 are closed submodules of $H_{\mathcal{A}}$, and N_1 is finitely generated. Similarly, we say that F is a lower semi-$\mathcal{A}$-Fredholm operator if all the above conditions hold except that, in this case, we assume that N_2 (rather than N_1) is finitely generated.

Set
$\mathcal{M}\Phi_+(H_{\mathcal{A}}) = \{\mathrm{F} \in B^a(H_{\mathcal{A}}) \mid \mathrm{F}$ is upper semi-$\mathcal{A}$-Fredholm$\}$,
$\mathcal{M}\Phi_-(H_{\mathcal{A}}) = \{\mathrm{F} \in B^a(H_{\mathcal{A}}) \mid \mathrm{F}$ is lower semi-$\mathcal{A}$-Fredholm$\}$,
$\mathcal{M}\Phi(H_{\mathcal{A}}) = \{\mathrm{F} \in B^a(H_{\mathcal{A}}) \mid \mathrm{F}$ is $\mathcal{A}$-Fredholm operator on $H_{\mathcal{A}}\}$.

Next, let $K^*(H_{\mathcal{A}})$ be the set of all adjointable compact operators in the sense of [10, Sec. 2.2], and let $K(H_{\mathcal{A}})$ be the set of all compact operators (not necessarily adjointable) in the sense of [4]. Denote by $\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ the class of operators in $B(H_{\mathcal{A}})$ that have the above $\mathcal{M}\Phi_+$-decomposition and are not necessarily adjointable. Hence, $\mathcal{M}\Phi_+(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}})$. Similarly, denote by $\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ the set of all operators in $B(H_{\mathcal{A}})$ that have the $\mathcal{M}\Phi_-$-decomposition and are not necessarily adjointable. Thus, $\mathcal{M}\Phi_-(H_{\mathcal{A}}) = \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}})$. Finally, let $\widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$ be the set of all $\mathcal{A}$-Fredholm operators on $H_{\mathcal{A}}$ in the sense of [4] that are not necessarily adjointable.

Remark 2.2. [3] Note that, if M and N are two arbitrary Hilbert C*-modules, then the above definition can be generalized to the classes $\mathcal{M}\Phi_+(M,N)$ and $\mathcal{M}\Phi_-(M,N)$.

Recall that, by [10, Definition 2.7.8] (originally given in [9]), if $\mathrm{F} \in \mathcal{M}\Phi(H_{\mathcal{A}})$ and

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{\mathrm{F}} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

is an $\mathcal{M}\Phi$ decomposition for F, then the index of F takes values in $K(\mathcal{A})$ and is defined by the index $\mathrm{F} = [N_1] - [N_2] \in K(\mathcal{A})$, where $[N_1]$ and $[N_2]$ stand for the isomorphism classes of N_1 and N_2, respectively. By [10, Definition 2.7.9], the index is well defined.

Clearly, any operator $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ is also left invertible in $B(H_{\mathcal{A}})/K(H_{\mathcal{A}})$, whereas any operator $G \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ is right invertible $B(H_{\mathcal{A}})/K(H_{\mathcal{A}})$. The converse also holds.

Proposition 2.3. *If F is left invertible in $B(H_{\mathcal{A}})/K(H_{\mathcal{A}})$, then $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$. If F is right invertible in $B(H_{\mathcal{A}})/K(H_{\mathcal{A}})$, then $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$.*

Proof. If $GF = id + K''$ for some $G \in B(H_{\mathcal{A}})$, $K'' \in K(H_{\mathcal{A}})$, then, following the proof of [4, Th. 5], we obtain (45) and (46) of [4]. By (46), it follows readily that $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$. Moreover, by the first half of the proof of [4, Th. 5], we also see that G has the matrix $\begin{pmatrix} G_1 & G_2 \\ 0 & G_4 \end{pmatrix}$ with respect to the decomposition $H_{\mathcal{A}} = M_3 \oplus N_3 \xrightarrow{G} M_2 \oplus N_2 = H_{\mathcal{A}}$, where G_1 is an isomorphism. Indeed, by (45) of [4], $M_3 = \operatorname{Im} P = \operatorname{Im} F K_1^{-1} p_2 G$. It follows that $M_3 = F(M_1)$. Since $GF_{|_{M_1}}$ is an isomorphism onto M_2, it follows that $G_{|_{F(M_1)}}$ is an isomorphism onto M_2. Then, considering the operator G and applying the above arguments, one derives the second statement in the proposition.

Corollary 2.4. *The sets $\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ and $\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ are closed under multiplication.*

Lemma 2.5. *Let $F, D \in B^a(H_{\mathcal{A}})$. Suppose that $\operatorname{Im} F, \operatorname{Im} D$, and $\operatorname{Im} DF$ are closed. Then there exist closed submodules X, W, M' such that $\operatorname{Im} F = W \oplus (\ker D \cap \operatorname{Im} F)$, $\operatorname{Im} D = \operatorname{Im} DF \oplus X$, and $\ker D = M' \oplus (\ker D \cap \operatorname{Im} F)$. Moreover, $H_{\mathcal{A}} = W \tilde{\oplus} S(X) \oplus \ker D$, where $S = D|_{\ker D^{\perp}}{}^{-1}$.*

Proof. Since $\operatorname{Im} F, \operatorname{Im} D$, and $\operatorname{Im} DF$ are closed, these images are orthogonally complementable by [10, Th. 2.3.3]. Moreover, $\ker D$ is also orthogonally complementable. The operator $D_{|_{\operatorname{Im} F}}$ can, therefore, be viewed as an adjointable operator from $\operatorname{Im} F$ to $\operatorname{Im} D$. Since $\operatorname{Im} D_{|_{\operatorname{Im} F}} = \operatorname{Im} DF$ is closed, $\operatorname{Im} D = \operatorname{Im} DF \oplus X$ for some closed submodule X by [10, Th. 2.3.3]. Moreover, $\operatorname{Im} F = W \oplus \ker D_{|_{\operatorname{Im} F}} = W \oplus (\ker D \cap \operatorname{Im} F)$ for some closed submodule W. Hence,

$$H_{\mathcal{A}} = W \oplus (\ker D \cap \operatorname{Im} F) \oplus \operatorname{Im} F^{\perp}.$$

Therefore,

$$\ker D = (\ker D \cap \operatorname{Im} F) \oplus M', \quad \text{where} \quad M' = \ker D \cap (W \oplus \operatorname{Im} F^{\perp}).$$

Now, $D_{|_W}$ is an isomorphism onto $\operatorname{Im} DF$. However, $D_{|_W} = DP_{|_W}$, where P stands for the orthogonal projection onto $\ker D^{\perp}$. Then it follows that $P_{|_W}$ must be bounded below, and hence, $P(W)$ is closed in $\ker D^{\perp}$. In addition, $P(W) = S(\operatorname{Im} DF)$, where $S = D_{|\ker D^{\perp}}{}^{-1}$ is the operator from $\operatorname{Im} D$ onto $\ker D^{\perp}$. Since $\operatorname{Im} D = \operatorname{Im} DF \oplus X$ and S is an isomorphism, we have $\ker D^{\perp} = S(\operatorname{Im} DF) \tilde{\oplus} S(X)$. Hence, $H_{\mathcal{A}} = S(\operatorname{Im} DF) \tilde{\oplus} S(X) \oplus \ker D$. However, $P_{|_W}$ is an isomorphism from W onto $S(\operatorname{Im} DF)$. Therefore, $H_{\mathcal{A}} = W \tilde{\oplus} S(X) \oplus \ker D$.

Lemma 2.6. *Let M and N be closed submodules of $H_{\mathcal{A}}$ such that $M \subseteq N$ and $H_{\mathcal{A}} = M \oplus M^{\perp}$. Then $N = M \oplus (N \cap M^{\perp})$.*

Proof. Since $H_{\mathcal{A}} = M \oplus M^{\perp}$, every $z \in N$ can be represented as the sum $z = x + y$, where $x \in M, y \in M^{\perp}$. Hence, $z - x \in N$ since $z \in N$ and $x \in M \subseteq N$. Thus, $y \in N \cap M^{\perp}$.

