ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 26, No. 4, 2019, pp. 428–453. © Pleiades Publishing, Ltd., 2019.

Convergence to Stationary States and Energy Current for Infinite Harmonic Crystals

T. V. Dudnikova

Keldysh Institute of Applied Mathematics, Russian Academy of Sciences
Miusskaya sq. 4, Moscow 125047, Russia,
E-mail: tdudnikov@mail.ru

Received June 16, 2019, Revised June 25, 2019, Accepted July 10, 2019

Abstract. We consider a d-dimensional harmonic crystal, $d \geqslant 1$, and study the Cauchy problem with random initial data. The distribution μ_t of the solution at time $t \in \mathbb{R}$ is studied. We prove the convergence of correlation functions of the measures μ_t to a limit for large times. The explicit formulas for the limiting correlation functions and for the energy current density (in the mean) are obtained in terms of the initial covariance. Furthermore, we prove the weak convergence of μ_t to a limit measure as $t \to \infty$. We apply these results to the case when initially some infinite "parts" of the crystal have Gibbs distributions with different temperatures. In particular, we find stationary states in which there is a constant nonzero energy current flowing through the crystal. We also study the initial boundary value problem for the harmonic crystal in the half-space with zero boundary condition and obtain similar results.

DOI 10.1134/S1061920819040034

1. INTRODUCTION

We study the Cauchy problem for a harmonic crystal in dimension d with n components, $d, n \geqslant 1$. We assume that the initial data $Y_0(x)$, $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, of the problem is a random element of the Hilbert space $\mathcal{H}_\alpha$ consisting of real sequences, see Definition 2.1 below. The distribution of $Y_0(x)$ is a probability measure μ_0 with zero mean value. We assume that the covariance $Q_0(x, y)$ of μ_0 decreases like $|x-y|^{-N}$ as $|x-y| \to \infty$ for some $N > d$. Furthermore, we impose the condition **S3** (see formulas (2.10)–(2.12) below) which means roughly that $Y_0(x)$ is close to different translation-invariant processes $Y_{\mathbf{n}}(x)$ with distributions $\mu_{\mathbf{n}}$ as $(-1)^{n_j} x_j \to +\infty$ for all $j = 1, \ldots, k$, with some $k \in \{1, \ldots, d\}$. Here $\mathbf{n}$ stands for the vector $\mathbf{n} = (n_1, \ldots, n_k)$, where all $n_j \in \{1, 2\}$. Given $t \in \mathbb{R}$, denote by μ_t the probability measure that gives the distribution of the solution $Y(x, t)$ to the dynamical equations with random initial data Y_0. We study the asymptotics of μ_t as $t \to \infty$. The first objective is to prove the convergence of the correlation functions of μ_t to a limit,

$$Q_t(x, y) \equiv \int_{\mathcal{H}_\alpha} (Y_0(x) \otimes Y_0(y))\, \mu_t(dY_0) \to Q_\infty(x, y), \quad t \to \infty, \quad x, y \in \mathbb{Z}^d. \tag{1.1}$$

The explicit formulas for the limit covariance Q_∞ are given in (2.14)–(2.19). They allow us to derive the expression for the limiting mean energy current density $\mathbf{J}_\infty$ in the terms of the initial covariance $Q_0(x, y)$.

We apply our results to a particular case when $\mu_{\mathbf{n}}$ are Gibbs measures with different temperatures $T_{\mathbf{n}} > 0$. Therefore, our model can be considered as a "system + 2^k reservoirs", where "reservoirs" consist of crystal particles lying in 2^k regions of the form $\{x \in \mathbb{Z}^d : (-1)^{n_j} x_j > a$ for all $j = 1, \ldots, k$, where $n_j = 1$ or $2\}$ with some $a > 0$, and the "system" is the remaining part of the crystal. At $t = 0$, the reservoirs have Gibbs distributions with corresponding temperatures $T_{\mathbf{n}}$, $\mathbf{n} = (n_1, \ldots, n_k)$. In the case of $d = 1$, a similar model was studied by Spohn and Lebowitz [24]. We show that the energy current density $\mathbf{J}_\infty$ is a constant vector satisfying formulas (4.4) and (4.5).

Furthermore, under additional symmetry conditions on the harmonic crystal, the coordinates of the energy current $\mathbf{J}_\infty \equiv (J^1_\infty, \dots, J^d_\infty)$ are of the form

$$J^l_\infty = \begin{cases} -c_l \sum{}' (T_{\mathbf{n}}|_{n_l=2} - T_{\mathbf{n}}|_{n_l=1}) & \text{for} \quad l = 1, \dots, k, \\ 0 & \text{for} \quad l = k+1, \dots, d, \end{cases} \tag{1.2}$$

with some constants $c_l > 0$. Here the summation $\sum{}'$ is taken over all n_j with $j \neq l$.

Our second result gives the (weak) convergence of the measures μ_t on the Hilbert space $\mathcal{H}_\alpha$ with $\alpha < -d/2$ to a limit measure μ_∞,

$$\mu_t \rightharpoondown \mu_\infty, \quad t \to \infty. \tag{1.3}$$

This means the convergence of the integrals

$$\int f(Y)\mu_t(dY) \to \int f(Y)\mu_\infty(dY) \quad \text{as} \quad t \to \infty$$

for any bounded continuous functional f on $\mathcal{H}_\alpha$. Furthermore, the limit measure μ_∞ is a translation-invariant Gaussian measure on $\mathcal{H}_\alpha$ and has the mixing property.

For infinite one-dimensional (1D) chains of harmonic oscillators, the results (1.1) and (1.3) were established by Boldrighini, Pellegrinotti and Triolo [1] and by Spohn and Lebowitz [24]. In earlier works, Lebowitz et al. [23, 4] and Nakazawa [21] analyzed the stationary energy current through the finite 1D chain of harmonic oscillators in contact with external heat reservoirs at different temperatures. For $d \geqslant 1$, the convergence (1.3) has been obtained for the first time by Lanford and Lebowitz [18] for initial measures which are absolutely continuous with respect to the canonical Gaussian measure. We consider a more general class of initial measures with the mixing condition and do not assume absolute continuity. The mixing condition was first introduced by Dobrushin and Sukhov for ideal gas [6]. Using the mixing condition, we have proved the convergence for the wave and Klein–Gordon equations (see [9] and the references therein) for non translation invariant initial measures μ_0. For multi-dimensional crystals, the results (1.1) and (1.3) were obtained in [7] for translation invariant measures μ_0. The present paper develops our previous work [8], where the assertions (1.1)–(1.3) were proved in the case $k = 1$.

In this paper, we also study the initial-boundary value problem for harmonic crystals in the half-space $\mathbb{Z}^d_+ = \{x \in \mathbb{Z}^d : x_1 > 0\}$ with *zero* boundary condition (as $x_1 = 0$) and obtain the results similar to (1.1) and (1.3). This generalizes the results of [10] to the more general class of initial measures. Furthermore, we calculate the limiting energy current density $\mathbf{J}_{+,\infty}(x_1)$, see formulas (6.12)–(6.15) below. In particular, if $d = 1$, then $\mathbf{J}_{+,\infty}(x_1) \equiv 0$. For any $d \geqslant 2$, $\mathbf{J}_{+,\infty}(0) = 0$. For $d \geqslant 2$ and $x_1 > 0$, the coordinates of $\mathbf{J}_{+,\infty}(x_1)$ are of a form similar to (1.2), but with positive functions $c_l = c_l(x_1)$ if $l = 2, \dots, k$, and vanish if $l = 1, k+1, \dots, d$. Moreover, $\mathbf{J}_{+,\infty}(x_1)$ tends to a limit as $x_1 \to +\infty$ (see formula (6.16)). For the 1D infinite chain of harmonic oscillators on the half-line with *nonzero* boundary condition, we prove the results (1.1) and (1.3) in [11] and show that there is a negative limiting energy current at origin (see [11, Remark 2.11]).

There is a large literature devoted to the study of the return to equilibrium, convergence to nonequilibrium states and heat conduction for nonlinear systems, see [2, 19, 25] and the survey book [20] for an extensive list of references. For instance, ergodic properties and long time behavior were studied for weak perturbation of the infinite chain of harmonic oscillators as a model of 1D harmonic crystals with defects by Fidaleo and Liverani [14] and for the finite chain of anharmonic oscillators coupled to a single heat bath by Jakšić and Pillet [15]. A finite chain of nonlinear oscillators coupled to two heat reservoirs was studied by Eckmann, Rey-Bellet and others [12, 13, 22]. For such systems, the existence of nonequilibrium states and the convergence to them were investigated in [12, 22]. In [13], Eckmann, Pillet, and Rey-Bellet showed that heat (in the mean) flows from the hot reservoir to the cold one. Fourier's law for a harmonic crystal with stochastic reservoirs was proved by Bonetto, Lebowitz, and Lukkarinen [3]. In the present paper, we find stationary nonequilibrium states in which there is a nonzero energy current flowing through the infinite d-dimensional harmonic crystal.

The paper is organized as follows. In Section 2, we impose the needed conditions on the model and on the initial measures μ_0 and state the main results. In Section 3, we construct examples of random initial data satisfying all assumptions imposed. The application to Gibbs initial measures and the derivation of the formula (1.2) are given in Section 4. In Section 5.1, uniform bounds for covariance of μ_t are obtained, and the proof of (1.3) is discussed. The asymptotics (1.1) is proved in Section 5.2. In Section 6, we study the initial-boundary value problem for harmonic crystals in the half-space and prove results similar to (1.1)–(1.3).

2. MAIN RESULTS

2.1. The Model

We consider a Bravais lattice in $\mathbb{R}^d$ with a unit cell which contains a finite number of atoms. For notational simplicity, the lattice is assumed to be simple hypercubic. Let $u(x)$ be the field of displacements of the crystal atoms in cell x ($x \in \mathbb{Z}^d$) from the equilibrium position. In the harmonic approximation, the field $u(x)$ is governed by equations of the following type (see, e.g., [18]):

$$\begin{cases} \ddot{u}(x,t) = -\sum_{y\in\mathbb{Z}^d} V(x-y)u(y,t), & x\in\mathbb{Z}^d, \quad t\in\mathbb{R}, \\ u|_{t=0} = u_0(x), & \dot{u}|_{t=0} = v_0(x). \end{cases} \tag{2.1}$$

Here $u(x,t) = (u_1(x,t),\dots,u_n(x,t)), u_0(x) = (u_{01}(x),\dots,u_{0n}(x)) \in \mathbb{R}^n$ and correspondingly for $v_0(x)$, $V(x)$ is the real interaction (or force) matrix, $(V_{kl}(x))$, $k,l = 1,\dots,n$. Physically $n = d \times$ (number of atoms in the unit cell). Here we take n to be an arbitrary positive integer. The dynamics (2.1) is invariant under lattice translations.

Let us denote $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot,t), \dot{u}(\cdot,t))$, $Y_0 = (Y_0^0, Y_0^1) \equiv (u_0(\cdot), v_0(\cdot))$. Then (2.1) takes the form of an evolution equation

$$\dot{Y}(t) = \mathcal{A}(Y(t)), \quad t\in\mathbb{R}; \quad Y(0) = Y_0. \tag{2.2}$$

This is a linear Hamiltonian system, since $\mathcal{A}(Y) = J\begin{pmatrix} \mathcal{V} & 0 \\ 0 & I \end{pmatrix} Y = J\nabla H(Y)$ and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Here $\mathcal{V}$ is a convolution operator with matrix kernel V, I is the unit matrix, and H is the Hamiltonian functional

$$H(Y) := \frac{1}{2}\langle v,v\rangle + \frac{1}{2}\langle u,\mathcal{V}u\rangle, \quad Y = (u,v), \tag{2.3}$$

where the kinetic energy is given by

$$(1/2)\langle v,v\rangle = (1/2)\sum_{x\in\mathbb{Z}^d} |v(x)|^2$$

and the potential energy by

$$(1/2)\langle u,\mathcal{V}u\rangle = (1/2)\sum_{x,y\in\mathbb{Z}^d} (u(x), V(x-y)u(y)),$$

$(\cdot\,,\cdot)$ stands for the real scalar product in Euclidean space $\mathbb{R}^n$ (or in $\mathbb{R}^d$).

We assume that the initial data Y_0 belongs to the phase space $\mathcal{H}_\alpha$, $\alpha\in\mathbb{R}$.

Definition 2.1. $\mathcal{H}_\alpha$ is the Hilbert space of pairs $Y \equiv (u(x), v(x))$ of $\mathbb{R}^n$-valued functions of $x \in \mathbb{Z}^d$ endowed with the norm

$$\|Y\|_\alpha^2 \equiv \sum_{x\in\mathbb{Z}^d} \langle x\rangle^{2\alpha}(|u(x)|^2 + |v(x)|^2) < \infty, \quad \langle x\rangle := \sqrt{1+|x|^2}.$$

We impose the following conditions **E1**–**E6** on the matrix V.

E1. There exist positive constants C and γ such that $\|V(x)\| \leqslant Ce^{-\gamma|x|}$ for $x \in \mathbb{Z}^d$, where $\|V(x)\|$ denotes the matrix norm. Let $\hat V(\theta)$ be the Fourier transform of $V(x)$, with the convention

$$\hat V(\theta) = F_{x\to\theta}[V(x)] \equiv \sum_{x\in\mathbb{Z}^d} e^{i(x,\theta)}V(x)\,, \quad \theta \in \mathbb{T}^d,$$

where $\mathbb{T}^d$ denotes the d-torus $\mathbb{R}^d/(2\pi\mathbb{Z})^d$.

E2. V is real and symmetric, i.e., $V_{lk}(-x) = V_{kl}(x) \in \mathbb{R}$, $k, l = 1, \ldots, n$, $x \in \mathbb{Z}^d$.

The conditions **E1** and **E2** imply that $\hat V(\theta)$ is a real-analytic Hermitian matrix-valued function in $\theta \in \mathbb{T}^d$.

E3. The matrix $\hat V(\theta)$ is nonnegative definite for every $\theta \in \mathbb{T}^d$.

Let us define the Hermitian nonnegative definite matrix,

$$\Omega(\theta) = (\hat V(\theta))^{1/2} \geqslant 0. \tag{2.4}$$

$\Omega(\theta)$ has the eigenvalues ("dispersion relations") $0 \leqslant \omega_1(\theta) < \omega_2(\theta) < \cdots < \omega_s(\theta)$, $s \leqslant n$, and the corresponding spectral projections $\Pi_\sigma(\theta)$ with multiplicity $r_\sigma = \operatorname{tr}\Pi_\sigma(\theta)$.

Lemma 2.2. (*see* [7, Lemma 2.2]) *Let conditions* **E1** *and* **E2** *be fulfilled. Then there exists a closed subset $\mathcal{C}_* \subset \mathbb{T}^d$ of zero Lebesgue measure such that the following assertions hold.* (i) *For any point $\Theta \in \mathbb{T}^d \setminus \mathcal{C}_*$, there exists a neighborhood $\mathcal{O}(\Theta)$ such that each band function $\omega_\sigma(\theta)$ can be chosen as real-analytic function in $\mathcal{O}(\Theta)$.* (ii) *The eigenvalue $\omega_\sigma(\theta)$ has constant multiplicity in $\mathbb{T}^d \setminus \mathcal{C}_*$.* (iii) *The following spectral decomposition holds,*

$$\Omega(\theta) = \sum_{\sigma=1}^{s} \omega_\sigma(\theta)\Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus \mathcal{C}_*, \tag{2.5}$$

where $\Pi_\sigma(\theta)$ is the orthogonal projection in $\mathbb{R}^n$. Π_σ is a real-analytic function on $\mathbb{T}^d \setminus \mathcal{C}_*$.

Below we denoted by $\omega_\sigma(\theta)$ the local real-analytic functions from Lemma 2.2 (i). The next condition on V is the following.

E4. For each $l = 1, \ldots, d$ and $\sigma = 1, \ldots, s$, $\partial_{\theta_l}\omega_\sigma(\theta)$ does not vanish identically on $\mathbb{T}^d \setminus \mathcal{C}_*$.

To prove the convergence (1.3), we need a stronger condition **E4'**.

E4'. For each $\sigma = 1, \ldots, s$, the determinant of the matrix of second partial derivatives of $\omega_\sigma(\theta)$ does not vanish identically on $\mathbb{T}^d \setminus \mathcal{C}_*$.

Write

$$\mathcal{C}_0 = \{\theta \in \mathbb{T}^d : \det \hat V(\theta) = 0\}, \quad \mathcal{C}_\sigma = \bigcup_{l=1}^{d} \{\theta \in \mathbb{T}^d \setminus \mathcal{C}_* : \partial_{\theta_l}\omega_\sigma(\theta) = 0\}, \ \ \sigma = 1, \ldots, s. \tag{2.6}$$

Then the Lebesgue measure of $\mathcal{C}_\sigma$ vanishes, $\sigma = 0, 1, \ldots, s$ (see [7, Lemma 2.3]).

