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On The Existence and Asymptotic Stability of Periodic Contrast Structures in Quasilinear Reaction-Advection-Diffusion Equations

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Abstract. We consider periodic solutions with internal transition and boundary layers (periodic contrast structures) for a singularly perturbed parabolic equation that is referred to in applications as reaction-advection-diffusion equation. An asymptotic approximation to such solutions is constructed and an existence theorem is proved. An efficient algorithm is developed for constructing an asymptotic approximation to the localization curve of the transition layer. To substantiate the asymptotic thus constructed, we use the asymptotic method of differential inequalities. Moreover, we assert that asymptotic stability of the solution in the sense of Lyapunov occurs.

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1. INTRODUCTION. STATEMENT OF THE PROBLEM.

We consider the boundary value problem for a singularly perturbed quasilinear equation with a small parameter ε

$$\begin{aligned} &N_\varepsilon(u) := \varepsilon^2\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\right) - \varepsilon A(u,x,t)\,\frac{\partial u}{\partial x} - F(u,x,t,\varepsilon) = 0, \\ &(x,t) \in D := \{(x,t) \in R^2 : -1 < x < 1, t \in R\}, \\ &u(-1,t,\varepsilon) = u^{(-)}(t), \quad u(1,t,\varepsilon) = u^{(+)}(t) \quad \text{for} \quad t \in R, \\ &u(x,t,\varepsilon) = u(x,t+T,\varepsilon) \quad \text{for} \quad t \in R, \quad -1 \leqslant x \leqslant 1 \end{aligned} \tag{1.1}$$

for $\varepsilon \in I_{\varepsilon_0} := \{0 < \varepsilon \leqslant \varepsilon_0\},\ 0 < \varepsilon_0 \ll 1$. The functions A, F, $u^{(-)}$ and $u^{(+)}$ are sufficiently smooth and T-periodic in t.

Equations of type (1.1) have many applications: they arise in different areas such as chemical and biological kinetics, population dynamics, and so on (see [1, 2] and references therein). In the case when the parameter is small, such problems are called *singularly perturbed reaction-diffusion-advection problems* and used in many practical applications in order to model narrow boundary and interior layers. The main difficulty in these problems is related to the determination of the location of the transitional layer in whose neighborhood the solution $u(x,t,\varepsilon)$ rapidly changes.

In [3], an approach was proposed for considering periodic boundary layers, including nonmonotonic layers. The present paper develops the result of the [4], applying a new method of searching for the transitional layer location (see, for example, [5–7]) and asserting the Lyapunov asymptotic stability of the periodical solution to problem (1.1). Thus, the results obtained in this paper are based on a further development of the asymptotic comparison principle.

The paper is organized as follows: in Section 2, we state our main result and, in Section 3, we present a method for constructing an asymptotic approximation of the solution with a boundary and an internal transition layer. In Section 4, we prove the existence theorem for the solution with the asymptotics constructed and, in Section 5, its asymptotic stability in the Lyapunov sense is proved. In Section 6, we apply our results to a class of problems with a solution-independent advection coefficient, which are important for applications.

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2. MAIN RESULTS

2.1. Assumptions

We define the associated system of equations

$$\frac{\partial \tilde{v}}{\partial \xi} = A(\tilde{u}, x, t)\tilde{v} + F(\tilde{u}, x, t, 0), \quad \frac{\partial \tilde{u}}{\partial \xi} = \tilde{v}, \quad -\infty < \xi < \infty, \tag{2.2}$$

where x and t are treated as parameters, and consider problem (1.1) under the following assumptions (similar to [5]).

Condition A0. The functions A, F, $u^{(-)}$ and $u^{(+)}$ are sufficiently smooth and T-periodic in t.