3. SEMI-FREDHOLM OPERATORS AND CLOSED RANGE OPERATORS OVER W^*-ALGEBRAS

We begin with the following proposition.

Proposition 3.1. *Let $F \in \widehat{\mathcal{M}\Phi}_l(l_2(\mathcal{A}))$ or $F \in \mathcal{M}\Phi_+(H_\mathcal{A})$. Then there exists a decomposition*

$$H_\mathcal{A} = M_0 \tilde{\oplus} M_1' \tilde{\oplus} \ker F \xrightarrow{F} N_0 \tilde{\oplus} N_1' \tilde{\oplus} N_1'{}' = H_\mathcal{A}$$

with respect to which F has the matrix $\begin{bmatrix} F_0 & 0 & 0 \\ 0 & F_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where F_0 is an isomorphism and M_1' and $\ker F$ are finitely generated. Moreover, $M_1' \cong N_1'$. If $F \in \widehat{\mathcal{M}\Phi}_l(H_\mathcal{A})$ and $\operatorname{Im} F$ is closed, then $\operatorname{Im} F$ is complementable in $H_\mathcal{A}$.

Proof. The first statement follows from the same arguments as in the proof of [10, Proposition 3.6.8]. The other statement follows from the decomposition of F in the first statement.

Proposition 3.2. *If $F \in \widehat{\mathcal{M}\Phi}_r(H_\mathcal{A})$ and $\operatorname{Im} F$ is closed and complementable in $H_\mathcal{A}$, then the decomposition given above exists for the operator F.*

Proof. Suppose that $F \in \widehat{\mathcal{M}\Phi}_r(H_\mathcal{A})$. Let $H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$ be an $\widehat{\mathcal{M}\Phi}_r$-decomposition for F such that N_2 is finitely generated. Since $\operatorname{Im} F$ is closed by assumption and $F(M_1) = M_2, F(N_1) \subseteq N_2$, it follows readily that $F(N_1)$ is closed. Since $\operatorname{Im} F$ is complementable by assumption, it follows that $F(N_1)$ is complementable in N_2. Therefore, $F(N_1)$ is finitely generated and projective as a direct summand of a finitely generated projective module N_2. Since the map $F_{|_{N_1}} : N_1 \to F(N_1)$ is an epimorphism, there exists a decomposition $N_1 = N_1' \tilde{\oplus} \ker F$, where $N_1' \cong F(N_1)$.

Corollary 3.3. 1. *If $F \in \mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})$, then there exists an $\epsilon > 0$ such that, for $D \in B^a(H_\mathcal{A})$ and $\|D\| < \epsilon$, the sum $(F + D)$ is in $\mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})$ and $\operatorname{Im}(F + D)^\perp$ is <u>not</u> finitely generated. If $F \in \widehat{\mathcal{M}\Phi}_l(H_\mathcal{A}) \setminus \widehat{\mathcal{M}\Phi}(H_\mathcal{A})$, then the complement of $\overline{\operatorname{Im} F}$ is not finitely generated.*

2. *If $F \in \mathcal{M}\Phi_-(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})$, then there exists an $\epsilon > 0$ such that, for $D \in B^a(H_\mathcal{A})$ and $\|D\| < \epsilon$, we have $(F + D) \in \mathcal{M}\Phi_-(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})$ and $\ker(F + D)$ is <u>not</u> finitely generated.*

Proof. 1. It was shown in [3, Th. 4.1] that there exists an $\epsilon > 0$ such that

$$(F + D) \in \mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A}),$$

whenever $\|D\| < \epsilon$.

Now, since

$$(F + D) \in \mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A}),$$

by Proposition 3.1, there exists a decomposition

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1' \oplus \ker(F + D)$$

$$\downarrow F + D$$

$$H_\mathcal{A} = M_2 \oplus \overline{(F + D)(N_1')} \oplus \operatorname{Im}(F + D)^\perp$$

with respect to which $(F + D)$ has the matrix

$$\begin{bmatrix} (F + D)_1 & 0 & 0 \\ 0 & (F + D)_4 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $(F+D)_1$ is an isomorphism and $N_1' \oplus \ker(F+D)$ is finitely generated; however,

$$\overline{(F+D)(N_1')} \oplus \operatorname{Im}(F+D)^{\perp}$$

is <u>not</u> finitely generated, since $(F+D) \notin \mathcal{M}\Phi(H_{\mathcal{A}})$. Now, since $\overline{(F+D)(N_1')} \cong N_1'$ by Proposition 3.1 and N_1' is finitely generated as a direct summand of the finitely generated submodule $N_1' \oplus \ker(F+D)$, it follows that $\operatorname{Im}(F+D)^{\perp}$ <u>cannot</u> be finitely generated, since

$$\overline{(F+D)(N_1')} \oplus \operatorname{Im}(F+D)^{\perp}$$

is <u>not</u> finitely generated. The proof is similar to the case in which

$$F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}}).$$

We simply observe that the proof of [3, Th. 4.1] does not require the adjointability of F and, moreover, Proposition 3.1 also applies to the case when $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$.

2. This can be proved by passing to adjoints and using [3, Corollary 2.11].

Lemma 3.4. *If $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \setminus \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$, $\operatorname{Im} F$ is closed and complementable, then the complement of $\operatorname{Im} F$ is not finitely generated.*

Theorem 3.5. *Let $F \in B^a(H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ if and only if $\ker(F-K)$ is finitely generated for all $K \in K^*(H_{\mathcal{A}})$.*

Proof. If $F \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})$, choose a sequence $\{x_k\} \subseteq H_{\mathcal{A}}$ and an increasing sequence $\{n_k\} \subseteq \mathbb{N}$ such that

$$x_k \in L_{n_k} \setminus L_{n_{k-1}} \quad \text{for all} \quad k \in \mathbb{N}, \quad \|x_k\| \leqslant 1 \quad \text{for all} \quad k \in \mathbb{N}$$

and

$$\|Fx_k\| \leqslant 2^{1-2k} \quad \text{for all} \quad k \in \mathbb{N}.$$

By [3, Lemma 3.2], such a sequence exists. Set

$$K_n x = \sum_{k=1}^{n} \langle x_k, x\rangle F x_k \quad \text{for} \quad x \in H_{\mathcal{A}}.$$

Then $K_n \in K^*(H_{\mathcal{A}})$ for all n. For $n > m$, we have

$$\|(K_n - K_m)x\| \leqslant \sum_{k=m+1}^{n} \|x_k\|\|x\|\|Fx_k\| \leqslant \|x\| \sum_{k=m+1}^{n} 2^{1-2(k+1)},$$

and thus, $K_n - K_m$ vanishes as $n, m \longrightarrow \infty$.

Let $K \in K^*(H_{\mathcal{A}})$ be the limit of $\{K_n\}$ in the operator norm. Clearly, then

$$Kx = \sum_{k=1}^{\infty} \langle x_k, x\rangle F x_k \quad \forall x \in H_{\mathcal{A}}.$$

Observe next that, by the construction of the sequence $\{x_k\}$,

$$\langle x_j, x_k\rangle = \delta_{j,k} \quad \forall j,k \quad \text{as} \quad x_k = L_{n_k} \setminus L_{n_{k-1}} \quad \forall k$$

and the sequence $\{n_n\}_k \subseteq \mathbb{N}$ is increasing. Thus, $\{x_k\} \subseteq \ker(F-K)$. Now, if $\ker(F-K)$ were finitely generated, then, by [10, Lemma 2.3.7], $\ker(F-K)$ would be an orthogonal direct summand in $H_{\mathcal{A}}$. Hence, by the proof of [10, Th. 2.7.5], there exists an $n \in \mathbb{N}$ such that $p_{n|_{\ker(F-K)}}$ is an isomorphism from $\ker(F-K)$ onto some direct summand of L_n (where p_n is the orthogonal projection onto L_n along $L_n^{\perp}$). In particular, $p_{n|_{\ker(F-K)}}$ is injective. However, since the sequence $\{n_k\}_k$ is increasing, we can find an n_{k_0} such that $n_k \geqslant n$ for all $k \geqslant k_0$. Now, by construction, $x_k \in L_{n_k} \setminus L_{n_{k-1}}$ for all k, and hence, $x_k \in L_n^{\perp}$ for all $k \geqslant k_0$, since $n_k > n$ for all $k > k_0$. Consequently, $p_n(x_k) = 0$ for all $k \geqslant k_0$. Since $\{x_k\}_{k \geqslant k_0} \subseteq \ker(F-K)$, we see that p_n is not injective, a contradiction. Thus, $\ker(F-K)$ is not finitely generated. On the other hand, if $F \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$, then

$$(F+K) \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \quad \forall K \in K^*(H_{\mathcal{A}}).$$