E5. For each $\sigma \neq \sigma'$, the identities $\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \equiv \text{const}_\pm$, $\theta \in \mathbb{T}^d \setminus \mathcal{C}_*$, do not hold with $\text{const}_\pm \neq 0$.

The condition **E5** can be weakened to the condition **E5'**, see Remark 2.7 below.

E6. $\|\hat V^{-1}(\theta)\| \in L^1(\mathbb{T}^d)$.

Example 2.3. For any $d,n\geqslant 1$, we consider the *nearest neighbor crystal* for which

$$\langle u,\mathcal{V}u\rangle=\sum_{l=1}^{n}\sum_{x\in\mathbb{Z}^d}\left(\sum_{i=1}^{d}\kappa_l|u_l(x+e_i)-u_l(x)|^2+m_l^2|u_l(x)|^2\right),\ \ \kappa_l>0,\ \ m_l\geqslant 0,$$

where $e_i=(\delta_{i1},\dots,\delta_{id})$. Then

$$V_{kl}(x)=0\quad\text{for}\quad k\neq l,\quad V_{ll}(x)=\begin{cases}-\kappa_l & \text{for}\ \ |x|=1,\\ 2d\kappa_l+m_l^2 & \text{for}\ \ \ x=0,\\ 0 & \text{for}\ \ |x|\geqslant 2,\end{cases}\quad l=1,\dots,n. \tag{2.7}$$

Hence, the eigenvalues of $\hat{V}(\theta)$ are

$$\tilde{\omega}_l(\theta)=\sqrt{2\kappa_l(1-\cos\theta_1)+\dots+2\kappa_l(1-\cos\theta_d)+m_l^2},\quad l=1,\dots,n. \tag{2.8}$$

These eigenvalues still have to be labelled according to magnitude and degeneracy as in Lemma 2.2. Clearly conditions **E1**–**E5** hold and $\mathcal{C}_*=\emptyset$. If all $m_l>0$, then the set $\mathcal{C}_0$ is empty and condition **E6** is fulfilled. Otherwise, if $m_l=0$ for some l, then $\mathcal{C}_0=\{0\}$. In this case, **E6** is equivalent to the condition $\tilde{\omega}_l^{-2}(\theta)\in L^1(\mathbb{T}^d)$ that holds if $d\geqslant 3$. Therefore, conditions **E1**–**E6** hold for (2.7) provided either (i) $d\geqslant 3$, or (ii) $d=1,2$ and all $m_l>0$.

The following proposition is proved in [18, p. 150; 1, p. 128].

Proposition 2.4. *Let conditions* **E1** *and* **E2** *hold, and choose some* $\alpha\in\mathbb{R}$. *Then for any* $Y_0\in\mathcal{H}_\alpha$, *there exists a unique solution* $Y(t)\in C(\mathbb{R},\mathcal{H}_\alpha)$ *to the Cauchy problem* (2.2); *the operator* $U(t):Y_0\mapsto Y(t)$ *is continuous in* $\mathcal{H}_\alpha$.

We assume that Y_0 in (2.2) is a *measurable random function* and denote by μ_0 a Borel probability measure on $\mathcal{H}_\alpha$ that gives the distribution of Y_0. The expectation with respect to μ_0 is denoted by $\mathbb{E}$. We impose the following conditions **S1**–**S3** on the initial measure μ_0.

S1. μ_0 has zero expectation value, $\mathbb{E}(Y_0(x))\equiv\int(Y_0(x))\,\mu_0(dY_0)=0$, $x\in\mathbb{Z}^d$.

S2. The initial correlation functions $Q_0^{ij}(x,y):=\mathbb{E}\left(Y_0^i(x)\otimes Y_0^j(y)\right)$, $x,y\in\mathbb{Z}^d$, satisfy the bound

$$|Q_0^{ij}(x,y)|\leqslant h(|x-y|),\quad\text{where}\ \ r^{d-1}h(r)\in L^1(0,+\infty). \tag{2.9}$$

Here for $a,b,c\in\mathbb{C}^n$, we denote by $a\otimes b$ the linear operator $(a\otimes b)c=a\sum_{j=1}^n b_jc_j$.

S3. Choose some $k\in\{1,\dots,d\}$. The initial covariance $Q_0(x,y)=(Q_0^{ij}(x,y))_{i,j=0,1}$ depends on the difference x_l-y_l for all $l=k+1,\dots,d$, i.e.,

$$Q_0(x,y)=q_0(\bar{x},\bar{y},\tilde{x}-\tilde{y}), \tag{2.10}$$

where $x=(x_1,\dots,x_d)\equiv(\bar{x},\tilde{x})$, $\bar{x}=(x_1,\dots,x_k)$, $\tilde{x}=(x_{k+1},\dots,x_d)$. Write

$$\mathcal{N}^k:=\{\mathbf{n}=(n_1,\dots,n_k),\ \text{where all}\ \ n_j\in\{1,2\}\}. \tag{2.11}$$

Suppose that for any $\varepsilon>0$, there exists an $N(\varepsilon)\in\mathbb{N}$ such that for any $\bar{y}\in\mathbb{Z}^k$ such that $(-1)^{n_j}y_j>N(\varepsilon)$ for each $j=1,\dots,k$, the following bound holds

$$|q_0(\bar{y}+\bar{z},\bar{y},\tilde{z})-q_{\mathbf{n}}(z)|<\varepsilon\quad\text{for any fixed}\ \ z=(\bar{z},\tilde{z})\in\mathbb{Z}^d\quad\text{and}\ \ \mathbf{n}\in\mathcal{N}^k. \tag{2.12}$$

Here $q_{\mathbf{n}}(z)$, $\mathbf{n} \in \mathcal{N}^k$, are the correlation matrices of some translation-invariant measures $\mu_{\mathbf{n}}$ with zero mean value in $\mathcal{H}_\alpha$.

In particular, if $k = 1$, then condition **S3** means that

$$Q_0(x,y) = q_0(x_1, y_1, \tilde{x} - \tilde{y}), \quad \text{where} \quad x = (x_1, \tilde{x}), \quad \tilde{x} = (x_2, \dots, x_d),$$

and

$$q_0(y_1 + z_1, y_1, \tilde{z}) \to \begin{cases} q_1(z) & \text{as} \quad y_1 \to -\infty, \\ q_2(z) & \text{as} \quad y_1 \to +\infty, \end{cases} \quad z = (z_1, \tilde{z}) \in \mathbb{Z}^d. \tag{2.13}$$

A measure μ is called *translation invariant* if $\mu(T_h B) = \mu(B)$ for $B \in \mathcal{B}(\mathcal{H}_\alpha)$ and $h \in \mathbb{Z}^d$, where $T_h Y(x) = Y(x - h)$, $x \in \mathbb{Z}^d$, $\mathcal{B}(\mathcal{H}_\alpha)$ stands for the Borel σ-algebra in $\mathcal{H}_\alpha$. Note that the initial measure μ_0 is not translation-invariant if $q_{\mathbf{n}} \neq q_{\mathbf{n}'}$ for some $\mathbf{n} \neq \mathbf{n}'$. Examples of μ_0 satisfying conditions **S1**–**S3** are given in Section 3.

2.2. Convergence of Correlations Functions

Definition 2.5. Let μ_t be a Borel probability measure in $\mathcal{H}_\alpha$ which gives the distribution of $Y(t)$, $\mu_t(B) = \mu_0(U(-t)B)$, $\forall B \in \mathcal{B}(\mathcal{H}_\alpha)$, $t \in \mathbb{R}$. The correlation functions of the measure μ_t are defined by

$$Q_t^{ij}(x,y) = \mathbb{E}\left(Y^i(x,t) \otimes Y^j(y,t)\right), \quad i,j = 0,1, \quad x,y \in \mathbb{Z}^d;$$

here $Y^i(x,t)$ are the components of the random solution $Y(t) = (Y^0(\cdot,t), Y^1(\cdot,t))$.

Denote by $\mathcal{Q}_t$ the quadratic form with matrix kernel $(Q_t^{ij}(x,y))_{i,j=0,1}$,

$$\mathcal{Q}_t(\Psi,\Psi) = \int |\langle Y, \Psi\rangle|^2 \, \mu_t(dY) = \sum_{i,j=0,1} \sum_{x,y \in \mathbb{Z}^d} \left(Q_t^{ij}(x,y), \Psi^i(x) \otimes \Psi^j(y)\right), \quad t \in \mathbb{R},$$

$\Psi = (\Psi^0, \Psi^1) \in \mathcal{S} := S \oplus S$, $S := S(\mathbb{Z}^d) \otimes \mathbb{R}^n$, where $S(\mathbb{Z}^d)$ denotes the space of real quickly decreasing sequences,

$$\langle Y, \Psi \rangle = \sum_{i=0,1} \sum_{x \in \mathbb{Z}^d} \left(Y^i(x), \Psi^i(x)\right).$$

Let us introduce the limiting correlation matrix $Q_\infty(x,y) = (Q_\infty^{ij}(x,y))_{i,j=0}^1$ as follows

$$Q_\infty(x,y) = q_\infty(x-y), \quad x,y \in \mathbb{Z}^d. \tag{2.14}$$

Here $q_\infty(x)$ has the form (in its Fourier transform)

$$\hat{q}_\infty(\theta) = \sum_{\sigma=1}^{s} \Pi_\sigma(\theta)(\mathbf{M}_{k,\sigma}^+(\theta) + i\,\mathbf{M}_{k,\sigma}^-(\theta))\Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus \mathcal{C}_*, \tag{2.15}$$

where $\Pi_\sigma(\theta)$ is the spectral projection from Lemma 2.2 (iii),

$$\mathbf{M}_{k,\sigma}^+(\theta) = \frac{1}{2^k} \sum_{\mathbf{n} \in \mathcal{N}^k} L_1^+(\hat{q}_{\mathbf{n}}(\theta)) \left[1 + S_{k,\mathbf{n}}^{\text{even}}(\omega_\sigma(\theta))\right], \quad \mathbf{M}_{k,\sigma}^-(\theta) = \frac{1}{2^k} \sum_{\mathbf{n} \in \mathcal{N}^k} L_2^-(\hat{q}_{\mathbf{n}}(\theta))\, S_{k,\mathbf{n}}^{\text{odd}}(\omega_\sigma(\theta)), \tag{2.16}$$

$$S_{k,\mathbf{n}}^{\text{even}}(\omega_\sigma) = \sum_{\text{even } m \in \{1,\dots,k\}} \sum_{(p_1,\dots,p_m) \in \mathcal{P}_m(k)} \operatorname{sgn}\left(\frac{\partial \omega_\sigma(\theta)}{\partial \theta_{p_1}}\right) \cdot \ldots \cdot \operatorname{sgn}\left(\frac{\partial \omega_\sigma(\theta)}{\partial \theta_{p_m}}\right) (-1)^{n_{p_1} + \dots + n_{p_m}},$$

$$S_{k,\mathbf{n}}^{\text{odd}}(\omega_\sigma) = \sum_{\text{odd } m \in \{1,\dots,k\}} \sum_{(p_1,\dots,p_m) \in \mathcal{P}_m(k)} \operatorname{sgn}\left(\frac{\partial \omega_\sigma(\theta)}{\partial \theta_{p_1}}\right) \cdot \ldots \cdot \operatorname{sgn}\left(\frac{\partial \omega_\sigma(\theta)}{\partial \theta_{p_m}}\right) (-1)^{n_{p_1} + \dots + n_{p_m}}, \tag{2.17}$$

The symbol $\mathcal{P}_m(k)$ denotes the collection of all m-combinations of the set $\{1,\dots,k\}$ (for instance, $\mathcal{P}_2(3)=\{(1,2),(2,3),(1,3)\}$),

$$L_1^{\pm}(\hat q_{\mathbf n}(\theta))=\frac{1}{2}\left(\hat q_{\mathbf n}(\theta)\pm C(\theta)\hat q_{\mathbf n}(\theta)C^*(\theta)\right),\quad L_2^{\pm}(\hat q_{\mathbf n}(\theta))=\frac{1}{2}\left(C(\theta)\hat q_{\mathbf n}(\theta)\pm \hat q_{\mathbf n}(\theta)C^*(\theta)\right), \tag{2.18}$$

$$C(\theta)=\begin{pmatrix}0 & \Omega(\theta)^{-1}\\ -\Omega(\theta) & 0\end{pmatrix},\quad C^*(\theta)=\begin{pmatrix}0 & -\Omega(\theta)\\ \Omega(\theta)^{-1} & 0\end{pmatrix}, \tag{2.19}$$

the matrices $q_{\mathbf n}$ are introduced in condition **S3**. In particular, if $k=1$, then formulas (2.16) become

$$\mathbf{M}_{1,\sigma}^{+}(\theta)=\frac{1}{2}\,L_1^{+}\left(\hat q_2(\theta)+\hat q_1(\theta)\right),\quad \mathbf{M}_{1,\sigma}^{-}(\theta)=\frac{1}{2}\,L_2^{-}\left(\hat q_2(\theta)-\hat q_1(\theta)\right)\operatorname{sgn}\left(\frac{\partial\omega_\sigma(\theta)}{\partial\theta_1}\right) \tag{2.20}$$

with the matrices q_1 and q_2 defined in (2.13). For $d=n=1$, formulas (2.20) were obtained in [1, p. 139]. For any $d,n\geqslant 1$ and $k=1$, these formulas were derived in [8].

Note that $\hat q_\infty\in L^1(\mathbb{T}^d)$ by Lemma 5.1 and condition **E6**. Moreover, by (2.15)–(2.19), the matrix $\hat q_\infty(\theta)$ satisfies the "equilibrium condition," i.e., $\hat q_\infty^{11}(\theta)=\hat V(\theta)\hat q_\infty^{00}(\theta)$, $\hat q_\infty^{10}(\theta)=-\hat q_\infty^{01}(\theta)$. We also have

$$(\hat q_\infty^{ii}(\theta))^*=\hat q_\infty^{ii}(\theta)\geqslant 0,\quad i=0,1,\quad (\hat q_\infty^{10}(\theta))^*=-\hat q_\infty^{10}(\theta).$$

The first result of the paper is the following theorem.

Theorem 2.6. *Let $d,n\geqslant 1$, $\alpha<-d/2$, and assume that conditions* **E1**–**E6** *and* **S1**–**S3** *hold. Then the convergence* (1.1) *is true, where Q_∞ is defined in* (2.14)–(2.19).

In Sec. 6, we study the initial boundary value problem for harmonic crystals with zero boundary condition and obtain results similar to Theorem 2.6, see Theorem 6.4 below.

Remark 2.7. Condition **E5** on the matrix V can be weakened. Namely, it suffices to impose the following restriction.

E5'. If for some $\sigma\neq\sigma'$, $\omega_\sigma(\theta)+\omega_{\sigma'}(\theta)\equiv \mathrm{const}_+$ with $\mathrm{const}_+\neq 0$, then

$$p_{\mathbf n,\sigma\sigma'}^{11}(\theta)-\omega_\sigma(\theta)\omega_{\sigma'}(\theta)p_{\mathbf n,\sigma\sigma'}^{00}(\theta)\equiv 0\quad\text{and}\quad \omega_\sigma(\theta)p_{\mathbf n,\sigma\sigma'}^{01}(\theta)+\omega_{\sigma'}(\theta)p_{\mathbf n,\sigma\sigma'}^{10}(\theta)\equiv 0.$$

If for some $\sigma\neq\sigma'$, $\omega_\sigma(\theta)-\omega_{\sigma'}(\theta)\equiv \mathrm{const}_-$ with $\mathrm{const}_-\neq 0$, then

$$p_{\mathbf n,\sigma\sigma'}^{11}(\theta)+\omega_\sigma(\theta)\omega_{\sigma'}(\theta)p_{\mathbf n,\sigma\sigma'}^{00}(\theta)\equiv 0\quad\text{and}\quad \omega_\sigma(\theta)p_{\mathbf n,\sigma\sigma'}^{01}(\theta)-\omega_{\sigma'}(\theta)p_{\mathbf n,\sigma\sigma'}^{10}(\theta)\equiv 0.$$

Here

$$p_{\mathbf n,\sigma\sigma'}^{ij}(\theta):=\Pi_\sigma(\theta)\hat q_{\mathbf n}^{ij}(\theta)\Pi_{\sigma'}(\theta),\ \ \theta\in\mathbb{T}^d,\ \ \sigma,\sigma'=1,\dots,s,\ \ i,j=0,1,\ \ \mathbf n\in\mathcal{N}^k. \tag{2.21}$$

This condition holds, for instance, for the canonical Gibbs measures $\mu_{\mathbf n}$ considered in Section 4.2.

Examples 2.8. We rewrite the formulas for q_∞ in some particular cases.