Condition A1. The so-called degenerate equation $F(u, x, t, 0)$ has exactly three roots $\varphi^{(-)}(x,t)$, $\varphi^{(0)}(x,t)$, $\varphi^{(+)}(x,t)$, moreover,

$$\varphi^{(-)}(x,t) < \varphi^{(0)}(x,t) < \varphi^{(+)}(x,t), \quad F_u(\varphi^{(\pm)}, x, t, 0) > 0, \quad F_u(\varphi^{(0)}, x, t, 0) < 0, \quad (x,t) \in \bar{D}.$$

Condition A1 implies that, on the phase plane $(\tilde{u}, \tilde{v})$ of the associated system (2.2), there are two equilibrium points $(\varphi^{(-)}(x,t), 0)$ and $(\varphi^{(+)}(x,t), 0)$ of saddle type. Outside small neighborhoods of these equilibrium points, the behavior of separatrices is defined by the following requirements.

Condition A2. For any fixed $(x,t) \in (-1; +1) \times R$, the separatrix issuing from the saddle $(\varphi^{(-)}(x,t), 0)$ can be represented in the form $\tilde{v} = \tilde{v}^-(\xi, x, t), \tilde{u} = \tilde{u}^-(\xi, x, t)$ and intersects the line $u = \varphi^{(0)}(x,t)$; in addition, the point of intersection corresponds to the value $\xi = 0$, while the saddle corresponds to the value $\xi = -\infty$. The separatrix entering the saddle $(\varphi^{(+)}(x,t), 0)$ can be represented in the form $\tilde{v} = \tilde{v}^+(\xi, x, t), \tilde{u} = \tilde{u}^+(\xi, x, t)$, and it intersects the line $u = \varphi^{(0)}(x,t)$, the point of intersection corresponds to the value $\xi = 0$, and the saddle corresponds to $\xi = \infty$.

The separatrices are described by (2.2) with the conditions at infinity

$$\tilde{u}^{\pm}(\pm\infty, x, t) = \varphi^{(\pm)}(x,t), \quad \tilde{v}^{\pm}(\pm\infty, x, t) = 0. \tag{2.3}$$

Without loss of generality, we assume that the separatrices lie in the upper half-plane. For each $(x,t) \in (-1; +1) \times R$, we define the function $H(x,t) := \tilde{v}^+(0, x, t) - \tilde{v}^-(0, x, t)$, where the $\tilde{v}^{\pm}(\xi, x, t)$ are solutions of system (2.2) satisfying condition (2.3).

Condition A3. The equation $H(x,t) = 0$ has a T-periodic solution $x = x_0(t)$, $-1 < x_0(t) < 1$; moreover, $\frac{\partial H}{\partial x}\big|_{x=x_0(t)} > 0, t \in R$.

Condition A3 implies that there exists a separatrix issuing from the saddle $(\varphi^{(-)}(x_0,t), 0)$ and entering the saddle $(\varphi^{(+)}(x_0,t), 0)$. Its existence provides the solvability of the boundary value problems

$$\frac{\partial^2 u^{\pm}}{\partial \xi^2} = A(u^{\pm}, x_0, t)\frac{\partial u^{\pm}}{\partial \xi} + F(u^{\pm}, x_0, t, 0),$$
$$u^{\pm}(0, x_0, t) = \varphi^{(0)}(x_0, t), \quad u^-(-\infty, x_0, t) = \varphi^{(-)}(x_0, t), \quad u^+(\infty, x_0, t) = \varphi^{(+)}(x_0, t),$$

which are used for the description of the internal layer in the zero approximation. To be definite, we assume that the separatrix lies in the upper half-plane $(\tilde{u}, \tilde{v})$. The requirement $\frac{\partial H}{\partial x}\big|_{x=x_0(t)} > 0$ eliminates the case in which the joining separatrix takes place for any dependence $x(t)$ or the branching takes place for $x(t) = x_0(t)$. This requirement permits one to prove the theorem on the existence of a stable solution.

According to Condition A3, the dependence of the phase picture of the system (2.2) (see also system (6.2) for the example in Section 6) on the parameter x must be like the one in Fig. 1.

We also need the condition for the boundary conditions to belong to the influence domain of the solutions $\varphi^{(\pm)}(x,t)$ of the degenerate equation.