Now, since $\mathcal{A}$ is a W^*-algebra by assumption, it follows that $\ker(F-K)$ must be finitely generated for all $K \in K^*(H_{\mathcal{A}})$, because

$$(F-K) \in \mathcal{M}\Phi_+(H_{\mathcal{A}}) \quad \text{for all} \quad K \in K^*(H_{\mathcal{A}}),$$

which holds by the same arguments as in the proof of [10, Lemma 2.7.13] and follows from Proposition 3.1.

Corollary 3.6. *Let $\mathcal{A}$ be a W^*-algebra and $F \in B^a(H_{\mathcal{A}})$. Then $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ if and only if $\operatorname{Im}(F-K^*)^{\perp}$ is finitely generated for all $K^* \in K^*(H_{\mathcal{A}})$.*

Proof. Suppose that $F \notin \mathcal{M}\Phi_-(H_{\mathcal{A}})$. By [3, Corollary 2.11], then $F^* \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})$. Hence, there exists some $K^* \in K^*(H_{\mathcal{A}})$ such that $\ker(F^*-K^*)$ is not finitely generated. However, $\ker(F^*-K^*) = \operatorname{Im}(F-K^*)^{\perp}$. On the other hand, if $F \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$, then $F^* \in \mathcal{M}\Phi_+(H_{\mathcal{A}})$ by [3, Corolary 2.11]. Hence, by Theorem 3.5, $\ker(F^*-K^*)$ is finitely generated for all $K^* \in K^*(H_{\mathcal{A}})$, and thus $\operatorname{Im}(F-K^*)^{\perp}$ is finitely generated for all $K^* \in K^*(H_{\mathcal{A}})$.

Let us introduce another class of operators on $H_{\mathcal{A}}$.

Definition 3.7. Let $F \in B(H_{\mathcal{A}})$. We say that $F \in \widehat{\mathcal{M}\Phi}_+(H_{\mathcal{A}})$ if there exist a closed submodule M and a finitely generated submodule N such that $H_{\mathcal{A}} = M \tilde{\oplus} N$ and $F|_M$ is bounded below.

Note that we do not assume that $F(M)$ is complementable in $H_{\mathcal{A}}$. Thus, $\widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}}) \subseteq \widehat{\mathcal{M}\Phi}_+(H_{\mathcal{A}})$, and the equality need not hold.

Lemma 3.8. *Let $F \in B(H_{\mathcal{A}})$. Then $F \in \widehat{\mathcal{M}\Phi}_+(H_{\mathcal{A}})$ if and only if $\ker(F-K)$ is finitely generated for all $K \in K^*(H_{\mathcal{A}})$.*

Proof. Let $H_{\mathcal{A}} = M \tilde{\oplus} N$ be an $\widehat{\mathcal{M}\Phi}_+$ decomposition for F defined above. Since N is finitely generated, we may choose an $n \in \mathbb{N}$ such that $H_{\mathcal{A}} = L_n^{\perp} \tilde{\oplus} P \tilde{\oplus} N$ for some finitely generated closed submodule P. Then F is bounded below on $L_n^{\perp} \tilde{\oplus} P$, and thus, there exists a $C > 0$ such that $\|Fx\| \geqslant C\|x\|$ for all $x \in L_n^{\perp} \oplus P$. Further, if $K \in K^*(H_{\mathcal{A}})$, then there exists some $m \geqslant n$ such that $\|K_{|_{L_m^{\perp}}}\| < C$ by Proposition 2.1.1 of [10]. Then $F-K$ is bounded below on $L_m^{\perp}$. Conversely, if $F \notin \widehat{\mathcal{M}\Phi}_+(H_{\mathcal{A}})$, then, in particular, F is not bounded below on $L_n^{\perp}$ for all n. We may, hence, repeat the construction of [3, Lemma 3.2] to obtain a sequence $\{x_k\}_k$ such that the proof of Theorem 3.5 applies. The operator K in this proof is adjointable and compact as the limit in operator norm of operators in $K^*(H_{\mathcal{A}})$. Indeed, the operators of the form $< x_k, \cdot > Fx_k$, where x_k's are fixed vectors, are adjointable, although the operator F need not be adjointable. This is because Fx_k's are then also fixed vectors, and hence, the nonadjointability of the operator F does not reflect on the adjointability of the operators $< x_k, \cdot > Fx_k$. Moreover, $\ker(F-K)$ is not finitely generated.

Set $\widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}}) = \{G \in B(H_{\mathcal{A}}) \mid$ there exist closed submodules M, N, M' of $H_{\mathcal{A}}$ such that $H_{\mathcal{A}} = M \tilde{\oplus} N$, N is finitely generated and $G_{|_{M'}}$, is an isomorphism onto $M\}$.

Remark 3.9. We do not require that M' be complementable in $H_{\mathcal{A}}$. Hence, we have only the inclusion $\widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}}) \subseteq \widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}})$, and the equality is not necessary.

Proposition 3.10. *Let $G \in \widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}})$. Then, for every $K \in K(H_{\mathcal{A}})$, there exists an inner product equivalent to the initial one and such that the orthogonal complement of $\overline{\operatorname{Im}(G+K)}$ with respect to this new inner product is finitely generated.*

Proof. Let $H_{\mathcal{A}} = M \tilde{\oplus} N$ be an $\widehat{\mathcal{M}\Phi}_-$ decomposition for G defined above, and let $M' \subseteq H_{\mathcal{A}}$ be such that $G_{|_{M'}}$ is an isomorphism onto M. Since N is finitely generated, there exists an $n \in \mathbb{N}$ such that $H_{\mathcal{A}} = L_n^{\perp} \oplus P \oplus N$ for some finitely generated submodule P. Denote by $\sqcap$ the projection onto $L_n^{\perp} \oplus P$ along N; it follows that $\sqcap_{|_M}$ is an isomorphism onto $L_n^{\perp} \oplus P$. Hence, $\sqcap G_{|M'}$ is an isomorphism of M' onto $L_n^{\perp} \oplus P$. If $K \in K(H_{\mathcal{A}})$, then there exists an $m \geqslant n$ such that $\|q_m K\| < \|(\sqcap G_{|M'})^{-1}\|^{-1}$. Let $M'' = (\sqcap G_{|M'})^{-1}(L_m^{\perp})$. Then $\sqcap G_{|M''} = q_m G_{|M''}$ and, moreover, $q_m(G-K)_{|M''}$ is an isomorphism onto $L_m^{\perp}$. Now $M' = M'' \tilde{\oplus} N''$, where $N'' = (\sqcap G_{|M'})^{-1}(P \oplus (L_m \setminus L_n))$. With respect to the decomposition

$$M' = M'' \tilde{\oplus} N'' \xrightarrow{G-K} L_m^{\perp} \oplus L_m = H_{\mathcal{A}},$$