(i) In the case when the initial covariance is translation invariant, i.e., $Q_0(x,y)=q_0(x-y)$, the matrix $\hat q_\infty$ is of the form

$$\hat q_\infty(\theta)=\sum_{\sigma=1}^{s}\Pi_\sigma(\theta)L_1^{+}(\hat q_0(\theta))\Pi_\sigma(\theta),\quad \theta\in\mathbb{T}^d\setminus\mathcal{C}_*. \tag{2.22}$$

(ii) Let the initial covariance Q_0 satisfy a stronger condition than (2.12). Namely, assume that Q_0 has the form (2.10) and for any $z=(\bar z,\tilde z)\in\mathbb{Z}^d$, $\lim_{|\bar y|\to\infty}q_0(\bar y+\bar z,\bar y,\tilde z)=q_*(z)$. Then the condition (2.12) is fulfilled with $q_{\mathbf n}(z)=q_*(z)$ for any $\mathbf n\in\mathcal{N}^k$. In this case, Theorem 2.6 holds, and $\hat q_\infty$ is of the form (2.22) with $\hat q_*$ instead of $\hat q_0$. Therefore, Theorem 2.6 generalizes the result of [7, Prop. 3.2], where the convergence (1.1) was proved in the case when $Q_0(x,y)=q_0(x-y)$.

2.3. Weak Convergence of Measures

To prove the convergence (1.3) of the measures μ_t, we impose the stronger condition **S4** on μ_0 than the bound (2.9). To formulate this condition, let us denote by $\sigma(\mathcal{A})$, $\mathcal{A}\subset\mathbb{Z}^d$, the σ-algebra in $\mathcal{H}_\alpha$ generated by $Y_0(x)$ with $x\in\mathcal{A}$. Define the Ibragimov mixing coefficient of a probability measure μ_0 on $\mathcal{H}_\alpha$ by the rule (cf [16, Definition 17.2.2])

$$\varphi(r)\equiv \sup_{\substack{\mathcal{A},\mathcal{B}\subset\mathbb{Z}^d:\\ \operatorname{dist}(\mathcal{A},\mathcal{B})\geqslant r}}\ \sup_{\substack{A\in\sigma(\mathcal{A}),\,B\in\sigma(\mathcal{B})\\ \mu_0(B)>0}} \frac{|\mu_0(A\cap B)-\mu_0(A)\mu_0(B)|}{\mu_0(B)}.$$

Definition 2.9. We say that the measure μ_0 satisfies a *strong uniform Ibragimov mixing condition* if $\varphi(r)\to 0$ as $r\to\infty$.

S4. The initial mean "energy" density is uniformly bounded:

$$\mathbb{E}[|u_0(x)|^2+|v_0(x)|^2]=\operatorname{tr}Q_0^{00}(x,x)+\operatorname{tr}Q_0^{11}(x,x)\leqslant e_0<\infty,\quad x\in\mathbb{Z}^d. \tag{2.23}$$

Moreover, μ_0 satisfies the strong uniform Ibragimov mixing condition and

$$\int_0^\infty r^{d-1}\varphi^{1/2}(r)\,dr<\infty.$$

Remark 2.10. By [16, Lemma 17.2.3], conditions **S1** and **S4** imply the bound (2.9) with $h(r)=Ce_0\varphi^{1/2}(r)$, where e_0 is a constant from the bound (2.23).

For a probability measure μ on $\mathcal{H}_\alpha$, we denote by $\hat\mu$ the characteristic functional (Fourier transform),

$$\hat\mu(\Psi)\equiv\int\exp(i\langle Y,\Psi\rangle)\,\mu(dY),\quad \Psi\in\mathcal{S}.$$

A measure μ is called *Gaussian* (of zero mean) if its characteristic functional has the form $\hat\mu(\Psi)=\exp\{-\mathcal{Q}(\Psi,\Psi)/2\}$, where $\mathcal{Q}$ is a real nonnegative quadratic form in $\mathcal{S}$.

Theorem 2.11. *Let $d,n\geqslant 1$, $\alpha<-d/2$, and assume that conditions* **E1**–**E3**, **E4'**, **E5'**, **E6**, **S1**, **S3**, *and* **S4** *are fulfilled. Then the following assertions hold.*

(i) *The measures μ_t weakly converge in the Hilbert space $\mathcal{H}_\alpha$,*

$$\mu_t\to\mu_\infty\quad as\quad t\to\infty. \tag{2.24}$$

The limit measure μ_∞ is a Gaussian translation-invariant measure on $\mathcal{H}_\alpha$. The characteristic functional of μ_∞ is of the form $\hat\mu_\infty(\Psi)=\exp\{-\mathcal{Q}_\infty(\Psi,\Psi)/2\}$, $\Psi\in\mathcal{S}$, where $\mathcal{Q}_\infty$ is the quadratic form with the matrix kernel $Q_\infty(x,y)$ defined in (2.14).

(ii) *The measure μ_∞ is time stationary, i.e., $[U(t)]^*\mu_\infty=\mu_\infty$, $t\in\mathbb{R}$.*

(iii) *The flow $U(t)$ is mixing with respect to the measure μ_∞, i.e., for any $f,g\in L^2(\mathcal{H}_\alpha,\mu_\infty)$,*

$$\lim_{t\to\infty}\int f(U(t)Y)g(Y)\,\mu_\infty(dY)=\int f(Y)\,\mu_\infty(dY)\int g(Y)\,\mu_\infty(dY).$$

In particular, the flow $U(t)$ is ergodic with respect to the measure μ_∞.

For harmonic crystals in the half-space, the convergence (2.24) also holds. For details, see Section 6. Assertion (i) of Theorem 2.11 follows from Propositions 2.12 and 2.13.

Proposition 2.12. *Let the conditions* **E1**–**E3**, **E6**, **S1** *and* **S2** *hold. Then the family of measures* $\{\mu_t,\, t\in\mathbb{R}\}$ *is weakly compact in* $\mathcal{H}_\alpha$ *with any* $\alpha<-d/2$, *and the following bound holds*

$$\sup_{t\in\mathbb{R}}\mathbb{E}\|U(t)Y_0\|^2_\alpha<\infty. \tag{2.25}$$

Proposition 2.13. *Let conditions* **E1**–**E3**, **E4'**, **E5'**, **E6**, **S1**, **S3**, *and* **S4** *hold. Then for every* $\Psi\in\mathcal{S}$, *the characteristic functionals of* μ_t *converge to a Gaussian functional,*

$$\hat\mu_t(\Psi):=\int e^{i\langle Y,\Psi\rangle}\mu_t(dY)\to\exp\left\{-\frac{1}{2}\mathcal{Q}_\infty(\Psi,\Psi)\right\},\quad t\to\infty.$$

Proposition 2.12 (Proposition 2.13) provides the existence (respectively, the uniqueness) of the limit measure μ_∞. Proposition 2.12 is proved in Section 5.1. Proposition 2.13 can be proved using the technique from [8]. Assertion (ii) of Theorem 2.11 follows from (2.24) since the group $U(t)$ is continuous in $\mathcal{H}_\alpha$ by Proposition 2.4. The ergodicity and mixing of the limit measures μ_∞ follow by the same arguments as in [7].

Lemma 2.14. *Let conditions* **E1**–**E4**, **E5'**, *and* **E6** *hold. Assume that the initial measure* μ_0 *is Gaussian and satisfies conditions* **S1**–**S3**. *Then all assertions of Theorem* 2.11 *remain valid.*

This lemma follows from Theorem 2.6 and Proposition 2.12.

3. EXAMPLES OF INITIAL MEASURES

Now we construct Gaussian initial measures μ_0 satisfying conditions **S1–S3**. For $k=1$ (see the condition **S3**), an example of μ_0 is given in [8]. For any $k\geqslant1$, the measure μ_0 can be constructed in the following way. At first, for simplicity, we assume that $u_0,v_0\in\mathbb{R}^1$ and define the correlation functions $q^{ij}_{\mathbf{n}}(x-y)$, $\mathbf{n}\in\mathcal{N}^k$, which are zero for $i\ne j$, while for $i=0,1$,

$$\hat q^{ii}_{\mathbf{n}}(\theta):=F_{z\to\theta}[q^{ii}_{\mathbf{n}}(z)]\in L^1(\mathbb{T}^d),\quad \hat q^{ii}_{\mathbf{n}}(\theta)\geqslant0. \tag{3.1}$$

Then, by the Minlos theorem [5], there exist Borel Gaussian measures $\mu_{\mathbf{n}}$ on $\mathcal{H}_\alpha$, $\alpha<-d/2$, with correlation functions $q^{ij}_{\mathbf{n}}(x-y)$, because

$$\int\|Y\|^2_\alpha\,\mu_{\mathbf{n}}(dY)=\sum_{x\in\mathbb{Z}^d}\langle x\rangle^{2\alpha}\,\mathrm{tr}\left(q^{00}_{\mathbf{n}}(0)+q^{11}_{\mathbf{n}}(0)\right)=C(\alpha,d)\int_{\mathbb{T}^d}\mathrm{tr}\left(\hat q^{00}_{\mathbf{n}}(\theta)+\hat q^{11}_{\mathbf{n}}(\theta)\right)d\theta<\infty.$$

Further, we take functions $\overline{\zeta}_{\mathbf{n}}\in C(\mathbb{Z}^k)$ such that

$$\overline{\zeta}_{\mathbf{n}}(\bar x)=\zeta_{n_1}(x_1)\cdot\ldots\cdot\zeta_{n_k}(x_k),\quad \bar x=(x_1,\dots,x_k),\quad \mathbf{n}=(n_1,\dots,n_k),\quad n_j\in\{1,2\},$$

where the sequences $\zeta_1(x)$ and $\zeta_2(x)$, $x\in\mathbb{Z}$, are defined by the rule

$$\zeta_1(x)=\begin{cases}1 & \text{for}\ \ x<-a,\\ 0 & \text{for}\ \ x>a,\end{cases}\qquad \zeta_2(x)=\begin{cases}1 & \text{for}\ \ x>a,\\ 0 & \text{for}\ \ x<-a,\end{cases}\qquad\text{with some}\ \ a>0. \tag{3.2}$$

Finally, define a Borel probability measure μ_0 as a distribution of the random function

$$Y_0(x)=\sum_{\mathbf{n}\in\mathcal{N}^k}\overline{\zeta}_{\mathbf{n}}(\bar x)Y_{\mathbf{n}}(x),\quad x=(\bar x,\tilde x)\in\mathbb{Z}^d,\quad \bar x=(x_1,\dots,x_k),\quad \tilde x=(x_{k+1},\dots,x_d), \tag{3.3}$$

where $Y_{\mathbf{n}}(x)$ are Gaussian independent functions in $\mathcal{H}_\alpha$ with distributions $\mu_{\mathbf{n}}$. Then, the correlation matrix of μ_0 is of the form

$$Q_0(x,y) = \sum_{\mathbf{n}\in\mathcal{N}^k} \overline{\zeta}_{\mathbf{n}}(\bar{x})\overline{\zeta}_{\mathbf{n}}(\bar{y})q_{\mathbf{n}}(x-y), \tag{3.4}$$

where $x = (\bar{x},\tilde{x})$, $y = (\bar{y},\tilde{y}) \in \mathbb{Z}^d$, and $q_{\mathbf{n}}(x-y)$ are the correlation matrices of the measures $\mu_{\mathbf{n}}$. Hence, $Q_0(x,y) = q_0(\bar{x},\bar{y},\tilde{x}-\tilde{y})$, and for every $z = (\bar{z},\tilde{z}) \in \mathbb{Z}^d$,

$$q_0(\bar{y}+\bar{z},\bar{y},\tilde{z}) = q_{\mathbf{n}}(z) \quad \text{if } (-1)^{n_j}y_j > a+|z_j|, \quad \forall j=1,\dots,k, \quad \mathbf{n}=(n_1,\dots,n_k).$$

Therefore, the measure μ_0 satisfies conditions **S1** and **S3**. If

$$|q_{\mathbf{n}}^{ii}(z)| \leqslant h(|z|), \quad \text{where} \quad r^{d-1}h(r) \in L^1(0,+\infty), \tag{3.5}$$

then μ_0 satisfies **S2** by (3.4). Now we give examples of $q_{\mathbf{n}}^{ii}$ satisfying (3.1) and (3.5).

Example 3.1. Put $q_{\mathbf{n}}^{ii}(z) = f(z_1)f(z_2)\cdot\ldots\cdot f(z_d)$ and construct sequences $f(z)$, $z\in\mathbb{Z}$, such that conditions (3.1) and (3.5) hold.

(i) Let $f(z) = N_0 - |z|$ for $|z| \leqslant N_0$ and $f(z) = 0$ for $|z| > N_0$ with some $N_0 > 0$. Then $q_{\mathbf{n}}^{ii}(z) = 0$ for $|z| \geqslant r_0 \equiv N_0\sqrt{d}$, and $\hat{f}(\theta) = (1-\cos N_0\theta)/(1-\cos\theta)$, $\theta\in\mathbb{T}^1$. Hence, (3.1) and (3.5) are fulfilled. Furthermore, the condition **S4** also follows with $\varphi(r) = 0$ for $r \geqslant r_0$. This example of the sequence f can be generalized as follows.

Let f be an even nonnegative sequence such that $f\in\ell^1$ and $\Delta_L f(z) \geqslant 0$ for any $z\geqslant 1$. Then $\hat{f}(\theta)\geqslant 0$ by [17, Th. 4.1 and 2.7], and (3.1) follows. If, in addition, $|f(z)| \leqslant C(1+|z|)^{-N}$ with $N > d$, then (3.5) holds.

(ii) Let $f(z) = (a+b|z|)\gamma^{|z|}$, $z\in\mathbb{Z}$, with $\gamma\in(0,1)$, $a>0$ and $b\geqslant 0$. Hence, (3.5) is fulfilled with $h(r) = C(1+r)^{-N}$ $(N>d)$. If $a \geqslant 2b\gamma/(1-\gamma)$, then $\Delta_L f(z)\geqslant 0$ for any $z\geqslant 1$ and $\hat{f}(\theta)\geqslant 0$ (see case (i)). If $2b\gamma/(1-\gamma^2) \leqslant a < 2b\gamma/(1-\gamma)$, then $\Delta_L f(1) < 0$. However, in this case, (3.1) also holds because

$$\begin{aligned}\hat{f}(\theta) &= \frac{a(1-\gamma^2)}{1-2\gamma\cos\theta+\gamma^2} + \frac{2b\gamma\left((1+\gamma^2)\cos\theta-2\gamma\right)}{(1-2\gamma\cos\theta+\gamma^2)^2} \\ &= \frac{[a(1-\gamma^2)+2b\gamma\cos\theta](1-\gamma)^2 + [a(1-\gamma^2)-2b\gamma](1-\cos\theta)2\gamma}{(1-2\gamma\cos\theta+\gamma^2)^2} \geqslant 0.\end{aligned}$$

Example 3.2. Let $u_0, v_0 \in\mathbb{R}^n$ with any $n\geqslant 1$, and

$$q_{\mathbf{n}}^{00}(z) = T_{\mathbf{n}}F^{-1}_{\theta\to z}[\hat{V}^{-1}(\theta)], \quad q_{\mathbf{n}}^{11}(z) = T_{\mathbf{n}}I, \quad z\in\mathbb{Z}^d,$$

with some constants $T_{\mathbf{n}} > 0$. Assume, in addition, that $\mathcal{C}_0 = \emptyset$ (see (2.6)), i.e.,

$$\det\hat{V}(\theta) \neq 0, \quad \forall\theta\in\mathbb{T}^d. \tag{3.6}$$

Hence,

$$|q_{\mathbf{n}}^{00}(z)| = T_{\mathbf{n}}\left|F^{-1}_{\theta\to z}[\hat{V}^{-1}(\theta)]\right| \sim (1+|z|)^{-N}, \quad \forall N\in\mathbb{N}, \tag{3.7}$$

and conditions (3.1) and (3.5) are fulfilled with $h(r) = (1+r)^{-N}$, where $N > d$.

Remark 3.3. Suppose that the initial covariance has the following particular form:

$$Q_0(x,y) = T(\bar{x}+\bar{y})r(x-y) \quad \text{or} \quad Q_0(x,y) = \sqrt{T(\bar{x})T(\bar{y})}\,r(x-y), \tag{3.8}$$

where $T(\bar{x})$ is a bounded nonnegative sequence on $\mathbb{Z}^k$, $r(x) = (r^{ij}(x))$ is a correlation matrix of some translation-invariant measure in $\mathcal{H}_\alpha$ with zero mean value, $|r^{ij}(x)| \leqslant h(|x|)$, where we assume that $r^{d-1}h(r)\in L^1(0,+\infty)$. Then, condition **S2** is fulfilled. For every $\mathbf{n}\in\mathcal{N}^k$, we assume that for any $\varepsilon > 0$ there exists an $N(\varepsilon)\in\mathbb{N}$ such that for any $\bar{x}\in\mathbb{Z}^k$: $(-1)^{n_j}x_j > N(\varepsilon)$ with any $j=1,\dots,k$, $|T(\bar{x}) - T_{\mathbf{n}}| < \varepsilon$. Hence, condition **S3** is fulfilled with $q_{\mathbf{n}}(z) := T_{\mathbf{n}}\,r(z)$.