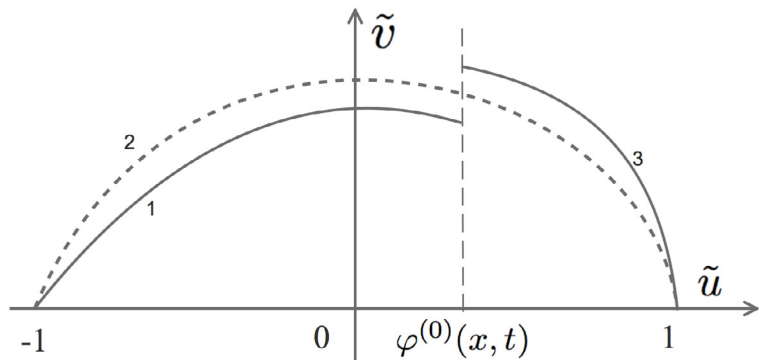

Fig. 1. The phase plane for the system (2.2) in the case $\varphi^{(\pm)}(x,t) = \pm 1$ (see the example in Section 6). The separatrices are depicted by dashed line for $x = x_0$ and by continious line for $x > x_0$.

Condition A4. On the phase plane of the system

$$\frac{\partial v}{\partial \tau^-} = A(u,-1,t)v + F(u,-1,t,0), \quad \frac{\partial u}{\partial \tau^-} = v, \quad \tau^- > 0,$$

t is a parameter, the line $u = u^{(-)}(t)$ intersects the separatrix entering the saddle $\varphi^-(-1,t)$ as $\tau^- \to \infty$. An absolutely analogous condition is required for the saddle $\varphi^+(1,t)$.

Condition A4 will be used for the construction of the asymptotics of the solution in a neighborhood of the domain's boundary.

2.2. Main theorem

Theorem 2.1. *Suppose that the assumptions A0 − A4 are satisfied. Then, for a sufficiently small ε, there exits a periodic solution of problem* (1.1) *such that*

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon) = \begin{cases} \varphi^{(-)}(x,t), & -1 < x < x_0(t), \quad t \in R, \\ \varphi^{(+)}(x,t), & x_0(t) < x < 1, \quad t \in R. \end{cases}$$

This solution is asymptotically stable with domain of attraction width at least $O(\varepsilon)$ and locally unique.

In Sections 4 and 5, we prove more precise results.

3. THE CONSTRUCTION OF THE FORMAL ASYMPTOTICS OF THE SOLUTION

In this section, we describe the construction of a formal asymptotic solution of the periodic boundary value problem (1.1). Later on, we will prove the existence of a solution to (1.1) near this formal asymptotic approximation.

3.1. Asymptotic Approximation

To characterize the location $x^*(t,\varepsilon)$ of the interior layer of the formal asymptotic solution in the (x,t)-plane, we introduce the curve $x = x^*(t,\varepsilon)$ as the locus of the intersection of the solution $u(x,t,\varepsilon)$ of (1.1) with the surface $\varphi^{(0)}(x,t)$. In this way, we decompose the periodic boundary value problem (1.1) with the interior layer near $x^*(t,\varepsilon)$ into two boundary value problems with boundary layers near $x = x^*(t,\varepsilon)$. For the following, we introduce the notation

$$\xi := \frac{x - x^*(t,\varepsilon)}{\varepsilon},$$
$$\overline{D}^{(-)} := \{(x,t) \in R^2 : -1 \leqslant x \leqslant x^*(t,\varepsilon),\, t \in R\},$$
$$\overline{D}^{(+)} := \{(x,t) \in R^2 : x^*(t,\varepsilon) \leqslant x \leqslant 1,\, t \in R\}.$$

First, we consider for small ε in $\overline{D}^{(-)}$ the boundary value problem

$$\begin{gathered}\varepsilon^2\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\right)-\varepsilon A(u,x,t)\frac{\partial u}{\partial x}-F(u,x,t,\varepsilon)=0 \quad \text{for} \quad (x,t)\in D^{(-)},\\ u(-1,t,\varepsilon)=u^{(-)}(t), \quad u(x^*(t,\varepsilon),t,\varepsilon)=\varphi^{(0)}(x^*(t,\varepsilon),t) \quad \text{for} \quad t\in R,\\ u(x,t,\varepsilon)=u(x,t+T,\varepsilon) \quad \text{for} \quad (x,t)\in\overline{D}^{(-)}.\end{gathered} \tag{3.4}$$