$G-K$ has the matrix $\begin{bmatrix} (G-K)_1 & (G-K)_2 \\ (G-K)_3 & (G-K)_4 \end{bmatrix}$, where $(G-K)_1 = q_m(G-K)_{|_{M''}}$ is an isomorphism. Hence, by the same arguments as in the proof of [10, Lemma 2.7.10], there exists an isomorphism $U\colon M' \longrightarrow M'$ and an isomorphism $V\colon H_{\mathcal{A}} \longrightarrow H_{\mathcal{A}}$ such that $G-K$ has the matrix $\begin{bmatrix} \overbrace{(G-K)_1} & 0 \\ 0 & \overbrace{(G-K)_4} \end{bmatrix}$ with respect to the decomposition

$$M' = U(M'') \tilde{\oplus} U(N'') \xrightarrow{G-K} V(L_m^{\perp}) \tilde{\oplus} V(L_m) = H_{\mathcal{A}},$$

where $\overbrace{(G-K)_1}$ is an isomorphism. Moreover, V is such that $V(L_m) = L_m$ by the construction in the proof of [10, Lemma 2.7.10]. Since $V(L_m^{\perp}) \subseteq \operatorname{Im}(G-K) \subseteq \overline{\operatorname{Im}(G-K)}$ and $H_{\mathcal{A}} = V(L_m^{\perp}) \tilde{\oplus} L_m$, we obtain $\overline{\operatorname{Im}(G-K)} = V(L_m^{\perp}) \tilde{\oplus} (L_m \cap \overline{\operatorname{Im}(G-K)})$. On $H_{\mathcal{A}}$, we may replace the inner product by an equivalent one in such a way that $V(L_m^{\perp})$ and L_m form an orthogonal direct sum with respect to this new inner product. Since L_m is finitely generated and $L_m \cap \overline{\operatorname{Im}(G-K)}$ is a closed submodule of L_m, we see by [10, Lemma 3.6.1] that

$$L_m = (L_m \cap \overline{\operatorname{Im}(G-K)})^{\perp^{\perp}} \oplus (L_m \cap \overline{\operatorname{Im}(G-K)})^{\perp}.$$

Then it follows that $(L_m \cap \overline{\operatorname{Im}(G-K)})^{\perp}$ is finitely generated. Since $\overline{\operatorname{Im}(G-K)} = V(L_m^{\perp}) \oplus (L_m \cap \overline{\operatorname{Im}(G-K)})$, we see that $\overline{\operatorname{Im}(G-K)}^{\perp}$ is finitely generated. Here, certainly, the orthogonal complement is taken with respect to the new inner product.

It follows from the proof of Proposition 3.10 that, if $\overline{\operatorname{Im}(F-K)}$ is complementable in $H_{\mathcal{A}}$ (for $F \in \widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}})$ and $K \in K(H_{\mathcal{A}})$), then the complement must be finitely generated.

Lemma 3.11. *Let $D \in B^a(H_{\mathcal{A}})$. Then $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$ if and only if there exist closed submodules M, N such that $H_{\mathcal{A}} = M \tilde{\oplus} N$, N is finitely generated and $M \subseteq \operatorname{Im} D$.*

Proof. If $D \in B^a(H_{\mathcal{A}})$, then such modules clearly exist by the $\mathcal{M}\Phi_-$ decomposition of D. Conversely, if such modules exist for $D \in B^a(H_{\mathcal{A}})$, then N is orthogonally complementable in $H_{\mathcal{A}}$ by [10, Lemma 2.3.7]. If P stands for the orthogonal projection onto $N^{\perp}$, then $P_{|_M}$ is an isomorphism onto $N^{\perp}$ since $M \tilde{\oplus} N = H_{\mathcal{A}}$. Hence, the operator PD is adjointable and $\operatorname{Im} PD = N^{\perp}$. By [10, Th. 2.3.3], $\ker PD$ is orthogonally complementable in $H_{\mathcal{A}}$. With respect to the decomposition

$$H_{\mathcal{A}} = \ker PD^{\perp} \oplus \ker PD \xrightarrow{D} N^{\perp} \oplus N = H_{\mathcal{A}},$$

D has the matrix $\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$, where $D_1 = PD_{|_{\ker PD^{\perp}}}$ is an isomorphism. Using the technique of diagonalization as in the proof of [10, Lemma 2.7.10] and the fact that N is finitely generated, we see that $D \in \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Corollary 3.12. $\widehat{\mathcal{M}\Phi}_-(H_{\mathcal{A}}) \cap B^a(H_{\mathcal{A}}) = \mathcal{M}\Phi_-(H_{\mathcal{A}})$.

Lemma 3.13. *Let $F, D \in B^a(H_{\mathcal{A}})$. Suppose that $\operatorname{Im} F$ and $\operatorname{Im} D$ are closed. If $\operatorname{Im} F + \ker D$ is closed, then $\operatorname{Im} F + \ker D$ is orthogonally complementable.*

Proof. Suppose that $\operatorname{Im} F + \ker D$ is closed. Since $\operatorname{Im} F \oplus \operatorname{Im} F^{\perp} = H_{\mathcal{A}}$ by [10, Th. 2.3.3], we have $\operatorname{Im} F + \ker D = \operatorname{Im} F \oplus M''$, where $M'' = (\operatorname{Im} F + \ker D) \cap \operatorname{Im} F^{\perp}$, since $\operatorname{Im} F \subseteq \operatorname{Im} F + \ker D$. This follows from Lemma 2.6. Let P be the orthogonal projection onto $\operatorname{Im} F^{\perp}$. Then $M'' = P(\operatorname{Im} F + \ker D) = P(\operatorname{Im} F) + P(\ker D) = P(\ker D)$. Thus, $\operatorname{Im}(P_{|_{\ker D}}) = M''$. Now, since $\operatorname{Im} D$ is closed, $\ker D$ is orthogonally complementable in $H_{\mathcal{A}}$ by [10, Th. 2.3.3] again. Hence, $P_{|_{\ker D}}$ is an adjointable operator from $\ker D$ to $\operatorname{Im} F^{\perp}$, and its image is closed. Applying [10, Th. 2.3.3] once again to the operator $P_{|_{\ker D}}$, we see that $\operatorname{Im} F^{\perp} = M'' \oplus N'', \ker D = \ker(P_{|_{\ker D}}) \oplus M' = (\ker D \cap \operatorname{Im} F) \oplus M'$ for some closed submodules N'', M'. Then $P_{|_{M'}}$ is an isomorphism onto M''. Then it follows that $H_{\mathcal{A}} = (\operatorname{Im} F \oplus N'')\tilde{\oplus} M'$. Moreover, since $H_{\mathcal{A}} = (\ker D \cap \operatorname{Im} F) \oplus M' \oplus \ker D^{\perp}$, we see that $\ker D \cap \operatorname{Im} F$ is orthogonally complementable in $H_{\mathcal{A}}$. Hence, $\operatorname{Im} F = (\ker D \cap \operatorname{Im} F) \oplus M$, where $M = \operatorname{Im} F \cap (\ker D \cap \operatorname{Im} F)^{\perp}$. Here we apply Lemma 2.6 again. Then we obtain $H_{\mathcal{A}} = ((\ker D \cap \operatorname{Im} F) \oplus M \oplus N'')\tilde{\oplus} M' = ((\ker D \cap \operatorname{Im} F)\tilde{\oplus} M' \tilde{\oplus} M)\tilde{\oplus} N'' = (\ker D + \operatorname{Im} F)\tilde{\oplus} N''$. Let $Q = (P_{|_{M'}})^{-1}$. Then Q is a bounded adjointable operator from M'' onto M'. Consider now the operator $\sqcap_{\operatorname{Im} F} + J_{M'} Q \sqcap_{M''}$ where $\sqcap_{\operatorname{Im} F}$ and $\sqcap_{M''}$ stand for the orthogonal projections onto $\operatorname{Im} F$ and M'', respectively, and $J_{M'}$ is the inclusion. Since M' is orthogonally complementable, it follows that $J_{M'}$ is adjointable. Hence, $\sqcap_{\operatorname{Im} F} + J_{M'} Q \sqcap_{M''} \in B^a(H_{\mathcal{A}})$. Moreover, $\operatorname{Im}(\sqcap_{\operatorname{Im} F} + J_{M'} Q \sqcap_{M''}) = \operatorname{Im} F \tilde{\oplus} M' = \operatorname{Im} F + \ker D$, which is closed by assumption. It follows from [10, Th. 2.3.3] that $\operatorname{Im} F + \ker D$ is orthogonally complementable.