4. ENERGY CURRENT

4.1. nonequilibrium States

First we derive an expression for the energy current density. Let $u(x,t)$ be a solution of Eq. (2.1) with finite energy (see (2.3)). For the half-space $\Omega_l := \{x \in \mathbb{Z}^d : x_l \geqslant 0\}$, we define the energy in the region Ω_l as

$$\mathcal{E}_l(t) := \frac{1}{2}\sum_{x\in\Omega_l}\left\{|\dot u(x,t)|^2 + \sum_{y\in\mathbb{Z}^d}(u(x,t), V(x-y)u(y,t))\right\}, \quad l=1,\dots,d.$$

Introduce new variables: $x = x' + me_l$, $y = y' + pe_l$, where $x', y' \in \mathbb{Z}^d$ with $x'_l = y'_l = 0$, $e_l = (\delta_{l1},\dots,\delta_{ld})$, $l = 1,\dots,d$. Using Eq. (2.1), we obtain $\dot{\mathcal{E}}_l(t) = \sum_{x'} J^l(x',t)$. Here $J^l(x',t)$ stands for the energy current density in the direction e_l:

$$J^l(x',t) := \frac{1}{2}\sum_{y'}\left\{\sum_{m\leqslant -1,\, p\geqslant 0}(\dot u(x'+me_l,t), V(x'+me_l-y'-pe_l)u(y'+pe_l,t)) - \sum_{m\geqslant 0,\, p\leqslant -1}(\dot u(x'+me_l,t), V(x'+me_l-y'-pe_l)u(y'+pe_l,t))\right\},$$

where $x', y' \in \mathbb{Z}^d$ with $x'_l = y'_l = 0$. Now let $u(x,t)$ be a random solution of (2.1) with the initial measure μ_0 satisfying conditions **S1**–**S3**. The convergence (1.1) yields

$$\begin{aligned}\mathbb{E}\left(J^l(x',t)\right) \to J^l_\infty :=& \frac{1}{2}\sum_{y'}\Bigg(\sum_{m\leqslant -1,\, p\geqslant 0}\operatorname{tr}[q^{10}_\infty(x'-y'+(m-p)e_l)V^T(x'-y'+(m-p)e_l)] \\ &- \sum_{m\geqslant 0,\, p\leqslant -1}\operatorname{tr}[q^{10}_\infty(x'-y'+(m-p)e_l)V^T(x'-y'+(m-p)e_l)]\Bigg) \\ =& -\frac{1}{2}\sum_{z\in\mathbb{Z}^d} z_l \operatorname{tr}[q^{10}_\infty(z)V^T(z)] \quad \text{as} \ \ t\to\infty.\end{aligned}$$

Applying the Fourier transform and the equality $\hat V^*(\theta) = \hat V(\theta)$, we obtain

$$J^l_\infty = -(2\pi)^{-d}\,\frac{1}{2}\int_{\mathbb{T}^d} i\ \operatorname{tr}\left[\hat q^{10}_\infty(\theta)\partial_{\theta_l}\hat V(\theta)\right] d\theta, \quad l=1,\dots,d.$$

Since $\Pi_\sigma(\theta)$ are orthogonal projections, $\Pi_\sigma(\theta)\left(\partial_{\theta_l}\Pi_{\sigma'}(\theta)\right)\Pi_\sigma(\theta) = 0$ for any $\sigma,\sigma' = 1,\dots,s$ and $l = 1,\dots,d$. Hence, applying the formula (2.15) and the following decomposition of $\hat V(\theta)$,

$$\hat V(\theta) = \sum_{\sigma=1}^{s}\Pi_\sigma(\theta)\omega^2_\sigma(\theta),$$

we obtain $\operatorname{tr}\left[\hat q^{10}_\infty(\theta)\sum_{\sigma=1}^s \omega^2_\sigma(\theta)\partial_{\theta_l}\Pi_\sigma(\theta)\right] = 0$ and

$$\begin{aligned}J^l_\infty =& -i(2\pi)^{-d}\sum_{\sigma=1}^{s}\int_{\mathbb{T}^d}\operatorname{tr}\left[\Pi_\sigma(\theta)\left(\mathbf{M}^+_{k,\sigma}(\theta)+i\mathbf{M}^-_{k,\sigma}(\theta)\right)^{10}\Pi_\sigma(\theta)\right]\omega_\sigma(\theta)\partial_{\theta_l}\omega_\sigma(\theta)\,d\theta \\ =& -(2\pi)^{-d}\frac{1}{2^k}\sum_{\sigma=1}^{s}\sum_{\mathbf{n}\in\mathcal{N}^k}\Bigg\{\frac{1}{2}\int_{\mathbb{T}^d}\operatorname{tr}\left[\omega^2_\sigma(\theta)p^{00}_{\mathbf{n},\sigma\sigma}(\theta)+p^{11}_{\mathbf{n},\sigma\sigma}(\theta)\right]S^{\rm odd}_{k,\mathbf{n}}(\omega_\sigma)\partial_{\theta_l}\omega_\sigma(\theta)\,d\theta \\ &+\int_{\mathbb{T}^d}\operatorname{Im}\left(\operatorname{tr}p^{01}_{\mathbf{n},\sigma\sigma}(\theta)\right)\left(1+S^{\rm even}_{k,\mathbf{n}}(\omega_\sigma)\right)\omega_\sigma(\theta)\partial_{\theta_l}\omega_\sigma(\theta)\,d\theta\Bigg\}, \quad l=1,\dots,d, \qquad (4.1)\end{aligned}$$

where the $p^{ij}_{\mathbf{n},\sigma\sigma}$ are introduced in (2.21). Here we use the equality $\overline{\operatorname{tr}p^{ij}_{\mathbf{n},\sigma\sigma}(\theta)} = \operatorname{tr}p^{ji}_{\mathbf{n},\sigma\sigma}(\theta)$.

Example 4.1. Let, for simplicity, all the functions $\mathrm{tr}[p^{ij}_{\mathbf{n},\sigma\sigma}(\theta)]$ and $\omega_\sigma(\theta)$ be even for every variable $\theta_1,\dots,\theta_d$. Then, $J^l_\infty = 0$ for $l > k$, and $J^l_\infty = C^l_1 - C^l_2$ for $l = 1,\dots,k$, where

$$C^l_j := (2\pi)^{-d}\,\frac{1}{2^k}\sum_{\sigma=1}^{s}\sum_{\mathbf{n}}{}'\,\frac{1}{2}\int_{\mathbb{T}^d}\mathrm{tr}\left[\omega^2_\sigma(\theta)p^{00}_{\mathbf{n},\sigma\sigma}(\theta)+p^{11}_{\mathbf{n},\sigma\sigma}(\theta)\right]|_{n_l=j}\,|\partial_{\theta_l}\omega_\sigma(\theta)|\,d\theta,\quad j=1,2.$$

Here the summation $\sum'_{\mathbf{n}}$ is taken over $n_1,\dots,n_{l-1},n_{l+1},\dots,n_k \in \{1,2\}$. Assume, in addition, that the initial correlation matrix Q_0 has the form (3.8). Then, $p^{ij}_{\mathbf{n},\sigma\sigma}(\theta) = T_{\mathbf{n}}p^{ij}_{\sigma\sigma}(\theta)$, where, by definition, $p^{ij}_{\sigma\sigma}(\theta) := \Pi_\sigma(\theta)\hat r^{ij}(\theta)\Pi_\sigma(\theta)$. In this case, $\mathbf{J}_\infty$ is of the form (1.2) with

$$c_l = \sum_{\sigma=1}^{s}(2\pi)^{-d}\,\frac{1}{2}\int_{\mathbb{T}^d}\mathrm{tr}\left[\omega^2_\sigma(\theta)p^{00}_{\sigma\sigma}(\theta)+p^{11}_{\sigma\sigma}(\theta)\right]|\partial_{\theta_l}\omega_\sigma(\theta)|\,d\theta.$$

We see that one can choose positive numbers $T_{\mathbf{n}}$ such that $\mathbf{J}_\infty \neq 0$.

Below we simplify formula (4.1) in the case when $\mu_{\mathbf{n}}$ are Gibbs measures corresponding to positive temperatures $T_{\mathbf{n}}$. Furthermore, under additional symmetry conditions on the eigenvalues $\omega_\sigma(\theta)$, we derive formula (1.2) for $\mathbf{J}_\infty$. Thus, there exist stationary nonequilibrium states (in fact, Gaussian measures μ_∞) in which there is a nonzero constant energy current passing through the points of the crystal.

4.2. Energy Current for Gibbs Measures

Formally, Gibbs measures g_β are $g_\beta(dY) = \frac{1}{Z}\,e^{-\beta H(Y)}\prod_{x\in\mathbb{Z}^d}dY(x)$, where $H(Y)$ is defined in (2.3), Z is normalization factor, $\beta = T^{-1}$, $T > 0$ is a corresponding absolute temperature. We introduce the Gibbs measures $g_\beta(dY)$ as the Gaussian measures in $\mathcal{H}_\alpha$, $\alpha < -d/2$, with zero mean and with the correlation matrices defined by their Fourier transform,

$$\hat q^{00}_\beta(\theta) = T\hat V^{-1}(\theta),\quad \hat q^{11}_\beta(\theta) = TI,\quad \hat q^{01}_\beta(\theta) = \hat q^{10}_\beta(\theta) = 0, \tag{4.2}$$

where I denotes unit matrix in $\mathbb{R}^n\times\mathbb{R}^n$. By the Minlos theorem [5], the Borel probability measures g_β exist in the spaces $\mathcal{H}_\alpha$. Indeed,

$$\int \|Y\|^2_\alpha\,g_\beta(dY) = \sum_{x\in\mathbb{Z}^d}\langle x\rangle^{2\alpha}\,\mathrm{tr}[q^{00}_\beta(0)+q^{11}_\beta(0)] < \infty,$$

since $\alpha < -d/2$ and

$$\mathrm{tr}[q^{00}_\beta(0)+q^{11}_\beta(0)] = (2\pi)^{-d}\int_{\mathbb{T}^d}\mathrm{tr}[\hat q^{00}_\beta(\theta)+\hat q^{11}_\beta(\theta)]\,d\theta = T(2\pi)^{-d}\int_{\mathbb{T}^d}\mathrm{tr}\,\hat V^{-1}(\theta)\,d\theta + Tn < \infty.$$

The last bound is obvious if $\mathcal{C}_0 = \emptyset$ and follows from the condition **E6** if $\mathcal{C}_0 \neq \emptyset$.

Let μ_0 be a Borel probability measure in $\mathcal{H}_\alpha$ giving the distribution of the random function Y_0 constructed in Section 3 (see formula (3.3)) with Gibbs measures $\mu_{\mathbf{n}} \equiv g_{\beta_{\mathbf{n}}}$ ($\beta_{\mathbf{n}} = 1/T_{\mathbf{n}}$, $T_{\mathbf{n}} > 0$) which have correlation matrices $q_{\mathbf{n}}(x) \equiv q_{\beta_{\mathbf{n}}}(x)$, where the matrix $q_\beta = (q^{ij}_\beta)_{i,j=0,1}$ is defined by (4.2). We impose, in addition, condition (3.6). Then, conditions **S1**–**S3** hold (see Example 3.2). We check that in the case of the Gibbs measures $\mu_{\mathbf{n}} \equiv g_{\beta_{\mathbf{n}}}$, condition **E5'** is fulfilled (see Remark 2.7). Indeed, by (4.2), we have

$$\left.\begin{aligned} p^{00}_{\mathbf{n},\sigma\sigma'}(\theta) &= \Pi_\sigma(\theta)\hat q^{00}_{\mathbf{n}}(\theta)\Pi_{\sigma'}(\theta) = T_{\mathbf{n}}\omega^{-2}_\sigma(\theta)\Pi_\sigma(\theta)\delta_{\sigma\sigma'}\\ p^{11}_{\mathbf{n},\sigma\sigma'}(\theta) &= \Pi_\sigma(\theta)\hat q^{11}_{\mathbf{n}}(\theta)\Pi_{\sigma'}(\theta) = T_{\mathbf{n}}\,\Pi_\sigma(\theta)\delta_{\sigma\sigma'}\end{aligned}\right|\quad \sigma,\sigma' = 1,\dots,s, \tag{4.3}$$

and $p^{ij}_{\mathbf{n},\sigma\sigma'}(\theta)=0$ for $i\ne j$. Hence, assertions (1.1) and (2.24) hold, see Lemma 2.14.

Now we rewrite the limit covariance $\hat q_\infty(\theta)$ and the limit mean energy current $\mathbf{J}_\infty$ in the case when $\mu_{\mathbf{n}}=g_{\beta_{\mathbf{n}}}$ are Gibbs measures. Applying (2.15), (2.16), and (2.18), we obtain

$$\hat q^{11}_\infty(\theta)=\hat V(\theta)\hat q^{00}_\infty(\theta)=\sum_{\sigma=1}^s \Pi_\sigma(\theta)\,\frac{1}{2^k}\sum_{\mathbf{n}\in\mathcal{N}^k} T_{\mathbf{n}}\left[1+S^{\rm even}_{k,\mathbf{n}}(\omega_\sigma(\theta))\right],$$
$$\hat q^{10}_\infty(\theta)=-\hat q^{01}_\infty(\theta)=-i\sum_{\sigma=1}^s \Pi_\sigma(\theta)\omega_\sigma^{-1}(\theta)\,\frac{1}{2^k}\sum_{\mathbf{n}\in\mathcal{N}^k} T_{\mathbf{n}}S^{\rm odd}_{k,\mathbf{n}}(\omega_\sigma(\theta)),$$

where the functions $S^{\rm even}_{k,\mathbf{n}}(\omega_\sigma)$ and $S^{\rm odd}_{k,\mathbf{n}}(\omega_\sigma)$ are defined in (2.17). Substituting $p^{ij}_{\mathbf{n},\sigma\sigma}(\theta)$ from (4.3) in the r.h.s. of (4.1), we obtain

$$\begin{aligned} J^l_\infty &=-\frac{1}{(2\pi)^d}\,\frac{1}{2^k}\sum_{\sigma=1}^s\sum_{\mathbf{n}\in\mathcal{N}^k}\int_{\mathbb{T}^d} r_\sigma\, T_{\mathbf{n}}\, S^{\rm odd}_{k,\mathbf{n}}(\omega_\sigma(\theta))\,\partial_{\theta_l}\omega_\sigma(\theta)\,d\theta \\ &=-\frac{1}{2^k}\sum_{\mathbf{n}\in\mathcal{N}^k} T_{\mathbf{n}}\left(\sum_{\text{odd } m\in\{1,\dots,k\}}\ \sum_{(p_1,\dots,p_m)\in\mathcal{P}_m(k)} c^l_{p_1\dots p_m}\,(-1)^{n_{p_1}+\dots+n_{p_m}}\right),\quad l=1,\dots,d, \end{aligned}\tag{4.4}$$

where $r_\sigma=\operatorname{tr}[\Pi_\sigma(\theta)]$ is the multiplicity of the eigenvalue ω_σ (see Lemma 2.2), the numbers $c^l_{p_1\dots p_m}$ are defined as follows

$$c^l_{p_1\dots p_m}:=\frac{1}{(2\pi)^d}\sum_{\sigma=1}^s\int_{\mathbb{T}^d} r_\sigma\,\mathrm{sign}\left(\frac{\partial\omega_\sigma(\theta)}{\partial\theta_{p_1}}\right)\cdot\ldots\cdot\mathrm{sign}\left(\frac{\partial\omega_\sigma(\theta)}{\partial\theta_{p_m}}\right)\frac{\partial\omega_\sigma(\theta)}{\partial\theta_l}\,d\theta.\tag{4.5}$$

Under the additional symmetry conditions (**SC**) on the interaction matrix V, the formulas (4.4) and (4.5) can be simplified.

SC. Suppose that one of the following conditions on ω_σ, $\sigma=1,\dots,s$, holds.

(a) Each $\omega_\sigma(\theta)$ is even for every variable $\theta_{k+1},\dots,\theta_d$, and, in addition, if $k\geqslant 2$, then each $\omega_\sigma(\theta)$ is even for some $k-1$ variables from the set $\{\theta_1,\dots,\theta_k\}$.

(b) Each $\omega_\sigma(\theta)$ is even for every variable $\theta_1,\dots,\theta_k$.

(c) For every $p=1,\dots,k$, $\mathrm{sgn}\,(\partial_{\theta_p}\omega_\sigma(\theta))$ depends only on the variable θ_p, and, in addition, if $k\geqslant 3$, then each $\omega_\sigma(\theta)$ is even for some $k-1$ variables from $\{\theta_1,\dots,\theta_k\}$.