Similarly, we consider in $\overline{D}^{(+)}$ the boundary value problem

$$\begin{gathered}\varepsilon^2\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\right)-\varepsilon A(u,x,t)\,\frac{\partial u}{\partial x}-F(u,x,t,\varepsilon)=0 \quad \text{for} \quad (x,t)\in\overline{D}^{(+)},\\ u(x^*(t,\varepsilon),t,\varepsilon)=\varphi(x^*(t,\varepsilon),t), \quad u(1,t,\varepsilon)=u^{(+)}(t) \quad \text{for} \quad t\in R,\\ u(x,t,\varepsilon)=u(x,t+T,\varepsilon) \quad \text{for} \quad (x,t)\in\overline{D}^{(+)}.\end{gathered} \tag{3.5}$$

We look for a formal asymptotic solution $U^{(\pm)}(x,t,\varepsilon)$ of this problem in the form

$$U^{(\pm)}(x,t,\varepsilon)=\bar{u}^{(\pm)}(x,t,\varepsilon)+Q^{(\pm)}(\xi,t,\varepsilon;x^*)+\Pi^{(\pm)}(\tau,t,\varepsilon), \tag{3.6}$$

where the regular part is

$$\bar{u}^{(\pm)}(x,t,\varepsilon)=\bar{u}_0^{(\pm)}(x,t)+\varepsilon\bar{u}_1^{(\pm)}(x,t)+\cdots+\varepsilon^n\bar{u}_n^{(\pm)}(x,t)+\cdots,$$

the boundary layer part in the neighborhood of $x=-1$ for $u^{(-)}$ and in the neighborhood of $x=1$ for $u^{(+)}$ is

$$\Pi^{(\pm)}(\tau,t,\varepsilon)=\Pi_0^{(\pm)}(\tau,t)+\varepsilon\Pi_1^{(\pm)}(\tau,t)+\cdots+\varepsilon^n\Pi_n^{(\pm)}(\tau,t)+\cdots,$$

where

$$\tau=\begin{cases}(x+1)/\varepsilon, & x\leqslant x^*(t,\varepsilon),\\ (x-1)/\varepsilon, & x\geqslant x^*(t,\varepsilon),\end{cases}$$

and the internal layer part $x^*(t,\varepsilon)$, $\xi=(x-x^*(t,\varepsilon))/\varepsilon$ is

$$Q^{(\pm)}(\xi,t,\varepsilon;x^*)=Q_0^{(\pm)}(\xi,t;x^*)+\varepsilon Q_1^{(\pm)}(\xi,t;x^*)+\cdots+\varepsilon^n Q_n^{(\pm)}(\xi,t;x^*)+\cdots;$$

its terms depend on x^*, but this is not an argument in this case. This dependence just reminds us about the fact that we don't expand $x^*(t,\varepsilon)$ into the asymptotic series yet. To determine the terms in the expansions (3.6), we use the standard procedure proposed by Vasil'eva (see, e.g., [8]). We substitute the asymptotic in the form (3.6) into the problems (3.4), (3.5). In particular, for the regular parts, we have for $(x,t)\in D^{(\pm)}$:

$$\overline{u}_0^{(\pm)}(x,t)=\varphi^{(\pm)}(x,t), \quad \overline{u}_1^{(\pm)}(x,t)=-\frac{\varphi_x^{(\pm)}(x,t)A(\varphi^{(\pm)}(x,t),x,t)+F_\varepsilon(\varphi^{(\pm)}(x,t),x,t,0)}{F_u(\varphi^{(\pm)}(x,t),x,t,0)}.$$

To determine the interior layer parts, we represent the differential operator

$$L_\varepsilon=\varepsilon^2\,\frac{\partial^2}{\partial x^2}-\varepsilon^2\,\frac{\partial}{\partial t}$$

when it acts on the interior layer functions by using the stretched variable ξ in the form:

$$L_\varepsilon=\frac{\partial^2}{\partial\xi^2}+\varepsilon\frac{\partial x^*(t,\varepsilon)}{\partial t}\,\frac{\partial}{\partial\xi}-\varepsilon^2\,\frac{\partial}{\partial t}.$$