Corollary 3.14. *Let $F, D \in B^a(H_{\mathcal{A}})$ and suppose that $\operatorname{Im} F$ and $\operatorname{Im} D$ are closed. Then $\operatorname{Im} DF$ is closed if and only if $\operatorname{Im} F + \ker D$ is orthogonally complementable.*

Proof. By [11, Corollary 1], $\operatorname{Im} DF$ is closed if and only if $\operatorname{Im} F + \ker D$ is closed. Now use Lemma 3.13.

Remark 3.15. The statement of Corollary 3.14 was already proved in [12]; however, we give here another (shorter) proof.

Recall the definition of the Dixmier angle between two Hilbert C^*-modules given in [12].

Definition 3.16. For two given closed submodules M, N of $H_{\mathcal{A}}$, write

$$c_0(M, N) = \sup\{\| < x, y > \| \mid x \in M, y \in N, \|x\|, \|y\| \leqslant 1\}.$$

We say then that the *Dixmier angle* between M and N is positive if $c_0(M, N) < 1$.

Lemma 3.17. *Let M and N be two closed orthogonally complementable submodules of $H_{\mathcal{A}}$ and suppose that $M \cap N = \{0\}$. Then $M + N$ is closed if the Dixmier angle between M and N is positive.*

Proof. Suppose that the Dixmier angle between M and N is positive. We wish to show first that, in this case, there exists a constant $C > 0$ such that, if $x \in M$ and $y \in N$ satisfy $\|x + y\| \leqslant 1$, then $\|x\| \leqslant C$. To this end, note first that, since M is orthogonally complementable in $H_{\mathcal{A}}$, there exist some $y' \in M, y'' \in M^{\perp}$ such that $y = y' + y''$ for $y \in N$. Now let $c_0(M, N) = \delta < 1$. Then

$$\sup\{\| < y, z > \| \mid z \in M, \|z\| = 1\} = \|y'\| \leqslant \|y\|\delta.$$

It follows that

$$\|y''\| = \|y - y'\| \geqslant \|y\| - \|y'\| \geqslant (1 - \delta)\|y\| = \frac{1 - \delta}{\delta}\delta\|y\| \geqslant \frac{1 - \delta}{\delta}\|y'\|.$$

Note now that $< x + y, x + y > = < x + y', x + y' > + < y'', y'' >$. By taking the supremum over all states on $\mathcal{A}$, we obtain $\|x + y\| \geqslant \max\{\|x + y'\|, \|y''\|\}$. Thus, if $\|x + y\| \leqslant 1$, then $\|x + y'\|, \|y''\| \leqslant 1$.

However, if $\|y''\| \leqslant 1$, then, by the above calculation, we see that $\|y'\| \leqslant \delta/(1-\delta)$. If, in addition, $\|x+y'\| \leqslant 1$, then $1 \geqslant \|x\| - \|y'\| \geqslant \|x\| - \delta/(1-\delta)$. Hence, $\|x\| \leqslant 1+\delta/(1-\delta)$, and thus, we may set $C = 1+\delta/(1-\delta)$. Assume now that $\{x_n+y_n\}_n$ is a Cauchy sequence in $M+N$ (here $x_n \in M, y_n \in N$ for all n). By the above arguments, then $\{x_n\}_n$ must be a Cauchy sequence in M. Indeed, if $\{x_n+y_n\}$ is a Cauchy sequence, then, for a given $\epsilon > 0$, there exists some $N_0 \in \mathbb{N}$ such that $\|(x_n-x_m)+(y_n-y_m)\| < \frac{\epsilon}{C}$ for all $n, m \geqslant N_0$. By the above arguments, then $\|x_n - x_m\| < \epsilon$ for all $n, m \geqslant N_0$. Since M is closed, $x_n \to x$ for some $x \in M$. However, then $\{y_n\}_n$ must be also convergent, and thus, $y_n \to y$ for some $y \in N$ since N is closed. Hence, $x_n + y_n$ converges to $x+y \in M+N$ as $n \to \infty$. Thus, $M+N$ is closed.

Corollary 3.18. *Suppose that* $F, D \in B^a(H_{\mathcal{A}})$ *and the sets* $\operatorname{Im} F$, $\operatorname{Im} D$ *are closed and* $\ker D \cap \in F$ *is orthogonally cpmplementable. Put* $M = \operatorname{Im} F \cap (\ker D \cap \operatorname{Im} F)^{\perp}$ *and* $M' = \ker D \cap (\ker D \cap \operatorname{Im} F)^{\perp}$. *Then* $\operatorname{Im} DF$ *is closed if the Dixmier angle between* M' *and* $\operatorname{Im} F$ *(or, equivalently, the Dixmier angle between* M *and* $\ker D$*) is positive.*

We now introduce the following notation: for two closed submodules N_1, N_2 of M, we write $N_1 \preceq N_2$ when N_1 is isomorphic to a closed submodule of N_2.

Proposition 3.19. *Let* $F, G \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ *with closed images. Suppose that* $\operatorname{Im} GF$ *is closed. Then* $\operatorname{Im} F$ *and* $\operatorname{Im} G$, $\operatorname{Im} GF$ *are complementable in* $H_{\mathcal{A}}$. *Moreover, if* $\operatorname{Im} F^0, \operatorname{Im} G^0$, *and* $\operatorname{Im} GF^0$ *denote the complements of* $\operatorname{Im} F, \operatorname{Im} G$, *and* $\operatorname{Im} GF$, *respectively, then*

$$\operatorname{Im} GF^0 \preceq \operatorname{Im} F^0 \oplus \operatorname{Im} G^0, \qquad \ker GF \preceq \ker G \oplus \ker F.$$

If $F, G \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ *and* $\operatorname{Im} F, \operatorname{Im} G$, *and* $\operatorname{Im} GF$ *are closed, then the statement above holds under the additional assumption that* $\operatorname{Im} F, \operatorname{Im} G$, *and* $\operatorname{Im} GF$ *are complementable in* $H_{\mathcal{A}}$.