For instance, conditions (a), (b), and (c) hold for the nearest neighbor crystal, see (2.8). Under these restrictions on ω_σ, all numbers $c^l_{p_1\dots p_m}$ in (4.5) are equal to zero except for the case when $m=1$ and $l=p_1\in\{1,\dots,k\}$. Write

$$c_l\equiv c^l_l=\frac{1}{(2\pi)^d}\sum_{\sigma=1}^s\int_{\mathbb{T}^d} r_\sigma\left|\frac{\partial\omega_\sigma(\theta)}{\partial\theta_l}\right|d\theta>0,\quad l=1,\dots,k.\tag{4.6}$$

Therefore,

$$J^l_\infty=\begin{cases} -c_l\,\frac{1}{2^k}\sum_{\mathbf{n}\in\mathcal{N}^k}(-1)^{n_l}T_{\mathbf{n}}=-c_l\,\frac{1}{2^k}\,\sum{}'\left(T_{\mathbf{n}}|_{n_l=2}-T_{\mathbf{n}}|_{n_l=1}\right), & l=1,\dots,k,\\ 0, & l=k+1,\dots,d,\end{cases}\tag{4.7}$$

where the summation $\sum{}'$ is taken over $n_1,\dots,n_{l-1},n_{l+1},\dots,n_k\in\{1,2\}$. In particular, if $k=1$, then the limiting energy current density is

$$\mathbf{J}_\infty=-\frac12(c_1\,(T_2-T_1)\,,0,\dots,0),\quad c_1>0.\tag{4.8}$$

In this case, the harmonic crystal can be considered as a "system + two reservoirs," where by "reservoirs" we mean the two parts of the crystal consisting of the particles with $x_1 \leqslant -a$ and with $x_1 \geqslant a$ ($a > 0$), and by a "system" the remaining ("middle") part (cf [24, Section 3]). Initially, the reservoirs are in thermal equilibrium with temperatures T_1 and T_2. Therefore, the formula (4.8) corresponds to the Second Law (see, for instance, [2, 24]), i.e., the heat flows (on average) from the "hot reservoir" to the "cold" one.

If $k = 2$, then our model can be represented as a "system + four reservoirs," where the reservoirs consist of particles lying in the four regions $\{x_1, x_2 \leqslant -a\}$, $\{x_1 \leqslant -a, x_2 \geqslant a\}$, $\{x_1 \geqslant a, x_2 \leqslant -a\}$, and $\{x_1, x_2 \geqslant a\}$. The initial states of the reservoirs are distributed according to Gibbs measures with corresponding temperatures T_{11}, T_{12}, T_{21}, and T_{22}. Formula (4.7) becomes

$$\mathbf{J}_\infty = -\frac{1}{4}\left(c_1\left(T_{21} - T_{11} + T_{22} - T_{12}\right), c_2\left(T_{12} - T_{11} + T_{22} - T_{21}\right), 0, \ldots, 0\right), \quad c_1, c_2 > 0.$$

Hence, $\mathbf{J}_\infty \neq 0$ for any positive values of T_{ij} except in the case when $T_{11} = T_{22}$ and $T_{12} = T_{21}$.

For any k, our model is a "system + 2^k reservoirs", where the "reservoirs" are the crystal particles with position in the regions $\{x \in \mathbb{Z}^d : (-1)^{n_j} x_j > a \ \ \forall j = 1, \ldots, k\}$. At $t = 0$, these reservoirs are assumed to be in thermal equilibrium with temperatures $T_{\mathbf{n}}$, $\mathbf{n} \in \mathcal{N}^k$. By virtue of formula (4.7), we can choose the temperatures $T_{\mathbf{n}}$ so that the limiting energy current density $\mathbf{J}_\infty$ is not zero.

5. CONVERGENCE OF COVARIANCE

5.1. Bounds of Correlation Matrices

By $l^p \equiv l^p(\mathbb{Z}^d) \otimes \mathbb{R}^n$, $p, d, n \geqslant 1$, we denote the space of sequences $f(x) = (f_1(x), \ldots, f_n(x))$ endowed with the norm

$$\|f\|_{l^p} = \Big(\sum_{x \in \mathbb{Z}^d} |f(x)|^p\Big)^{1/p}.$$

Lemma 5.1. (i) *Let conditions* **S1** *and* **S2** *hold. Then for any* $\Phi, \Psi \in l^2$,

$$|\langle Q_0(x,y), \Phi(x) \otimes \Psi(y)\rangle| \leqslant C\|\Phi\|_{l^2}\|\Psi\|_{l^2}. \tag{5.1}$$

(ii) *Let conditions* **S1**–**S3** *hold. Then* $q_{\mathbf{n}}^{ij} \in \ell^1$. *Hence,* $\hat{q}_{\mathbf{n}}^{ij} \in C(\mathbb{T}^d)$, $i, j = 0, 1$.

Proof. (i) It follows from the bound (2.9) that

$$\sum_{y \in \mathbb{Z}^d} |Q_0^{ij}(x,y)| \leqslant \sum_{z \in \mathbb{Z}^d} h(|z|) < \infty.$$

Similarly,

$$\sum_{x \in \mathbb{Z}^d} |Q_0^{ij}(x,y)| \leqslant C < \infty \quad \text{for all} \quad y \in \mathbb{Z}^d.$$

This implies the bound (5.1) by the Shur lemma.

(ii) The bound (2.9) and condition (2.12) imply the same bound for $q_{\mathbf{n}}^{ij}(z)$, i.e., $|q_{\mathbf{n}}^{ij}(z)| \leqslant h(|z|)$, where $r^{d-1}h(r) \in L^1(0, +\infty)$. Hence, $q_{\mathbf{n}}^{ij} \in l^1$.

Lemma 5.2. *Let conditions* **E1**–**E3**, **E6**, **S1**, **S2** *hold, and* $\alpha < -d/2$. *Then the bound* (2.25) *holds.*

This lemma can be proved by a same way as in [8]. We repeat the proof, since some notation and some technical bounds obtained in the proof will be applied in Section 5.2 below.

Proof. Note first that

$$\mathbb{E}\|Y(\cdot,t)\|_\alpha^2 = \sum_{x\in\mathbb{Z}^d} \langle x\rangle^{2\alpha}(\operatorname{tr} Q_t^{00}(x,x) + \operatorname{tr} Q_t^{11}(x,x)), \quad \text{where} \quad \alpha < -d/2.$$

Hence, to prove (2.25), it suffices to check that $\sup_{t\in\mathbb{R}}\sup_{x,y\in\mathbb{Z}^d}\|Q_t(x,y)\| \leqslant C < \infty$. Applying the Fourier transform to (2.2), we obtain

$$\dot{\hat{Y}}(t) = \hat{\mathcal{A}}(\theta)\hat{Y}(t), \quad t\in\mathbb{R}, \quad \hat{Y}(0) = \hat{Y}_0. \tag{5.2}$$

Here we denote $\hat{\mathcal{A}}(\theta) = \begin{pmatrix} 0 & 1 \\ -\hat{V}(\theta) & 0 \end{pmatrix}$, $\theta\in\mathbb{T}^d$. Therefore, the solution $\hat{Y}(\theta,t)$ of (5.2) admits the representation $\hat{Y}(\theta,t) = \hat{\mathcal{G}}_t(\theta)\hat{Y}_0(\theta)$ with $\hat{\mathcal{G}}_t(\theta) := \exp(\hat{\mathcal{A}}(\theta)t)$. In the coordinate space, we have

$$Y(x,t) = \sum_{x'\in\mathbb{Z}^d} \mathcal{G}_t(x-x')Y_0(x'), \quad x\in\mathbb{Z}^d. \tag{5.3}$$

The Green function $\mathcal{G}_t(x)$ has the form (in its Fourier transform)

$$\hat{\mathcal{G}}_t(\theta) = \begin{pmatrix} \cos\Omega t & \sin\Omega t\ \Omega^{-1} \\ -\sin\Omega t\ \Omega & \cos\Omega t \end{pmatrix}, \tag{5.4}$$

where $\Omega = \Omega(\theta)$ is the Hermitian matrix defined by (2.4). Then

$$\hat{\mathcal{G}}_t(\theta) = \cos\Omega t\, I + \sin\Omega t\, C(\theta), \tag{5.5}$$

where $C(\theta)$ is defined by (2.19). The representation (5.3) gives

$$\begin{aligned} Q_t^{ij}(x,y) = \mathbb{E}(Y^i(x,t)\otimes Y^j(y,t)) &= \sum_{x',y'\in\mathbb{Z}^d}\sum_{k,l=0,1} \mathcal{G}_t^{ik}(x-x')Q_0^{kl}(x',y')\mathcal{G}_t^{jl}(y-y') \\ &= \langle Q_0(x',y'), \Phi_x^i(x',t)\otimes\Phi_y^j(y',t)\rangle, \end{aligned} \tag{5.6}$$

where $\Phi_x^i(x',t) := (\mathcal{G}_t^{i0}(x-x'), \mathcal{G}_t^{i1}(x-x'))$, $x'\in\mathbb{Z}^d$, $i=0,1$. Note that the Parseval formula, (5.4), and condition **E6** imply

$$\|\Phi_x^i(\cdot,t)\|_{l^2}^2 = (2\pi)^{-d}\int_{\mathbb{T}^d} |\hat{\Phi}_x^i(\theta,t)|^2\,d\theta = (2\pi)^{-d}\int_{\mathbb{T}^d} (|\hat{\mathcal{G}}_t^{i0}(\theta)|^2 + |\hat{\mathcal{G}}_t^{i1}(\theta)|^2)\,d\theta \leqslant C_0 < \infty.$$

Then the bound (5.1) gives

$$|Q_t^{ij}(x,y)| = |\langle Q_0(x',y'), \Phi_x^i(x',t)\otimes\Phi_y^j(y',t)\rangle| \leqslant C\|\Phi_x^i(\cdot,t)\|_{l^2}\,\|\Phi_y^j(\cdot,t)\|_{l^2} \leqslant C_1 < \infty,$$

where the constant C_1 does not depend on $x,y\in\mathbb{Z}^d$, $t\in\mathbb{R}$.

Proposition 2.12 follows from the bound (2.25) by the Prokhorov Theorem [26, Lemma II.3.1] using the method of [26, Th. XII.5.2], since the embedding $\mathcal{H}_\alpha\subset\mathcal{H}_\beta$ is compact if $\alpha>\beta$.

5.2. Proof of Theorem 2.6

To prove Theorem 2.6, it suffices to check that for all $\Psi\in\mathcal{S}$,

$$\mathcal{Q}_t(\Psi,\Psi)\to\mathcal{Q}_\infty(\Psi,\Psi),\quad t\to\infty. \tag{5.7}$$

In the cases when $k=0$ and $k=1$, the convergence (5.7) was proved in [7] and [8], respectively. We derive (5.7) for any $k\geqslant 1$.

Definition 5.3. (i) The critical set is $\mathcal{C}:=\mathcal{C}_*\cup\mathcal{C}_0\cup_\sigma\mathcal{C}_\sigma$ with $\mathcal{C}_*$ as in Lemma 2.2 and sets $\mathcal{C}_0$ and $\mathcal{C}_\sigma$ defined by (2.6).

(ii) $\mathcal{S}^0:=\{\Psi\in\mathcal{S}:\hat\Psi(\theta)=0\ \text{ in a neighborhood of }\ \mathcal{C}\}$.

Obviously mes $\mathcal{C}=0$. Write the inner product $\langle Y(\cdot,t),\Psi\rangle$ in the form $\langle Y(\cdot,t),\Psi\rangle=\langle Y_0,\Phi(\cdot,t)\rangle$, where $\Phi(x,t):=F^{-1}_{\theta\to x}[\hat{\mathcal{G}}^*_t(\theta)\hat\Psi(\theta)]$. Therefore,

$$\mathcal{Q}_t(\Psi,\Psi)=\mathbb{E}|\langle Y(\cdot,t),\Psi\rangle|^2=\langle Q_0(x,y),\Phi(x,t)\otimes\Phi(y,t)\rangle, \tag{5.8}$$

where the Parseval identity and (5.4) yield

$$\|\Phi(\cdot,t)\|^2_{l^2}=(2\pi)^{-d}\int_{\mathbb{T}^d}\|\hat{\mathcal{G}}^*_t(\theta)\|^2|\hat\Psi(\theta)|^2d\theta\leqslant C\int_{\mathbb{T}^d}\big(1+\|V^{-1}(\theta)\|\big)\,|\hat\Psi(\theta)|^2d\theta=:C\|\Psi\|^2_V. \tag{5.9}$$

By (5.1), (5.8), and (5.9), the uniform bounds hold, $\sup_{t\in\mathbb{R}}|\mathcal{Q}_t(\Psi,\Psi)|\leqslant C\|\Psi\|^2_V$, $\Psi\in\mathcal{S}$. Therefore, it suffices to prove the convergence (5.7) for $\Psi\in\mathcal{S}^0$ only.

We define a matrix $Q_*(x,y)$, $x,y\in\mathbb{Z}^d$, as follows

$$\begin{aligned}Q_*(x,y)&=\frac{1}{2^k}\sum_{\mathbf{n}\in\mathcal{N}^k}q_{\mathbf{n}}(x-y)(1+(-1)^{n_1}\operatorname{sgn}y_1)\cdot\ldots\cdot(1+(-1)^{n_k}\operatorname{sgn}y_k)\\&=\frac{1}{2^k}\sum_{\mathbf{n}\in\mathcal{N}^k}q_{\mathbf{n}}(x-y)\left[1+\sum_{m=1}^k\ \sum_{(p_1,\ldots,p_m)\in\mathcal{P}_m(k)}(-1)^{n_{p_1}+\cdots+n_{p_m}}\operatorname{sgn}y_{p_1}\cdot\ldots\cdot\operatorname{sgn}y_{p_m}\right]\end{aligned} \tag{5.10}$$

with the matrices $q_{\mathbf{n}}(x)$ introduced in condition **S3**. For instance, for $k=1$,

$$Q_*(x,y)=\frac12\left(q_1(x-y)+q_2(x-y)\right)+\frac12\left(q_2(x-y)-q_1(x-y)\right)\operatorname{sgn}y_1.$$

Note that $Q_*(x,y)=q_{\mathbf{n}}(x-y)$ in every region $\{(x,y)\in\mathbb{Z}^{2d}:(-1)^{n_1}y_1>0,\ \ldots,(-1)^{n_k}y_k>0\}$, $\mathbf{n}=(n_1,\ldots,n_k)\in\mathcal{N}^k$. Denote $Q_r(x,y)=Q_0(x,y)-Q_*(x,y)$. Therefore, the convergence (5.7) follows from (5.8) and the following proposition.

Proposition 5.4. *For any $\Psi\in\mathcal{S}^0$, the following assertions hold.*

(a) $\lim_{t\to\infty}\langle Q_*(x,y),\Phi(x,t)\otimes\Phi(y,t)\rangle=\langle q_\infty(x-y),\Psi(x)\otimes\Psi(y)\rangle$,

(b) $\lim_{t\to\infty}\langle Q_r(x,y),\Phi(x,t)\otimes\Phi(y,t)\rangle=0$.

At first, we prove the auxiliary lemma.

Lemma 5.5. *Let $q(x) = \big(q^{ij}(x)\big)_{i,j=0,1}$, $x \in \mathbb{Z}^d$, be $2n\times 2n$ matrix with $n\times n$ entries $q^{ij}(x)$ satisfying the bound $|q^{ij}(x)| \leqslant h(|x|)$, where $r^{d-1}h(r) \in L^1(0,+\infty)$. Assume that either condition* **E5** *holds or condition* **E5'** *is fulfilled with the matrices $\hat q^{ij}(\theta)$ instead of $\hat q^{ij}_{\mathbf{n}}(\theta)$. Then for any $\Psi \in \mathcal{S}^0$,*

$$\lim_{t\to\infty}\langle q(x-y), \Phi(x,t)\otimes\Phi(y,t)\rangle = \langle q^0_\infty(x-y), \Psi(x)\otimes\Psi(y)\rangle, \qquad (5.11)$$

where

$$\hat q^0_\infty(\theta) = \sum_{\sigma=1}^{s} \Pi_\sigma(\theta) L_1^+\left(\hat q(\theta)\right)\Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d\setminus\mathcal{C}_*.$$

Moreover, for any $k \in \{1,\dots,d\}$,

$$\lim_{t\to\infty}\langle q(x-y)\operatorname{sgn} y_1\cdot\ldots\cdot\operatorname{sgn} y_k, \Phi(x,t)\otimes\Phi(y,t)\rangle = \langle q^k_\infty(x-y), \Psi(x)\otimes\Psi(y)\rangle, \qquad (5.12)$$

where the matrix $q^k_\infty(x)$ has the form (in its Fourier transform)

$$\hat q^k_\infty(\theta) = \sum_{\sigma=1}^{s}\Pi_\sigma(\theta)L_k\left(\hat q(\theta)\right)\Pi_\sigma(\theta)\operatorname{sgn}(\partial_{\theta_1}\omega_\sigma(\theta))\cdot\ldots\cdot\operatorname{sgn}(\partial_{\theta_k}\omega_\sigma(\theta)), \quad \theta\in\mathbb{T}^d\setminus\mathcal{C}_*.$$

Here

$$L_k\left(\hat q(\theta)\right) = \begin{cases} L_1^+\left(\hat q(\theta)\right) & \text{if } k \text{ is even},\\ i\,L_2^-\left(\hat q(\theta)\right) & \text{if } k \text{ is odd},\end{cases} \qquad (5.13)$$

where the expressions L_1^+ and L_2^- are introduced in (2.18).