For the zero-th order internal layer functions $Q_0^{(-)}$ and $Q_0^{(+)}$, we obtain the boundary value problems

$$\begin{gathered}\frac{\partial^2 Q_0^{(\pm)}(\xi,t;x^*)}{\partial\xi^2} - A(\varphi^{(\pm)}(x^*,t)+Q_0^{(\pm)},x^*,t)\frac{\partial Q_0^{(\pm)}(\xi,t;x^*)}{\partial\xi} = F(\varphi^{(\pm)}(x^*,t)+Q_0^{(\pm)},x^*,t,0),\\ Q_0^{(\pm)}(0,t;x^*)+\bar{u}_0^{(\pm)}(x^*,t)=\varphi^{(0)}(x^*,t),\\ Q_0^{(\pm)}(\pm\infty,t;x^*)=0.\end{gathered} \tag{3.7}$$

We set

$$\tilde{u}(\xi,x^*,t) := \begin{cases}\varphi^{(-)}(x^*(t,\varepsilon),t)+Q_0^{(-)}(\xi,t;x^*), & \xi<0,\ t\in R,\\ \varphi^{(0)}(x^*(t,\varepsilon),t), & \xi=0,\ t\in R,\\ \varphi^{(+)}(x^*(t,\varepsilon),t)+Q_0^{(+)}(\xi,t;x^*), & \xi>0,\ t\in R.\end{cases}$$

Now we can rewrite problems (3.7) in the form

$$\begin{gathered}\frac{\partial^2\tilde{u}^{(\pm)}}{\partial\xi^2} - A(\tilde{u}^{(\pm)},x^*,t)\frac{\partial\tilde{u}^{(\pm)}}{\partial\xi} = F(\tilde{u}^{(\pm)},x^*,t,0),\\ \tilde{u}^{(\pm)}(0,x^*,t)=\varphi^{(0)}(x^*,t), \qquad \tilde{u}^{(\pm)}(\pm\infty,x^*,t)=\varphi^{(\pm)}(x^*,t).\end{gathered} \tag{3.8}$$

Problems (3.8) are uniquely solvable by virtue of Condition A2.

Therefore, we see that the boundary value problems (3.7) have unique solutions satisfying the estimate (see [9])

$$|Q_0^{(\pm)}(\xi,t)| \leqslant c\,\exp(-\kappa|\xi|) \quad \text{for} \quad \xi\in R^{(\pm)},\ t\in R.$$

In the next approximation, we obtain the linear problems

$$\begin{gathered}\frac{\partial^2 Q_1^{(\pm)}(\xi,t;x^*)}{\partial\xi^2} - A(\tilde{u}^{(\pm)},x^*,t)\frac{\partial Q_1^{(\pm)}(\xi,t;x^*)}{\partial\xi}\\ = \left(\frac{\partial A}{\partial u}(\tilde{u}^{(\pm)},x^*,t)\frac{\partial Q_0^{(\pm)}(\xi,t;x^*)}{\partial\xi} + \frac{\partial F}{\partial u}(\tilde{u}^{(\pm)},x^*,t,0)\right)Q_1^{(\pm)}(\xi,t;x^*) + r_1(\xi,t)\\ Q_1^{(\pm)}(0,t;x^*)+\bar{u}_1^{(\pm)}(x^*,t)=0,\\ Q_1^{(\pm)}(\pm\infty,t;x^*)=0,\end{gathered} \tag{3.9}$$