Proof. Since $F \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$, it follows that F has the decomposition given in Proposition 3.1. Then $N_1' = \overline{F(M_1')}$, where we use the notation of Proposition 3.1. Since $\operatorname{Im} F$ is closed by assumption, we have now $N_1' = F(M_1')$. Hence, $\operatorname{Im} F = N_0 \tilde{\oplus} N_1'$, and thus $N_1'' = \operatorname{Im} F^0$. Since $\operatorname{Im} G, \operatorname{Im} GF$ are closed, it follows that $\operatorname{Im} G^0$ and $\operatorname{Im} GF^0$ exist by the same arguments, because $G, GF \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$. Here we use the fact that $GF \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$ by Corollary 2.4, since $F, G \in \widehat{\mathcal{M}\Phi}_l(H_{\mathcal{A}})$. Now, since $\ker G$ is self-dual (it is finitely generated) and since $\ker G \cap \operatorname{Im} F$ is the kernel of the projection onto $\operatorname{Im} F^0$ along $\operatorname{Im} F$ restricted to $\ker G$, we may derive by [10, Corollary 3.6.4] that $\ker G = (\ker G \cap \operatorname{Im} F) \oplus M'$ for some closed submodule M'. Hence $H_{\mathcal{A}} = (\ker G \cap \operatorname{Im} F) \tilde{\oplus} M' \tilde{\oplus} \ker G^0$, and thus $\ker G \cap \operatorname{Im} F$ is complementable in $H_{\mathcal{A}}$. By arguments similar to those used in Lemma 2.6, $\ker G \cap \operatorname{Im} F$ is complementable in $\operatorname{Im} F$ as a submodule of $\operatorname{Im} F$. Thus, $\operatorname{Im} F = (\ker G \cap \operatorname{Im} F) \tilde{\oplus} M$, where M is the intersection of $\operatorname{Im} F$ with the complement of $\ker G \cap \operatorname{Im} F$ in $H_{\mathcal{A}}$. Note also that $\ker G$ and $\ker F$ are complementable in $H_{\mathcal{A}}$ by Proposition 3.1. Since $\operatorname{Im} G$, $\operatorname{Im} GF$ are both complementable in $H_{\mathcal{A}}$ and $\operatorname{Im} GF \subseteq \operatorname{Im} G$, it follows that we have $\operatorname{Im} G = \operatorname{Im} GF \tilde{\oplus} X$, where $X = \operatorname{Im} G \cap \operatorname{Im} GF^0$ by arguments similar to those used in Lemma 2.6. Thus, we have $H_{\mathcal{A}} = (\ker G \cap \operatorname{Im} F) \tilde{\oplus} M' \tilde{\oplus} \ker G^0 = (\ker G \cap \operatorname{Im} F) \oplus M \tilde{\oplus} \operatorname{Im} F^0 = \operatorname{Im} GF \tilde{\oplus} X \tilde{\oplus} \operatorname{Im} G^0 = H_{\mathcal{A}}$ (where $\ker G^0$ is a complemented closed submodule of $\ker G$). Let $\sqcap \in B(H_{\mathcal{A}})$ be the projection onto $\ker G^0$ along $\ker G$. We see that $G_{|_M}$ is an isomorphism onto $\operatorname{Im} GF$ and, moreover, $G_{|_M} = G \sqcap_{|_M}$. Since $G_{|_{\ker G^0}}$ and $G\sqcap_{|_M}$ are isomorphisms, it follows that $\sqcap_{|_M}$ is an isomorphism. Let $S = (G_{|_{\ker G^0}})^{-1}$. Then $\sqcap(M) = S(\operatorname{Im} GF)$. Since $\ker G^0 = S(\operatorname{Im} GF) \tilde{\oplus} S(X)$, it follows that $H_{\mathcal{A}} = M \tilde{\oplus} S(X) \tilde{\oplus} \ker G = M \tilde{\oplus} S(X) \tilde{\oplus} M' \tilde{\oplus} (\ker G \cap \operatorname{Im} F)$. But, we also have $\operatorname{Im} F = (\ker G \cap \operatorname{Im} F) \oplus M$. It follows that $\operatorname{Im} F^0 \cong S(X) \tilde{\oplus} M' \cong X \oplus M'$. However, it follows from the above expression that $\operatorname{Im} GF^0 \cong X \oplus \operatorname{Im} G^0 \preceq X \oplus M' \oplus \operatorname{Im} G^0 \cong \operatorname{Im} F^0 \oplus \operatorname{Im} G^0$. If $F, G \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ and $\operatorname{Im} F, \operatorname{Im} G$, and $\operatorname{Im} GF$ are closed and complementable in $H_{\mathcal{A}}$, we can apply the same proof as above, and we only need to explain first why $\ker G \cap \operatorname{Im} F$ is complementable. This can be done in the following way. Since $F \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$, and $\operatorname{Im} F$ is closed and complementable, $\ker F$ is complementable in $H_{\mathcal{A}}$ by Proposition 3.2. Hence, $\ker F = \ker F \tilde{\oplus} W$, where W is the intersection of $\ker GF$ and the complement of $\ker F$, which follows again by arguments similar to those used in Lemma 2.6. Thus, $F_{|_W}$ is an isomorphism onto $\ker G \cap \operatorname{Im} F$. Further, again since $GF \in \widehat{\mathcal{M}\Phi}_r(H_{\mathcal{A}})$ and $\operatorname{Im} GF$ is closed and complementable, we see that $H_{\mathcal{A}} = \ker GF \tilde{\oplus} M$ for some closed submodule M. Hence, $H_{\mathcal{A}} = \ker F \tilde{\oplus} W \tilde{\oplus} M$. On $W \tilde{\oplus} M$, we have an isomorphism onto $\operatorname{Im} F$, and thus

$$\operatorname{Im} F = F(W) \tilde{\oplus} F(M) = (\ker G \cap \operatorname{Im} F) \tilde{\oplus} F(M).$$

Therefore,

$$H_{\mathcal{A}} = (\ker G \cap \operatorname{Im} F) \tilde{\oplus} F(M) \tilde{\oplus} \operatorname{Im} F^0,$$

where $\operatorname{Im} F^0$ stands for the complement of $\operatorname{Im} F$. It follows that $\ker G \cap \operatorname{Im} F$ is complementable. In order to derive that $\ker DF \preceq (\ker D \oplus \ker F)$, one can proceed in exactly the same way as in the proof of [16, Th. 1.2.4] to obtain $\ker DF = \ker F \tilde{\oplus} W$, where $W \cong (\ker D \cap \operatorname{Im} F)$. The rest follows.

Lemma 3.20. *Let $F, D \in B^a(H_{\mathcal{A}})$. Suppose that $\operatorname{Im} F, \operatorname{Im} D$, and $\operatorname{Im} DF$ are closed. Then*

$$\operatorname{Im} DF^{\perp} \preceq \operatorname{Im} F^{\perp} \oplus \operatorname{Im} D^{\perp}, \qquad \ker DF \preceq \ker D \oplus \ker F.$$

Proof. The statement can be proved in exactly the same way as in Proposition 3.19, since $\operatorname{Im} F$, $\operatorname{Im} D$, and $\operatorname{Im} DF$ are then orthogonally complementable in $H_{\mathcal{A}}$ by [10, Th. 2.3.3]. Again, we only need to argue that $\ker D \cap \operatorname{Im} F$ is orthogonally complementable in $H_{\mathcal{A}}$. Now $D_{|_{\operatorname{Im} F}}$ is an adjointable operator from $\operatorname{Im} F$ to $H_{\mathcal{A}}$ since $D \in B^a(H_{\mathcal{A}})$ and $\operatorname{Im} F$ is orthogonally complementable in $H_{\mathcal{A}}$. Moreover, $\operatorname{Im} D_{|_{\operatorname{Im} F}} = \operatorname{Im} DF$, which is closed by assumption. It follows from [10, Th. 2.3.3] that $\ker D_{|_{\operatorname{Im} F}} = \ker D \cap \operatorname{Im} F$ is orthogonally complementable in $\operatorname{Im} F$. Thus, we have $\operatorname{Im} F = (\ker D \cap \operatorname{Im} F) \oplus M$ for some closed submodule M. Hence, $H_{\mathcal{A}} = (\ker D \cap \operatorname{Im} F) \oplus M \oplus \operatorname{Im} F^{\perp}$.

Lemma 3.21. *Let $F, G \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$ and suppose that $\operatorname{Im} G$ and $\operatorname{Im} F$ are closed. Then $\operatorname{Im} GF$ is closed if and only if $\operatorname{Im} F + \ker G$ is closed and complementable.*

Proof. If $\operatorname{Im} F + \ker G$ is closed, then $\operatorname{Im} GF$ is closed by [11, Corollary 1]. Conversely, assume that $\operatorname{Im} GF$ is closed. Now, by Corollary 2.4, GF is in $\widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$, since so are G and F. Then, by Proposition 3.1, $\operatorname{Im} GF$ is complementable. Moreover, $\ker G \cap \operatorname{Im} F$ is complementable in $\operatorname{Im} F$ by the same arguments as above, because $F, G \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$. Thus, we may write $\operatorname{Im} F$ in the form $\operatorname{Im} F = (\ker G \cap \operatorname{Im} F) \tilde{\oplus} \tilde{M}$. Then $G_{|_{\tilde{M}}}$ is an isomorphism onto $\operatorname{Im} GF$. Let $\operatorname{Im} GF^0$ and $\operatorname{Im} G^0$ be the complements of $\operatorname{Im} GF$ and $\operatorname{Im} G$, respectively. Then, since $\operatorname{Im} GF \subseteq \operatorname{Im} G$, $\operatorname{Im} G = \operatorname{Im} GF \tilde{\oplus} (\operatorname{Im} GF^0 \cap \operatorname{Im} G)$ by the proof of Lemma 2.6. Hence, $H_{\mathcal{A}} = \operatorname{Im} GF \tilde{\oplus} (\operatorname{Im} GF^0 \cap \operatorname{Im} G) \tilde{\oplus} \operatorname{Im} G^0$. Moreover, since $G \in \widehat{\mathcal{M}\Phi}(H_{\mathcal{A}})$ and $\operatorname{Im} G$ is closed by assumption, $\ker G$ is complementable in $H_{\mathcal{A}}$ by Proposition 3.1. Let $\ker G^0$ be the complement of $\ker G$; then $G_{|_{\ker G^0}}$ is an isomorphism onto $\operatorname{Im} G$. Combining all these facts together, we are then in the position to apply the same arguments as in the proof of Lemma 2.5 to show that $H_{\mathcal{A}} = \tilde{M} \tilde{\oplus} S'(\operatorname{Im} GF^0 \cap \operatorname{Im} G) \tilde{\oplus} \ker G$, where $S' = (G_{|_{\ker G^0}})^{-1}$. Hence, $\tilde{M} \tilde{\oplus} \ker G$ is closed and complementable in $H_{\mathcal{A}}$. However, $\tilde{M} \tilde{\oplus} \ker G = \tilde{M} \tilde{\oplus} (\ker G \cap \operatorname{Im} F) \tilde{\oplus} \tilde{M}' = \ker G + \operatorname{Im} F$.