Proof. Using the Parseval identity, we have

$$\begin{aligned} I_t &:= \langle q(x-y)\operatorname{sgn} y_1\cdot\ldots\cdot\operatorname{sgn} y_k, \Phi(x,t)\otimes\Phi(y,t)\rangle\\ &= (2\pi)^{-2d}\int_{\mathbb{T}^{2d}}\Big(F_{\substack{x\to\theta\\ y\to-\theta'}}[q(x-y)\operatorname{sgn} y_1\cdot\ldots\cdot\operatorname{sgn} y_k], \hat\Phi(\theta,t)\otimes\overline{\hat\Phi(\theta',t)}\Big)\,d\theta\,d\theta'.\end{aligned}$$

Note that $F_{y\to\theta}(\operatorname{sgn} y) = i\,\mathrm{PV}(1/\mathrm{tg}(\theta/2))$, $\theta\in\mathbb{T}^1$, $y\in\mathbb{Z}^1$, where PV stands for the Cauchy principal part. Hence,

$$\begin{aligned} &F_{\substack{x\to\theta\\ y\to-\theta'}}[q(x-y)\operatorname{sgn} y_1\cdot\ldots\cdot\operatorname{sgn} y_k]\\ &\qquad = (2\pi)^{d-k}\delta(\tilde\theta-\tilde\theta')\,\hat q(\theta)\,i^k\,\mathrm{PV}\left(\frac{1}{\mathrm{tg}((\theta_1-\theta_1')/2)}\right)\cdot\ldots\cdot\mathrm{PV}\left(\frac{1}{\mathrm{tg}((\theta_k-\theta_k')/2)}\right),\end{aligned}$$

where $\tilde\theta = (\theta_{k+1},\dots,\theta_d)$. We choose a finite partition of unity

$$\sum_{m=1}^{M} g_m(\theta) = 1, \quad \theta\in\operatorname{supp}\hat\Psi, \qquad (5.14)$$

where g_m are nonnegative functions from $C_0^\infty(\mathbb{T}^d)$, which vanish in a neighborhood of the set $\mathcal{C}$ introduced in Definition 5.3 (i). Using the equality $\hat\Phi(\theta,t) = \hat{\mathcal{G}}_t^*(\theta)\hat\Psi(\theta)$, formula (5.5), the decomposition (2.5), and the partition (5.14), we obtain

$$\begin{aligned} I_t &= (2\pi)^{-d-k}\,i^k\,\mathrm{PV}\int_{\mathbb{T}^{d+k}}\frac{1}{\mathrm{tg}((\theta_1-\theta_1')/2)}\cdot\ldots\cdot\frac{1}{\mathrm{tg}((\theta_k-\theta_k')/2)}\\ &\quad\times\Big(\hat{\mathcal{G}}_t(\theta)\hat q(\theta)\hat{\mathcal{G}}_t^*(\theta'), \overline{\hat\Psi}(\theta)\otimes\hat\Psi(\theta')\Big)\big|_{\theta'=(\bar\theta',\tilde\theta)}\,d\bar\theta d\bar\theta' d\tilde\theta\\ &= (2\pi)^{-d-k}\,i^k\sum_{m,m'}\sum_{\sigma,\sigma'=1}^{s}\mathrm{PV}\int_{\mathbb{T}^{d+k}} g_m(\theta)g_{m'}(\theta')\frac{1}{\mathrm{tg}((\theta_1-\theta_1')/2)}\cdot\ldots\cdot\frac{1}{\mathrm{tg}((\theta_k-\theta_k')/2)}\\ &\quad\times\Big(\Pi_\sigma(\theta)\hat{\mathcal{G}}_{t,\sigma}(\theta)\hat q(\theta)\hat{\mathcal{G}}^*_{t,\sigma'}(\theta')\Pi_{\sigma'}(\theta'), \overline{\hat\Psi}(\theta)\otimes\hat\Psi(\theta')\Big)\big|_{\theta'=(\bar\theta',\tilde\theta)}\,d\bar\theta d\bar\theta' d\tilde\theta.\end{aligned} \qquad (5.15)$$

Here we denote

$$\hat{\mathcal{G}}_{t,\sigma}(\theta) = \cos\omega_\sigma(\theta)t\, I + \sin\omega_\sigma(\theta)t\, C_\sigma(\theta), \quad C_\sigma(\theta) = \begin{pmatrix} 0 & 1/\omega_\sigma(\theta) \\ -\omega_\sigma(\theta) & 0 \end{pmatrix}. \qquad (5.16)$$

By Lemma 2.2, we can choose the supports of g_m so small that the eigenvalues $\omega_\sigma(\theta)$ and the matrices $\Pi_\sigma(\theta)$ are real-analytic functions inside $\operatorname{supp} g_m$ for every m. (We do not label the functions by the index m so as not to overburden the notation.) Changing variables $\theta'_j \to \xi_j = \theta'_j - \theta_j$, $j = 1, \dots, k$, in the inner integrals in the r.h.s. of (5.5), we obtain

$$\begin{aligned} I_t =& (2\pi)^{-d-k}\,(-i)^k \sum_{m,m'} \sum_{\sigma,\sigma'=1}^{s} \int_{\mathbb{T}^d} \Big(g_m(\theta)\overline{\hat{\Psi}(\theta)}\Pi_\sigma(\theta)\hat{\mathcal{G}}_{t,\sigma}(\theta)\hat{q}(\theta) \\ &\times \mathrm{PV}\int_{\mathbb{T}^k} \frac{1}{\mathrm{tg}(\xi_1/2)} \cdot \ldots \cdot \frac{1}{\mathrm{tg}(\xi_k/2)}\, g_{m'}(\theta')\hat{\mathcal{G}}^*_{t,\sigma'}(\theta')\Pi_{\sigma'}(\theta')\hat{\Psi}(\theta')|_{\theta'=(\bar\theta+\bar\xi,\tilde\theta)} d\bar\xi \Big)\, d\theta. \end{aligned} \qquad (5.17)$$

It follows from Definition 5.3 that $\partial_{\theta'_j}\omega_{\sigma'}(\theta') \neq 0$ for $\theta' \in \operatorname{supp} g_{m'} \subset \operatorname{supp}\hat{\Psi}$. The next lemma follows from [1, Prop. A.4 (i), (ii)].

Lemma 5.6. *Let $\chi(\theta) \in C^1(\mathbb{T}^d)$ and $\partial_{\theta_1}\omega_\sigma(\theta) \neq 0$ for $\theta \in \operatorname{supp}\chi$. Then for $\theta \in \operatorname{supp}\chi$,*

$$P_\sigma(\theta,t) := \mathrm{PV}\int_{\mathbb{T}^1} \frac{e^{\pm i\omega_\sigma(\theta_1+\xi,\tilde\theta)t}}{\mathrm{tg}(\xi/2)}\chi(\theta_1+\xi,\tilde\theta)\, d\xi = e^{\pm i\omega_\sigma(\theta)t}\operatorname{sgn}(\partial_{\theta_1}\omega_\sigma(\theta)) + o(1) \quad \text{as } t \to +\infty,$$

where $\tilde\theta = (\theta_2, \dots, \theta_d)$. Moreover, $\sup_{t\in\mathbb{R},\,\theta\in\mathbb{T}^d} |P_\sigma(\theta,t)| < \infty$. Furthermore, yields (5.16), *we have*

$$\mathrm{PV}\int_{\mathbb{T}^1} \frac{1}{\mathrm{tg}(\xi/2)}\hat{\mathcal{G}}^*_{t,\sigma}(\theta_1+\xi,\tilde\theta)\chi(\theta_1+\xi,\tilde\theta)\, d\xi = 2\pi\,\chi(\theta)\, C^*_\sigma(\theta)\hat{\mathcal{G}}^*_{t,\sigma}(\theta)\operatorname{sgn}(\partial_{\theta_1}\omega_\sigma(\theta)) + o(1)$$

as $t \to +\infty$.

Applying Lemma 5.6 to the inner integrals w.r.t. $\xi_1, \dots, \xi_k$ in (5.17), we obtain

$$I_t = (2\pi)^{-d}\,(-i)^k \sum_{m} \sum_{\sigma,\sigma'=1}^{s} \int_{\mathbb{T}^d} g_m(\theta)(\Pi_\sigma(\theta)R^k_t(\theta)_{\sigma\sigma'}\Pi_{\sigma'}(\theta), \hat{\Psi}(\theta)\otimes\overline{\hat{\Psi}}(\theta))\, d\theta + o(1), \qquad (5.18)$$

where we denote $R^k_t(\theta)_{\sigma\sigma'} := \hat{\mathcal{G}}_{t,\sigma}(\theta)\hat{q}(\theta)\,(C^*_{\sigma'}(\theta))^k\,\hat{\mathcal{G}}^*_{t,\sigma'}(\theta)$. Note that

$$(C^*_{\sigma'}(\theta))^k = (-1)^l \quad \text{if} \quad k = 2l, \qquad \text{and} \qquad (C^*_{\sigma'}(\theta))^k = (-1)^l C^*_{\sigma'}(\theta) \ \text{ if } k = 2l+1 \ \text{ (with any } l \geqslant 0).$$

Using (5.16), we can write

$$R^k_t(\theta)_{\sigma\sigma'} = \begin{cases} (-1)^l \sum_\pm \left(\cos\left(\omega^\pm_{\sigma\sigma'}(\theta)t\right)\, L^\mp_1(\hat q) + \sin\left(\omega^\pm_{\sigma\sigma'}(\theta)t\right)\, L^\pm_2(\hat q)\right), & k = 2l, \\ (-1)^l \sum_\pm \left(\pm\cos\left(\omega^\pm_{\sigma\sigma'}(\theta)t\right)\, L^\pm_2(\hat q) \mp \sin\left(\omega^\pm_{\sigma\sigma'}(\theta)t\right)\, L^\mp_1(\hat q)\right), & k = 2l+1, \end{cases} \qquad (5.19)$$

where $\omega^\pm_{\sigma\sigma'}(\theta) \equiv \omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta)$. The oscillatory integrals in (5.18) with $\omega^\pm_{\sigma\sigma'}(\theta) \not\equiv \mathrm{const}$ vanish as $t \to \infty$ by the Lebesgue–Riemann theorem, since all integrands in (5.18) are summable. Furthermore, the identities $\omega^\pm_{\sigma\sigma'}(\theta) \equiv \mathrm{const}_\pm$ with the $\mathrm{const}_\pm \neq 0$ are impossible by **E5**. If we impose condition **E5'** (with $\hat q^{ij}(\theta)$ instead of $\hat q^{ij}_{\mathbf{n}}(\theta)$), then the case $\omega^\pm_{\sigma\sigma'}(\theta) \equiv \mathrm{const}_\pm$ (with $\mathrm{const}_\pm \neq 0$) is possible. However, in this case,

$$\Pi_\sigma(\theta)L^\mp_1(\hat q(\theta))\Pi_{\sigma'}(\theta) \equiv 0 \quad \text{and} \quad \Pi_\sigma(\theta)L^\pm_2(\hat q(\theta))\Pi_{\sigma'}(\theta) \equiv 0,$$

which implies that $\Pi_\sigma(\theta)R_t^k(\theta)_{\sigma\sigma'}\Pi_{\sigma'}(\theta)\equiv 0$. Thus, only the integrals with $\omega^-_{\sigma\sigma'}(\theta)\equiv 0$ contribute to the limit, since $\omega^+_{\sigma\sigma'}(\theta)\equiv 0$ would imply $\omega_\sigma(\theta)\equiv\omega_{\sigma'}(\theta)\equiv 0$ which is impossible by **E4**. Therefore, using (5.18) and (5.19), we obtain

$$I_t=(2\pi)^{-d}\sum_m\sum_{\sigma=1}^s\int_{\mathbb{T}^d} g_m(\theta)(\Pi_\sigma(\theta)L_k(\hat q(\theta))\Pi_\sigma(\theta),\hat\Psi(\theta)\otimes\overline{\hat\Psi}(\theta))\,d\theta+o(1),\quad t\to\infty,$$

where L_k is defined in (5.13). The convergence (5.12) is proved. The convergence (5.11) can be derived in a similar way.

Now Proposition 5.4 (a) follows from the decomposition (5.10), formulas (2.15)–(2.18) and Lemma 5.5 with the matrices $q(x)\equiv q_{\mathbf{n}}(x)$. The assertion (b) was proved in [1, p. 140] for $d=n=1$ and in [8] for any $d,n\geqslant 1$ and $k=1$. For any k, this assertion can be proved using the methods of [8, Lemma 8.4].

6. HARMONIC CRYSTALS IN THE HALF-SPACE

In this section, we consider the dynamics of the harmonic crystals in the integer half-space $\mathbb{Z}^d_+=\{x\in\mathbb{Z}^d:\,x_1>0\}$, $d\geqslant 1$,

$$\ddot u(x,t)=-\sum_{y\in\mathbb{Z}^d_+}(V(x-y)-V(x-y_-))\,u(y,t),\quad x\in\mathbb{Z}^d_+,\quad t\in\mathbb{R},\tag{6.1}$$

$y_-:=(-y_1,y_2,\dots,y_d)$, with zero boundary condition (as $x_1=0$)

$$u(x,t)|_{x_1=0}=0,\tag{6.2}$$

and with the initial data (as $t=0$)

$$u(x,0)=u_0(x),\quad \dot u(x,0)=v_0(x),\quad x\in\mathbb{Z}^d_+.\tag{6.3}$$

The matrix $V(x)$ satisfies conditions **E1**–**E4**. In addition, we assume that

$$V(x_-)=V(x).\tag{6.4}$$

This condition is fulfilled, for instance, for the nearest neighbor crystal (2.7). Condition **E6** imposed on $V(x)$ in Section 2.1 can be weakened as follows.

E6'. $\displaystyle\int_{\mathbb{T}^d}\sin^2(\theta_1)\|\hat V^{-1}(\theta)\|\,d\theta<\infty.$

Assume that the initial datum $Y_0=(u_0,v_0)$ of the problem (6.1)–(6.3) belongs to the phase space $\mathcal{H}_{\alpha,+}$, $\alpha\in\mathbb{R}$.

Definition 6.1. Denote by $\mathcal{H}_{\alpha,+}$ the Hilbert space of $\mathbb{R}^n\times\mathbb{R}^n$-valued functions of $x\in\mathbb{Z}^d_+$ endowed with the norm $\|Y\|^2_{\alpha,+}=\sum_{x\in\mathbb{Z}^d_+}\langle x\rangle^{2\alpha}|Y(x)|^2<\infty$.

To coordinate the boundary and initial conditions, we assume that $u_0(x)=v_0(x)=0$ for $x_1=0$. Write $Y(t)=(u(\cdot,t),\dot u(\cdot,t))$.

Lemma 6.2. (*see* [10, Corollary 2.4]) *Let conditions* (6.4), **E1**, *and* **E2** *hold, and choose some* $\alpha\in\mathbb{R}$. *Then for any* $Y_0\in\mathcal{H}_{\alpha,+}$, *there exists a unique solution* $Y(t)\in C(\mathbb{R},\mathcal{H}_{\alpha,+})$ *to problem* (6.1)–(6.3). *The operator* $U_+(t):Y_0\mapsto Y(t)$ *is continuous in* $\mathcal{H}_{\alpha,+}$.

Below, we assume that $\alpha<-d/2$ if condition **E6** holds, and $\alpha<-d/2-1$ if condition **E6'** holds.

We assume that Y_0 is a measurable random function with values in $(\mathcal{H}_{\alpha,+},\mathcal{B}(\mathcal{H}_{\alpha,+}))$ and denote by μ_0^+ a Borel probability measure on $\mathcal{H}_{\alpha,+}$ giving the distribution of Y_0. Let $\mathbb{E}_+$ stand for the integral w.r.t. μ_0^+. Denote by $Q_0^+(x,y)$ the correlation matrix of μ_0^+,

$$Q_0^+(x,y)=\mathbb{E}_+(Y_0(x)\otimes Y_0(y))\equiv\int(Y_0(x)\otimes Y_0(y))\,\mu_0^+(dY_0),\quad x,y\in\mathbb{Z}^d_+.$$

In particular, $Q_0^+(x,y)=0$ for $x_1=0$ or $y_1=0$. Assume that μ_0^+ satisfies conditions **S1** and **S2** stated in Section 2.1 (but with $\mathbb{Z}^d_+$ and $\mathbb{E}_+$ instead of $\mathbb{Z}^d$ and $\mathbb{E}$). Condition **S3** needs some modification.

S3. Choose some $k\in\{1,\dots,d\}$. The initial covariance $Q_0^+(x,y)$ is $Q_0^+(x,y)=q_0^+(\bar x,\bar y,\tilde x-\tilde y)$, $x,y\in\mathbb{Z}^d_+$, where $x=(\bar x,\tilde x)$, $\bar x=(x_1,\dots,x_k)$, $\tilde x=(x_{k+1},\dots,x_d)$. Write (cf. (2.11))

$$\mathcal{N}_+^k:=\{\mathbf{n}=(n_1,n_2,\dots,n_k),\ \ n_1=2,\ \ n_j\in\{1,2\}\ \text{ for all }\ j=2,\dots,k\}.$$

Suppose that for any $\varepsilon>0$ there exists an $N(\varepsilon)\in\mathbb{N}$ such that for any $\bar y=(y_1,\dots,y_k)\in\mathbb{Z}^k$: $y_1>N(\varepsilon)$ and $(-1)^{n_j}y_j>N(\varepsilon)$, $\forall j=2,\dots,k$, the following bound holds (cf (2.12))

$$\left|q_0^+(\bar y+\bar z,\bar y,\tilde z)-q_{\mathbf{n}}(z)\right|<\varepsilon\quad\text{for any fixed }\ z=(\bar z,\tilde z)\in\mathbb{Z}^d.$$

Here $q_{\mathbf{n}}(z)$, $\mathbf{n}\in\mathcal{N}_+^k$, are the correlation matrices of some translation-invariant measures $\mu_{\mathbf{n}}$ with zero mean value in $\mathcal{H}_\alpha$.

In particular, if $k=1$, then $Q_0^+(x,y)=q_0^+(x_1,y_1,\tilde x-\tilde y)$, $\tilde x=(x_2,\dots,x_d)$, and (cf. (2.13))

$$q_0^+(y_1+z_1,y_1,\tilde z)\to q_2(z)\quad\text{as}\ \ y_1\to+\infty,\quad z=(z_1,\tilde z)\in\mathbb{Z}^d. \tag{6.5}$$

Example 6.3. The examples of initial measures μ_0^+ satisfying conditions **S1**–**S3** can be constructed by a similar way as for μ_0 in Section 3. Indeed, let us define a Borel probability measure μ_0^+ as a distribution of the random function (cf. (3.3))

$$Y_0(x)=\sum_{\mathbf{n}\in\mathcal{N}_+^k}\overline{\zeta}_{\mathbf{n}}(\bar x)Y_{\mathbf{n}}(x),\quad x=(\bar x,\tilde x)\in\mathbb{Z}^d_+,\quad \bar x=(x_1,\dots,x_k),\quad \tilde x=(x_{k+1},\dots,x_d),$$

where $\overline{\zeta}_{\mathbf{n}}(\bar x)=\zeta_2(x_1)\zeta_{n_2}(x_2)\cdot\ldots\cdot\zeta_{n_k}(x_k)$, $\mathbf{n}\in\mathcal{N}_+^k$, the sequences ζ_1 and ζ_2 are defined in (3.2), $Y_{\mathbf{n}}(x)$ are Gaussian independent vectors in $\mathcal{H}_{\alpha,+}$ with distributions $\mu_{\mathbf{n}}$. The Gaussian measures $\mu_{\mathbf{n}}$ satisfying conditions **S1** and **S2** are constructed in Section 3. Then, the measure μ_0^+ satisfies **S1**–**S3**.

We define μ_t^+, $t\in\mathbb{R}$, as a Borel probability measure in $\mathcal{H}_{\alpha,+}$ which gives the distribution of the random solution $Y(t)$, $\mu_t^+(B)=\mu_0^+(U_+(-t)B)$, $B\in\mathcal{B}(\mathcal{H}_{\alpha,+})$, $t\in\mathbb{R}$. Denote by

$$Q_t^+(x,y)=\int(Y(x)\otimes Y(y))\ \mu_t^+(dY),\quad x,y\in\mathbb{Z}^d_+,$$

the covariance of μ_t^+. The mixing condition **S4** (see Section 2.3) for μ_0^+ is formulated as for the measure μ_0, but with sets $\mathcal{A}$ and $\mathcal{B}$ from $\mathbb{Z}^d_+$ instead of $\mathbb{Z}^d$.

Introduce the limiting correlation matrix $Q_\infty^+(x,y)$. It has the form

$$Q_\infty^+(x,y)=q_\infty^+(x-y)-q_\infty^+(x-y_-)-q_\infty^+(x_--y)+q_\infty^+(x_--y_-),\quad x,y\in\mathbb{Z}^d_+. \tag{6.6}$$

Here $q_\infty^+(x)$ is defined as $q_\infty(x)$ (see formulas (2.15)–(2.19)), but with $\mathcal{N}_+^k$ instead of $\mathcal{N}^k$. For example, if $k=1$, then $\hat q_\infty^+(\theta)$ has the form (2.15) with matrices (cf. (2.20))

$$\mathbf{M}^+_{1,\sigma}(\theta)=\frac{1}{2}L_1^+\left(\hat q_2(\theta)\right),\quad \mathbf{M}^-_{1,\sigma}(\theta)=\frac{1}{2}L_2^-\left(\hat q_2(\theta)\right)\operatorname{sgn}\left(\partial_{\theta_1}\omega_\sigma(\theta)\right),$$

where $\hat q_2(\theta)$ is the Fourier transform of the matrix $q_2(z)$ introduced in (6.5).

Theorem 6.4. (i) *Let conditions* (6.4), **E1**–**E4**, **E5'**, **E6'**, *and* **S1**–**S3** *be fulfilled. Then for any* $x,y\in\mathbb{Z}^d_+$, $Q_t^+(x,y)\to Q_\infty^+(x,y)$ *as* $t\to\infty$. (ii) *Let conditions* (6.4), **E1**–**E3**, **E4'**, **E5'**, **E6'**, **S1**, **S3**, *and* **S4** *be fulfilled. Then the measures* μ_t^+ *weakly converge in the Hilbert space* $\mathcal{H}_{\alpha,+}$ *as* $t\to\infty$. *The limiting measure* μ_∞^+ *is a Gaussian measure on* $\mathcal{H}_{\alpha,+}$ *with the covariance* $Q_\infty^+(x,y)$ *defined in* (6.6).

Theorem 6.4 can be proved using the technique of Theorems 2.6, 2.11 and [10, Th. A]. Below we note only some features in the proof of Theorem 6.4.

6.1. The Proof

Lemma 6.5. *Let conditions* (6.4), **E1**–**E3**, **E6'**, **S1**, *and* **S2** *be fulfilled. Then the following uniform bound holds,* $\sup_{t\in\mathbb{R}}\mathbb{E}_+\left(\|Y(t)\|^2_{\alpha,+}\right)<\infty$.

Proof. By $l^2_+\equiv l^2(\mathbb{Z}^d_+)\otimes\mathbb{R}^n$, $d,n\geqslant 1$, we shall denote the Hilbert space of sequences $f(x)=(f_1(x),\dots,f_n(x))$ endowed with the norm

$$\|f\|_{l^2_+}=\sqrt{\sum_{x\in\mathbb{Z}^d_+}|f(x)|^2}.$$

Let $\langle\cdot\,,\cdot\rangle_+$ stand for the inner product in $\ell^2_+\times\ell^2_+$. At first, by conditions **S1** and **S2**, we have (cf. (5.1))

$$|\langle Q^+_0(x,y),\Phi(x)\otimes\Psi(y)\rangle_+|\leqslant C\|\Phi\|_{l^2_+}\|\Psi\|_{l^2_+}\quad\text{for any}\quad\Phi,\Psi\in\ell^2_+\times\ell^2_+.\tag{6.7}$$

Second, the solutions of problem (6.1)–(6.3) has the form

$$Y(x,t)=\sum_{x'\in\mathbb{Z}^d_+}\mathcal{G}_{t,+}(x,x')Y_0(x'),\quad\text{where}\ \ \mathcal{G}_{t,+}(x,x')=\mathcal{G}_t(x-x')-\mathcal{G}_t(x-x'_-),\tag{6.8}$$

with $\mathcal{G}_t(x)$ defined in (5.4). As for (5.6), we have

$$\left(Q^+_t(x,y)\right)^{ij}=\left\langle Q^+_0(x',y'),\Phi^i_x(x',t)\otimes\Phi^j_y(y',t)\right\rangle_+,\quad x,y\in\mathbb{Z}^d_+,\tag{6.9}$$

where $\Phi^i_x(x',t):=\left(\mathcal{G}^{i0}_{t,+}(x,x'),\mathcal{G}^{i1}_{t,+}(x,x')\right)$, $i=0,1$. By the Parseval identity, formula (5.4), and condition **E6'**, we have

$$\begin{aligned}\|\Phi^i_x(\cdot,t)\|^2_{l^2}=&(2\pi)^{-d}\int_{\mathbb{T}^d}\left|\hat\Phi^i_x(\theta,t)\right|^2d\theta=\ (2\pi)^{-d}4\int_{\mathbb{T}^d}\sin^2(\theta_1x_1)\left(|\hat{\mathcal{G}}^{i0}_t(\theta)|^2+|\hat{\mathcal{G}}^{i1}_t(\theta)|^2\right)d\theta\\ &\leqslant\int_{\mathbb{T}^d}\sin^2(\theta_1x_1)\left(C_1+C_2\|\hat V^{-1}(\theta)\|\right)d\theta\leqslant C_3+C_4|x_1|^2,\end{aligned}\tag{6.10}$$

where the constants C_3 and C_4 do not depend on $t\in\mathbb{R}$ and $x\in\mathbb{Z}^d$, and $C_4=0$ if condition **E6** holds. Hence, (6.7), (6.9), and (6.10) imply

$$|\left(Q^+_t(x,y)\right)^{ij}|\leqslant C\|\Phi^i_x(\cdot,t)\|_{\ell^2_+}\|\Phi^j_y(\cdot,t)\|_{\ell^2_+}\leqslant\ C\sqrt{C_3+C_4|x_1|^2}\sqrt{C_3+C_4|y_1|^2},\quad x,y\in\mathbb{Z}^d_+.$$

Therefore, the choice of α implies the following bound

$$\begin{aligned}\mathbb{E}_+(\|Y(\cdot,t)\|^2_{\alpha,+})=&\sum_{x\in\mathbb{Z}^d_+}\langle x\rangle^{2\alpha}\,\mathrm{tr}\,\left(\left(Q^+_t(x,x)\right)^{00}+\left(Q^+_t(x,x)\right)^{11}\right)\\ \leqslant&C\sum_{x\in\mathbb{Z}^d_+}\langle x\rangle^{2\alpha}\left(C_3+C_4|x_1|^2\right)<\infty.\end{aligned}$$

By the Prokhorov Theorem, Lemma 6.5 implies that the family of measures $\{\mu^+_t,t\in\mathbb{R}\}$ is weakly compact on the space $\mathcal{H}_{\alpha,+}$.

Remark 6.6. Suppose that $V(x)$ is of the particular form (2.7). If (i) $d\geqslant 3$ or (ii) $d=1,2$ and all m_l are positive, then $V(x)$ satisfies condition **E6** and the results of Theorem 6.4 are valid with any $\alpha<-d/2$. If $d=1,2$ and $m_l=0$ for some l, then the condition **E6'** holds. In this case, we can apply Feiér's theorem (see, e.g., [17]) and obtain $\|\Phi_x^i(\cdot,t)\|_{l^2}^2 \leqslant C_3 + C_4|x_1|$ (cf. the bound (6.10)). Then the assertions of Theorem 6.4 remain valid with any $\alpha < -(d+1)/2$.

Proof of Theorem 6.4 (i). At first, using (6.8), we decompose the covariance $Q_t^+(x,y)$ into a sum of four terms:

$$Q_t^+(x,y) = \sum_{x',y'\in\mathbb{Z}_+^d} \mathcal{G}_{t,+}(x,x')Q_0^+(x',y')\mathcal{G}_{t,+}^T(y,y') = R_t(x,y) - R_t(x,y_-) - R_t(x_-,y) + R_t(x_-,y_-),$$

where $(\,)^T$ denotes matrix transposition,

$$R_t(x,y) := \sum_{x',y'\in\mathbb{Z}_+^d} \mathcal{G}_t(x-x')Q_0^+(x',y')\mathcal{G}_t^T(y-y').$$

Therefore, Theorem 6.4 (i) follows from the following convergence

$$R_t(x,y) \to q_\infty^+(x-y) \quad \text{as} \ \ t\to\infty, \quad x,y\in\mathbb{Z}^d. \tag{6.11}$$

To prove (6.11), let us define $\bar{Q}_0^+(x,y)$ to be equal to $Q_0^+(x,y)$ for $x,y\in\mathbb{Z}_+^d$, and to 0, otherwise. Denote by $Q_*^+(x,y)$ the matrix defined as $Q_*(x,y)$ (see (5.10)), but with the summation over $\mathcal{N}_+^k$ instead of $\mathcal{N}^k$. Put $Q_r^+(x,y) = \bar{Q}_0^+(x,y) - Q_*^+(x,y)$. Then (6.11) follows from the two next assertions. For any $x,y\in\mathbb{Z}^d$,

$$\sum_{x',y'\in\mathbb{Z}^d} \mathcal{G}_t(x-x')Q_*^+(x',y')\mathcal{G}_t^T(y-y') \to q_\infty^+(x-y), \quad t\to\infty,$$
$$\sum_{x',y'\in\mathbb{Z}^d} \mathcal{G}_t(x-x')Q_r^+(x',y')\mathcal{G}_t^T(y-y') \to 0, \quad t\to\infty.$$

The proof of these assertions similar to the proof of Proposition 5.4.

6.2. Energy Current in the Half-Space

Here we calculate the limiting energy current density $\mathbf{J}_{+,\infty} = (J_{+,\infty}^1,\ldots,J_{+,\infty}^d)$.

Lemma 6.7. *If $d=1$, then $\mathbf{J}_{+,\infty}=0$. If $d\geqslant 2$, then the coordinates on the energy current density $\mathbf{J}_{+,\infty}\equiv\mathbf{J}_{+,\infty}(x_1)$, $x_1\geqslant 0$, are*

$$J_{+,\infty}^1(x_1)\equiv 0,\ \ J_{+,\infty}^l(x_1) = -\frac{2i}{(2\pi)^d}\int_{\mathbb{T}^d} \sin^2(\theta_1 x_1)\,\mathrm{tr}\left[\left(\hat{q}_\infty^+(\theta)\right)^{10}\partial_{\theta_l}\hat{V}(\theta)\right]d\theta,\ \ l=2,\ldots,d, \tag{6.12}$$

with q_∞^+ from (6.6). *In particular,* $\mathbf{J}_{+,\infty}(0)=0$.

To prove (6.12), we first formally derive the expression of the energy current for the finite energy solutions $u(x,t)$. We define the energy in the region $\Omega_l := \{x\in\mathbb{Z}_+^d : x_l\geqslant 0\}$ as

$$\mathcal{E}_+^l(t) := \frac{1}{2}\sum_{x\in\Omega_l}\left\{|\dot{u}(x,t)|^2 + \sum_{y\in\mathbb{Z}_+^d}(u(x,t),(V(x-y)-V(x-y_-))u(y,t))\right\}, \quad l=1,\ldots,d.$$

Then, using Eq. (6.1) and conditions (6.12) and **E2**, we obtain

$$\dot{\mathcal{E}}_+^1(t)=0, \quad \dot{\mathcal{E}}_+^l(t) = \sum_{x'\in\mathbb{Z}_+^d} J_+^l(x',t), \quad l=2,\ldots,d.$$

Here $J^l_+(x',t)$ stands for the energy current density in the direction $e_l=(0,\delta_{l2},\dots,\delta_{ld})$,

$$J^l_+(x',t):=\frac{1}{2}\sum_{y'\in\mathbb{Z}^d_+}\left\{\sum_{m\leqslant -1,\,p\geqslant 0}A^l_{mp}(x',y',t)-\sum_{m\geqslant 0,\,p\leqslant -1}A^l_{mp}(x',y',t)\right\},$$

where $A^l_{mp}(x',y',t):=(\dot u(x,t),(V(x-y)-V(x-y_-))\,u(y,t))$ for $x\equiv x'+me_l$, $y\equiv y'+pe_l$, $x',y'\in\mathbb{Z}^d_+$ with $x'_l=y'_l=0$, $l=2,\dots,d$.