Lemma 3.22. *Let $F \in \mathcal{M}\Phi(M)$ be such that $\operatorname{Im} F$ is closed, where M is a Hilbert W^*-module. Then there exists an $\epsilon > 0$ such that, for every $D \in B^a(M)$ with $\|D\| < \epsilon$, we have*

$$\ker(F + D) \preceq \ker F \quad \operatorname{Im}(F + D)^{\perp} \preceq \operatorname{Im} F^{\perp}.$$

Proof. Since $F \in \mathcal{M}\Phi(M)$ has closed image, it has the matrix $\begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$ with respect to the decomposition $M = \ker F^{\perp} \tilde{\oplus} \ker F \xrightarrow{F} \operatorname{Im} F \tilde{\oplus} \operatorname{Im} F^{\perp} = M$, where F_1 is an isomorphism by [10, Th. 2.3.3]. By the proof of [10, Lemma 2.7.10], there exists an $\epsilon > 0$ such that, if $\|F - \tilde{D}\| < \epsilon$ for some $\tilde{D} \in B^a(M)$, then $\tilde{D}$ has the matrix $\begin{bmatrix} \tilde{D}_1 & 0 \\ 0 & \tilde{D}_4 \end{bmatrix}$ with respect to the decomposition

$$M = U_1(\ker F^{\perp}) \tilde{\oplus} U_1(\ker F) \xrightarrow{\tilde{D}} U_2^{-1}(\operatorname{Im} F) \tilde{\oplus} U_2^{-1}(\operatorname{Im} F^{\perp}) = M,$$

where U_1, U_2 and $\tilde{D}_1$ are isomorphisms. Then it follows that $\ker \tilde{D} \subseteq U_1(\ker F) \cong \ker F$. Set $D = \tilde{D} - F$; then $\tilde{D} = F + D$. Hence, $\ker(F + D) \preceq \ker F$. Note now that $U_2^{-1}(\operatorname{Im} F) \subseteq \operatorname{Im} \tilde{D}$. Hence, $\operatorname{Im} \tilde{D}^{\perp} \cap U_2^{-1}(\operatorname{Im} F) = \{0\}$, and therefore, $P_{U_2^{-1}(\operatorname{Im} F^{\perp})_{|_{\operatorname{Im} \tilde{D}^{\perp}}}}$ is injective, where $P_{U_2^{-1}(\operatorname{Im} F^{\perp})}$ stands for the projection onto $U_2^{-1}(\operatorname{Im} F^{\perp})$ along $U_2^{-1}(\operatorname{Im} F)$. Since $\tilde{D} \in \mathcal{M}\Phi(M)$, it follows that $\operatorname{Im} \tilde{D}^{\perp}$ is finitely generated, and hence, self-dual. By [10, Corollary 3.6.7], it follows that $\operatorname{Im} \tilde{D}^{\perp}$ is isomorphic to a direct summand of $U_2^{-1}(\operatorname{Im} F^{\perp})$. Since $U_2^{-1}(\operatorname{Im} F^{\perp}) \cong \operatorname{Im} F^{\perp}$, it follows that $\operatorname{Im} \tilde{D}^{\perp} \preccurlyeq \operatorname{Im} F^{\perp}$.

Remark 3.23. Lemma 3.22 is also valid for the case in which $F \in \widehat{\mathcal{M}\Phi}(M)$ with closed image because, in this case, by Proposition 3.1, there exists a decomposition

$$M = \ker F^0 \tilde{\oplus} \ker F \xrightarrow{F} \operatorname{Im} F \tilde{\oplus} \operatorname{Im} F^0 = M,$$

and $F_{|_{\ker F^0}}$ is an isomorphism onto $\operatorname{Im} F$. Proceeding as the proof of Lemma 3.22, we see that $\ker(F + G) \preceq \ker F$ and $\operatorname{Im}(F + G)^{\perp} \preceq \operatorname{Im} F^0$ if $G \in B(H_{\mathcal{A}})$ is such that $\|G\|$ is sufficiently small. If $\overline{\operatorname{Im}(F + G)}$ is complementable, then $\operatorname{Im}(F + G)^0 \preceq \operatorname{Im} F^0$ (where $\operatorname{Im}(F + G)^0$ stands for the complement of $\overline{\operatorname{Im}(F + G)}$).

Definition 3.24. Let M be a countably generated Hilbert W^*- module. For $F \in \mathcal{M}\Phi(M)$, we say that F *satisfies condition* $(*)$ if

(1) $\operatorname{Im} F^n$ is closed for all n,

(2) $F(\bigcap_{n=1}^{\infty} \operatorname{Im}(F^n)) = \bigcap_{n=1}^{\infty} \operatorname{Im}(F^n)$.

If we have a decreasing sequence of complementable submodules $N_k's$, then their intersection (for C^*-algebras) is not complementable in general; however, it is complementable for W^*-algebras. This is true because one can define a w^*-(or weak) direct sum of submodules, in contrast to the standard l_2 construction. Let $N_{k-1} = N_k \oplus L_k$. Then we can define $w^* - \oplus_k L_k$ as the set of sequences (x_k), $x_k \in L_k$, such that the sum $\sum_{k=1}^{\infty} \langle x_k, x_k \rangle$ is convergent in A with respect to the *-strong topology, rather than to the norm topology. Then it can readily be seen that $N_0 = \bigcap_{k=1}^{\infty} N_k \oplus (w^* - \oplus_k L_k)$.

Note that, if M is an ordinary Hilbert space, then (*) is always satisfied for any $F \in \mathcal{M}\Phi(M)$ by [16, Th. 1.1.9]. There are also other examples of Hilbert W^*-modules or which condition (*) is automatically satisfied for an $\mathcal{A}$-Fredholm operator F as long as F has closed image.