Let $u(x,t)$ be a random solution to problem (6.1)–(6.3) with initial measure μ_0^+ satisfying **S1**–**S3**. Using Theorem 6.4 (i), we can write

$$\mathbb{E}_+\left(J^l_+(x',t)\right)\to J^l_{+,\infty}:=\frac{1}{2}\sum_{y'\in\mathbb{Z}^d_+}\left\{\sum_{m\leqslant -1,\,p\geqslant 0}B^l_{mp}(x',y')-\sum_{m\geqslant 0,\,p\leqslant -1}B^l_{mp}(x',y')\right\}\quad\text{as}\quad t\to\infty,$$

where $B^l_{mp}(x',y'):=\operatorname{tr}\left[(Q^+_\infty(x,y))^{10}\left(V^T(x-y)-V^T(x-y_-)\right)\right]$, $x\equiv x'+me_l$, $y\equiv y'+pe_l$, $x',y'\in\mathbb{Z}^d_+$ with $x'_l=y'_l=0$. Applying (6.6), (6.4) and the Parseval identity, we obtain

$$\begin{aligned}J^l_{+,\infty}&=-\frac{1}{2}\operatorname{tr}\sum_{y\in\mathbb{Z}^d}y_l\left(\left(q^+_\infty(x'+y)\right)^{10}-\left(q^+_\infty(x'_-+y)\right)^{10}\right)\left(V^T(x'+y)-V^T(x'+y_-)\right)\\&=-\frac{i}{2}(2\pi)^{-d}\operatorname{tr}\int_{\mathbb{T}^d}\left(e^{-i(x',\theta)}-e^{-i(x'_-,\theta)}\right)\left(\hat q^+_\infty(\theta)\right)^{10}\left(e^{i(x',\theta)}-e^{i(x'_-,\theta)}\right)\partial_{\theta_l}\hat V^*(\theta)\,d\theta.\end{aligned}$$

Using equalities $e^{i\,x_1\theta_1}-e^{-i\,x_1\theta_1}=2i\sin(\theta_1x_1)$ and $\hat V^*(\theta)=\hat V(\theta)$, we obtain (6.12).

Let $\mu_{\mathbf n}=g_{\beta_{\mathbf n}}$, $\mathbf n\in\mathcal N^k_+$, be the Gibbs measures constructed in Section 4.2 with temperatures $T_{\mathbf n}>0$. The correlation matrices of $\mu_{\mathbf n}$ are $q_{\mathbf n}(x-y)\equiv q_{\beta_{\mathbf n}}(x-y)$, see (4.2). We impose, in addition, condition (3.6) on the matrix V, which implies the bound (3.7) for $q^{00}_{\mathbf n}$. Then, condition **S2** is fulfilled. In this case,

$$\left(\hat q^+_\infty(\theta)\right)^{10}=-i\sum_{\sigma=1}^{s}\omega_\sigma^{-1}(\theta)\Pi_\sigma(\theta)\left(\frac{1}{2^k}\sum_{\mathbf n\in\mathcal N^k_+}T_{\mathbf n}S^{\rm odd}_{k,\mathbf n}(\omega_\sigma(\theta))\right),$$

where the function $S^{\rm odd}_{k,\mathbf n}(\omega_\sigma)$ is defined in (2.17). Hence, for $l=2,\dots,d$ (cf. (4.4)),

$$\begin{aligned}J^l_{+,\infty}(x_1)&=-\frac{4}{(2\pi)^d}\sum_{\sigma=1}^{s}\int_{\mathbb{T}^d}r_\sigma\sin^2(\theta_1x_1)\left(\frac{1}{2^k}\sum_{\mathbf n\in\mathcal N^k_+}T_{\mathbf n}S^{\rm odd}_{k,\mathbf n}(\omega_\sigma(\theta))\right)\frac{\partial\omega_\sigma(\theta)}{\partial\theta_l}\,d\theta\\&=-\sum_{\text{odd}\,m\in\{1,\dots,k\}}\ \sum_{(p_1,\dots,p_m)\in\mathcal P_m(k)}c^l_{p_1\dots p_m}(x_1)\,\frac{1}{2^{k-1}}\sum_{\mathbf n\in\mathcal N^k_+}(-1)^{n_{p_1}+\dots+n_{p_m}}T_{\mathbf n},\end{aligned}\tag{6.13}$$

where the functions $c^l_{p_1\dots p_m}(x_1)$, $x_1\geqslant 0$, are defined as follows (cf. (4.5))

$$c^l_{p_1\dots p_m}(x_1):=\frac{2}{(2\pi)^d}\sum_{\sigma=1}^{s}\int_{\mathbb{T}^d}r_\sigma\sin^2(\theta_1x_1)\operatorname{sgn}\left(\frac{\partial\omega_\sigma(\theta)}{\partial\theta_{p_1}}\right)\cdot\ldots\cdot\operatorname{sgn}\left(\frac{\partial\omega_\sigma(\theta)}{\partial\theta_{p_m}}\right)\frac{\partial\omega_\sigma(\theta)}{\partial\theta_l}\,d\theta.$$

Write

$$c_l(x_1)\equiv c^l_l(x_1)=\frac{2}{(2\pi)^d}\sum_{\sigma=1}^{s}\int_{\mathbb{T}^d}r_\sigma\sin^2(\theta_1x_1)\left|\frac{\partial\omega_\sigma(\theta)}{\partial\theta_l}\right|d\theta>0,\quad l=2,\dots,k.\tag{6.14}$$

Let us apply condition **SC** to $\omega_\sigma(\theta)$. Then, $\mathbf{J}_{+,\infty}(x_1)\equiv 0$ if $k=1$. For $k\geqslant 2$, we obtain

$$J^l_{+,\infty}(x_1)=\begin{cases}-c_l(x_1)\dfrac{1}{2^{k-1}}\displaystyle\sum_{\mathbf{n}\in\mathcal{N}^k_+}{}'(T_{\mathbf{n}}|_{n_l=2}-T_{\mathbf{n}}|_{n_l=1}), & l=2,\dots,k,\\ 0, & l=1,\ l=k+1,\dots,d,\end{cases} \tag{6.15}$$

where the summation $\sum'$ is taken over $n_2,\dots,n_{l-1},n_{l+1},\dots,n_k\in\{1,2\}$. Therefore, in the case of $d,k\geqslant 2$, we can choose positive numbers $T_{\mathbf{n}}$ so that $\mathbf{J}_{+,\infty}(x_1)\neq 0$ for $x_1>0$.

Using the formula $2\sin^2(\theta_1x_1)=1-\cos(2\theta_1x_1)$ and the Lebesgue–Riemann theorem, we see that $c_l(x_1)\to c_l$ as $x_1\to+\infty$, where the positive constant c_l is defined in (4.6). Hence, for $l=2,\dots,k$,

$$J^l_{+,\infty}(x_1)\to -c_l\,\frac{1}{2^{k-1}}\sum{}'\left(T_{\mathbf{n}}|_{n_l=2}-T_{\mathbf{n}}|_{n_l=1}\right)\quad\text{as } x_1\to+\infty. \tag{6.16}$$

Consider some particular cases of the formula (6.13).

Example 6.8. Let $k=1$ and μ_0^+ satisfy condition **S3** with a Gibbs measure $\mu_2\equiv g_\beta$, $\beta=1/T_2$. For instance, the initial data Y_0 has the form $Y_0(x)=\zeta_2(x_1)Y_2(x)$, where ζ_2 is defined in (3.2) and $Y_2(x)$ has the Gibbs distribution g_β. Hence, $J^1_{+,\infty}\equiv 0$, and

$$J^l_{+,\infty}(x_1)=-\frac{2T}{(2\pi)^d}\sum_{\sigma=1}^s\int_{\mathbb{T}^d}r_\sigma\sin^2(\theta_1x_1)\,\mathrm{sgn}\left(\frac{\partial\omega_\sigma(\theta)}{\partial\theta_1}\right)\frac{\partial\omega_\sigma(\theta)}{\partial\theta_l}\,d\theta,\quad l=2,\dots,d.$$

If condition **SC** holds, then $\mathbf{J}_{+,\infty}(x_1)=0$ for any $x_1\geqslant 0$.

Example 6.9. Let $d\geqslant 2$, $k=2$ and μ_0^+ satisfy condition **S3** with Gibbs measures

$$\mu_{\mathbf{n}}\equiv g_{\beta_{\mathbf{n}}},\quad \beta_{\mathbf{n}}=1/T_{\mathbf{n}},\quad \mathbf{n}=(n_1,n_2)\in\mathcal{N}^2_+=\{(2,1);(2,2)\}.$$

For instance, the initial data Y_0 is of a form

$$Y_0(x)=\zeta_2(x_1)(\zeta_1(x_2)Y_{21}(x)+\zeta_2(x_2)Y_{22}(x)),\quad x\in\mathbb{Z}^d_+,$$

where $\zeta_n(x)$ is defined in (3.2), $Y_{21}(x)$ and $Y_{22}(x)$ are independent vectors in $\mathcal{H}_\alpha$ with Gibbs distributions μ_{21} and μ_{22} corresponding to positive temperatures T_{21} and T_{22}, respectively. Hence, our model can be considered as a "system + two reservoirs," where the "reservoirs" consist of crystal particles lying in the two regions $\{x\in\mathbb{Z}^d_+ : x_2<-a\}$ and $\{x\in\mathbb{Z}^d_+ : x_2>a\}$, $a>0$. It follows from Lemma 6.7 and formula (6.13) that $J^1_{+,\infty}(x_1)\equiv 0$ and

$$\begin{aligned}J^l_{+,\infty}(x_1)=-\frac{1}{(2\pi)^d}\sum_{\sigma=1}^s\int_{\mathbb{T}^d}r_\sigma\sin^2(\theta_1x_1)\Big[&\mathrm{sign}\left(\frac{\partial\omega_\sigma(\theta)}{\partial\theta_1}\right)(T_{21}+T_{22})\\ &+\mathrm{sgn}\left(\frac{\partial\omega_\sigma(\theta)}{\partial\theta_2}\right)(T_{22}-T_{21})\Big]\frac{\partial\omega_\sigma(\theta)}{\partial\theta_l}\,d\theta,\quad l=2,\dots,d.\end{aligned}$$

Under condition **SC** on the eigenvalues $\omega_\sigma(\theta)$, we obtain

$$\mathbf{J}_{+,\infty}(x_1)=-\frac{1}{2}(0,c_2(x_1)(T_{22}-T_{21}),0,\dots,0)$$

with $c_2(x_1)$ introduced in (6.14). Moreover,

$$\mathbf{J}_{+,\infty}(x_1)\to-\frac{1}{2}(0,c_2(T_{22}-T_{21}),0,\dots,0)\quad\text{as } x_1\to+\infty,$$

where the positive constant c_2 is defined in (4.6).

Remark 6.10. In [11], we considered the 1D chain of harmonic oscillators on the half-line with *nonzero* boundary condition and studied the following initial boundary value problem:

$$\ddot{u}(x,t)=\begin{cases}(\Delta_L-m^2)u(x,t), \qquad x\geqslant 1, \quad t>0,\\ \ddot{u}(0,t)=-\kappa u(0,t)-m^2u(0,t)-\gamma\dot{u}(0,t)+u(1,t)-u(0,t), \quad t>0,\\ u(x,0)=u_0(x), \quad \dot{u}(x,0)=v_0(x), \quad x\geqslant 0.\end{cases}$$

Here $u(x,t)\in\mathbb{R}$, $m\geqslant 0$, $\gamma\geqslant 0$, Δ_L denotes the second derivative on $\mathbb{Z}$. We impose some restrictions on the coefficients m,κ,γ of the system. In particular, if $\gamma>0$, then either $m>0$ or $\kappa>0$. If $\gamma=0$, then $\kappa\in(0,2)$. We obtain results similar to (1.1) and (1.3). Furthermore, the limiting energy current at the origin equals $J_\infty:=-\gamma\lim_{t\to\infty}\mathbb{E}\,(\dot{u}(0,t))^2$. Hence, in the case when $\gamma>0$, $J_\infty\neq 0$ (cf. Example 6.8) if $\int(Y^1(0))^2\,\mu_\infty(dY)\neq 0$ (the limit measures μ_∞ satisfying the last condition are constructed in [11]).

ACKNOWLEDGMENT

This work was done with the financial support from the Russian Science Foundation (Grant no. 19-71-30004).

REFERENCES

1. C. Boldrighini, A. Pellegrinotti, and L. Triolo, "Convergence to Stationary States for Infinite Harmonic Systems," J. Stat. Phys. **30**(1), 123–155 (1983).
2. F. Bonetto, J. L. Lebowitz, and L. Rey-Bellet, "Fourier Law: a Challenge to Theorists," in *Mathematical Physics 2000* (A. Fokas et al. (Eds), Imperial College Press, London, 2000), pp. 128–150.
3. F. Bonetto, J. L. Lebowitz, and J. Lukkarinen, "Fourier's Law for a Harmonic Crystal with Self-Consistent Stochastic Reservoirs," J. Statist. Phys. **116**(1–4), 783–813 (2004).
4. A. Casher and J. L. Lebowitz, "Heat Flow in Regular and Disordered Harmonic Chains," J. Math. Phys. **12**(8), 1701–1711 (1971).
5. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer, New York, 1981).
6. R. L. Dobrushin and Yu. M. Sukhov, "On the Problem of the Mathematical Foundation of the Gibbs Postulate in Classical Statistical Mechanics," in *Mathematical Problems in Theoretical Physics, Lecture Notes in Physics* (Springer-Verlag, Berlin, 1978), Vol. 80, pp. 325–340.
7. T. V. Dudnikova, A. I. Komech, and H. Spohn, "On the Convergence to Statistical Equilibrium for Harmonic Crystals," J. Math. Phys. **44**(6), 2596–2620 (2003).
8. T. V. Dudnikova, A. I. Komech, and N. J. Mauser, "On Two-Temperature Problem for Harmonic Crystals," J. Statist. Phys. **114**(3–4), 1035–1083 (2004).
9. T. V. Dudnikova and A. I. Komech, "On a Two-Temperature Problem for the Klein–Gordon Equation," Teor. Veroyattist. i Primenen. **50**, 675–710 (2005) (English transl. in Theory Probab. Appl. **50**(4), 582–611 (2006)).
10. T. V. Dudnikova, "On the Asymptotical Normality of Statistical Solutions for Harmonic Crystals in Half-Space," Russian J. Math. Phys. **15**(4), 460–472 (2008).
11. T. V. Dudnikova, "On Convergence to Equilibrium for One-Dimensional Chain of Harmonic Oscillators on the Half-Line," J. Math. Phys. **58**(4), 043301 (2017).
12. J.-P. Eckmann, C.-A. Pillet, and L. Rey-Bellet, "nonequilibrium Statistical Mechanics of Anharmonic Chains Coupled to Two Heat Baths at Different Temperatures," Comm. Math. Phys. **201**(3), 657–697 (1999).
13. J.-P. Eckmann, C.-A. Pillet, and L. Rey-Bellet, "Entropy Production in Nonlinear, Thermally Driven Hamiltonian Systems," J. Statist. Phys. **95**(1–2), 305–331 (1999).
14. F. Fidaleo and C. Liverani, "Ergodic Properties for a Quantum Nonlinear Dynamics," J. Statist. Phys. **97**(5–6), 957–1009 (1999).
15. V. Jakšić and C.-A. Pillet, "Ergodic Properties of Classical Dissipative Systems. I," Acta Math. **181**(2), 245–282 (1998).
16. I. A. Ibragimov and Yu. V. Linnik, *Independent and Stationary Sequences of Random Variables* (Ed. by J.F.C. Kingman, Wolters–Noordhoff, Groningen, 1971).
17. Y. Katznelson, *An Introduction in Harmonic Analysis*, 3rd edition (Cambridge University Press, 2004).

18. O. E. Lanford III and J. L. Lebowitz, “Time Evolution and Ergodic Properties of Harmonic Systems,” in *Dynamical Systems, Theory and Applications. Lecture Notes in Physics* (Springer-Verlag, Berlin, 1975), Vol. 38, pp. 144–177.
19. S. Lepri, R. Livi, and A. Politi, “Thermal Conduction in Classical Low-Dimensional Lattices,” Phys. Rep. **377**(1), 1–80 (2003).
20. S. Lepri (ed.): *Thermal Transport in Low Dimensions: From Statistical Physics to Nanoscale Heat Transfer* (Lecture Notes in Physics, Springer, 2016), Vol. 921.
21. H. Nakazawa, “On the Lattice Thermal Conduction,” Supplement of the Progress of Theor. Phys. **45**, 231–262 (1970).
22. L. Rey-Bellet and L. E. Thomas, “Exponential Convergence to nonequilibrium Stationary States in Classical Statistical Mechanics,” Comm. Math. Phys. **225**(2), 305–329 (2002).
23. Z. Rieder, J. L. Lebowitz, and E. Lieb, “Properties of a Harmonic Crystal in a Stationary Nonequilibrium State,” J. Math. Phys. **8**(5), 1073–1078 (1967).
24. H. Spohn and J. L. Lebowitz, “Stationary nonequilibrium States of Infinite Harmonic Systems,” Comm. Math. Phys. **54**(2), 97–120 (1977).
25. H. Spohn, *Large Scale Dynamics of Interacting Particles* (Texts and Monographs in Physics, Springer-Verlag, Heidelberg, 1991).
26. M. I. Vishik and A. V. Fursikov, *Mathematical Problems of Statistical Hydromechanics* (Kluwer Academic, New York, 1988).