Example 3.25. Let $\mathcal{A}$ be a commutative von Neumann algebra with a cyclic vector, i.e., $\mathcal{A} \cong L^{\infty}(X, \mu)$, where X is a compact topological space and u is a Borel probability measure. Consider $\mathcal{A}$ as a Hilbert module over itself. If F is an $\mathcal{A}$-linear operator on $\mathcal{A}$, we can readily see that $\operatorname{Im}(F^k) = \operatorname{Span}_{\mathcal{A}}\{(F(1))^k\}$ for all k. Let $S = (F(1)^{-1}(\{0\}))^c$. Then one can show that $\operatorname{Im} F = \operatorname{Im} F^k = \operatorname{Span}_{\mathcal{A}}\{\chi_S\}$ for all k if we assume that $F(1)$ is bounded away from 0 on S, and hence, invertible on S. However, if F is $\mathcal{A}$-Fredholm with closed image, then this is the case. Indeed,

$$\ker F = \{f \in \mathcal{A} \mid f_{|_S} = 0\mu \quad \text{a.e. on} \quad S = \operatorname{Span}_{\mathcal{A}}\{\chi_{S^c}\}, \text{and thus}, \ker F^{\perp} = \operatorname{Span}_{\mathcal{A}}\{\chi_S\}.$$

Since F is then bounded below on $\ker F^{\perp}$, we have $\|F(f)\|_{\infty} = \|fF(1)\|_{\infty} \geqslant C\|f\|_{\infty}$ for all f vanishing μ-almost everywhere on S^c and for some constant $C > 0$. However, if

$$\mu\left(\left(F(1)^{-1}\left(B(0, \frac{1}{n}\right)\right)\right) \cap S) > 0 \quad \forall n,$$

then, letting $f_n = \chi_{((F(1)^{-1}(B(0,\frac{1}{n})))\cap S)}$, we see that $\|f_n\|_{\infty} = 1$, for all n and

$$F(f_n) = f_n F(1) = \chi_{((F(1)^{-1}(B(0,\frac{1}{n})))\cap S)} F(1).$$

It follows that F is not hounded below on $(\ker F)^{\perp}$, a contradiction. Note now that

$$\mathrm{Im}(F) = \mathrm{Im}(F^k) = \mathrm{Span}_{\mathcal{A}}\{\chi_s\} = (\ker F)^{\perp} \quad \forall k,$$

and thus, $F(\mathrm{Im}^{\infty}(F)) = \mathrm{Im}^{\infty}(F)$, where $\mathrm{Im}^{\infty}(F)$ stands for $\bigcap_{k=1}^{\infty} \mathrm{Im}(F^k)$.

Recall that, for a W^*-algebra $\mathcal{A}$, $G(\mathcal{A})$ stands for the set of all invertible elements in $\mathcal{A}$ and $Z(\mathcal{A}) = \{\beta \in \mathcal{A} \mid \beta\alpha = \alpha\beta \text{ for all } \alpha \in \mathcal{A}\}$. Then the following theorem holds.

Theorem 3.26. *Let $F \in \mathcal{M}\Phi(\tilde{M})$, where $\tilde{M}$ is a countably generated Hilbert $\mathcal{A}$-module. Suppose that F satisfies (*). Then there exists an $\epsilon > 0$ such that, if $\alpha \in Z(\mathcal{A}) \cap G(\mathcal{A})$ and $\|\alpha\| < \epsilon$, then $[\ker(F - \alpha I)] + [N_1] = [\ker F]$ and $[\mathrm{Im}(F - \alpha I)^{\perp}] + [N_1] = [\mathrm{Im}(F)^{\perp}]$ for some chosen finitely generated closed submodule N_1.*

Proof. Since $F \in \mathcal{M}\Phi(\tilde{M})$ has closed image, it follows from Lemma 3.22 that there exists an $\epsilon_1 > 0$ for which, if $\|\alpha\| < \epsilon_1, \alpha \in Z(\mathcal{A}) \cap G(\mathcal{A})$, then

$$\ker(F - \alpha I) \preceq \ker F, \quad \mathrm{Im}(F - \alpha I)^{\perp} \preceq \mathrm{Im}\, F^{\perp}$$

and $\mathrm{index}(F - \alpha I) = \mathrm{index} F$ by the proof of [10, Lemma 2.7.10]. Now, by the same arguments as in the proof of [16, Th. 1.7.7], since $\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$, we have

$$\ker(F - \alpha I) \subseteq \mathrm{Im}^{\infty}(F) := \bigcap_{n=1}^{\infty} \mathrm{Im}(F^n).$$

Since $\mathrm{Im}^{\infty}(F)$ is orthogonally complementable in $\tilde{M}$, the orthogonal projection $P_{\mathrm{Im}^{\infty}(F)^{\perp}}$ onto $\mathrm{Im}^{\infty}(F)^{\perp}$ along $\mathrm{Im}^{\infty}(F)$ exists, and

$$(\ker F \cap \mathrm{Im}^{\infty}(F)) = \ker P_{\mathrm{Im}^{\infty}(F)^{\perp}|_{\ker F}}.$$

Since $\ker F$ is self-dual (it is finitely generated), it follows, from [10, Corollary 3.6.4] that, $\ker F \cap \mathrm{Im}^{\infty}(F)$ is an orthogonal direct summand in $\ker F$, and thus, $\ker F = (\ker F \cap \mathrm{Im}^{\infty}(F)) \oplus N_1$ for some closed submodule N_1. Therefore, $\ker F_0 = \ker F \cap M$ is finitely generated as a direct summand in $\ker F$ (which is finitely generated itself). Since $\ker F \cap M$ is finitely generated, $\ker F \cap M$ is orthogonally complementable in M by [10, Lemma 2.3.7], and thus, $M = (\ker F \cap M) \oplus M'$ for some closed submodule M'. On M', the mapping F_0 is an isomorphism from M' onto M, and thus, $F_0 \in \mathcal{M}\Phi(M)$ (recall that $M = (\ker F \cap M) \oplus M'$), and $\ker F_0 = \ker F \cap M$, where the latter is finitely generated). By Lemma 3.22, there exists in $\epsilon_2 > 0$ such that, if $\|\alpha\| < \epsilon_2$,$\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$, then $\ker(F_0 - \alpha I_{|M}) \preceq \ker F_0$ and $\mathrm{Im}(F_0 - \alpha I_{|M})^{\perp} \preceq \mathrm{Im}\, F_0^{\perp}$ in M while $\mathrm{index}(F_0 - \alpha I) = \mathrm{index} F_0 = [\ker F_0]$ since F_0 is surjective. Since $\mathrm{Im}\, F_0^{\perp} = \{0\}$ (in M), because F_0 is surjective, we have

$$\mathrm{Im}(F_0 - \alpha I)^{\perp} = 0 \quad \text{for all} \quad \|\alpha\| < \epsilon_2, \quad \alpha \in G(\mathcal{A}) \cap Z(\mathcal{A}),$$

since $\mathrm{Im}(F_0 - \alpha I_{|M})^{\perp} \preceq \mathrm{Im}\, F_0^{\perp}$ for all $\|\alpha\| < \epsilon_2, \alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$. Recall that $\ker(F - \alpha I) \subseteq \mathrm{Im}^{\infty}(F) = M$. Therefore,

$$[\ker(F - \alpha I)] = [\ker(F_0 - \alpha I_{|M})] = \mathrm{index}(F_0 - \alpha I_{|M}) = \mathrm{index} F_0 = [\ker F_0].$$

This holds whenever $\|\alpha\| < \epsilon_2$, $\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$. Now, $\ker F_0 = \ker F \cap M$ and $\ker F = (\ker F \cap M) \oplus N_1$. Therefore, if $\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$ and $\|\alpha\| < \epsilon_2$, then $[\ker F] = [\ker F \cap M] + [N_1] = [\ker F_0] + [N_1] = [\ker(F - \alpha I)] + [N_1]$ whenever $\|\alpha\| < \epsilon_2$,$\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$. If, in addition, $\|\alpha\| < \epsilon_1$, then, as we have seen at the beginning of this proof, by the choice of ϵ_1, we have $\mathrm{index}(F - \alpha I) = \mathrm{index} F$. Thus, if $\|\alpha\| < \min\{\epsilon_1, \epsilon_2\}$ for $\alpha \in G(\mathcal{A}) \cap Z(\mathcal{A})$, then $\mathrm{index}(F - \alpha I) = \mathrm{index} F$, and $[\ker F] = [\ker(F - \alpha I)] + [N_1]$. It follows that $[\mathrm{Im}\, F^{\perp}] = [\mathrm{Im}(F - \alpha I)^{\perp}] + [N_1]$.

Remark 3.27. If $\mathcal{A}$ is a factor, then Theorem 3.26 is of interest in the case of finite factors, since $K(\mathcal{A})$ is trivial otherwise.

ACKNOWLEDGMENT

I am especially grateful to my research supervisor Professor Vladimir M. Manuilov for carefully reading my paper and for inspiring comments and suggestions that led to an improved presentation of the text. I am also grateful to Professor Dragan S. Djordjevic for suggesting the research topic of the paper and for introducing the relevant reference books to me.

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