where

$$\begin{aligned}r_1(\xi,t,\varepsilon) =& A(\tilde{u}^{(\pm)},x^*,t)\frac{\partial\bar{u}_0}{\partial x}(x^*,t) + \frac{\partial F}{\partial\varepsilon}(\tilde{u}^{(\pm)},x^*,t,0) - \frac{\partial Q_0^{(\pm)}}{\partial\xi}(\xi,t;x^*)\frac{\partial x*}{\partial t}\\ &+ \left(\frac{\partial A}{\partial u}(\tilde{u}^{(\pm)},x^*,t)\frac{\partial Q_0^{(\pm)}(\xi,t;x^*)}{\partial\xi} + \frac{\partial F}{\partial u}(\tilde{u}^{(\pm)},x^*,t,0)\right)\left(\bar{u}_1^{(\pm)} + \frac{\partial\bar{u}_0^{(\pm)}}{\partial x}\xi\right)\\ &+ \left(\frac{\partial A}{\partial x}(\tilde{u}^{(\pm)},x^*,t)\frac{\partial Q_0^{(\pm)}(\xi,t;x^*)}{\partial\xi} + \frac{\partial F}{\partial x}(\tilde{u}^{(\pm)},x^*,t,0)\right)\xi.\end{aligned} \tag{3.10}$$

Problems (3.7) are linear inhomogeneous problems with exponentially decaying inhomogeneous terms. The solutions of these problems can be given explicitly

$$\begin{aligned}Q_1^{(\pm)}(\xi,x^*,t) =& -\bar{u}_1^{(\pm)}(x^*,t)\frac{\tilde{v}^{\pm}(\xi,x^*,t)}{\tilde{v}^{\pm}(0,x^*,t)}\\ &- \tilde{v}^{\pm}(\xi,x^*,t)\int_0^{\xi}\frac{1}{p^{\pm}(s,x^*,t)(\tilde{v}^{\pm}(s,x^*,t))^2}\int_s^{\pm\infty}p^{\pm}(\eta,x^*,t)\tilde{v}^{\pm}(\eta,x^*,t)r_1(\eta,t)\,d\eta\,ds,\end{aligned} \tag{3.11}$$

where

$$\tilde{v}^{\pm}(\xi, x^*, t) = \frac{\partial \tilde{u}^{(\pm)}(\xi, x^*, t)}{\partial \xi}, \quad p^{\pm}(\xi, x^*, t) = \exp\left(-\int_0^{\xi} A(\tilde{u}^{(\pm)}(y, x^*, t), x^*, t)dy\right).$$

For $Q_1^{(\pm)}$, the following estimate can be proved

$$|Q_1^{(\pm)}(\xi, x^*, t)| < C_1 e^{-k_1|\xi|}, \tag{3.12}$$

where C_1, k_1 are positive constants.

The next orders of the Q-functions are defined in a similar way, the Π-functions are constructed in standard way (see [4]) and not considered in this paper.

Since $A, B, u^{(\pm)}$ are sufficiently smooth, the formal asymptotics can be constructed to any order n. From these constructions, it follows that the corresponding approximations satisfy (1.1) with discrepancy of order ε^{n+1}.

3.2. Localization of the Interior Layer

One of the key problems is the construction of the asymptotic approximation of the transition layer curve $x^*(t, \varepsilon)$. To this end, we use the condition of C^1-matching. The continuity of the asymptotic approximation on the curve $x^*(t, \varepsilon)$ holds at the expense of matching the asymptotic expansions of $U^{(-)}$ and $U^{(+)}$ by using their construction. We will also require the continuity of the first derivatives of the asymptotic expansions on this curve (the condition of C^1-matching).

We consider the difference

$$\varepsilon\frac{\partial U^{(+)}(x^*, t, \varepsilon)}{\partial x} - \varepsilon\frac{\partial U^{(-)}(x^*, t, \varepsilon)}{\partial x} = \frac{\partial Q_0^{(+)}(0, t; x^*)}{\partial \xi} - \frac{\partial Q_0^{(-)}(0, t; x^*)}{\partial \xi}$$
$$+ \varepsilon\left(\frac{\partial \bar{u}_0^{(+)}(x^*, t)}{\partial x} + \frac{\partial Q_1^{(+)}(0, t; x^*)}{\partial \xi} - \left(\frac{\partial \bar{u}_0^{(-)}(x^*, t)}{\partial x} + \frac{\partial Q_1^{(-)}(0, t; x^*)}{\partial \xi}\right)\right) + \cdots. \tag{3.13}$$

Then the condition of C^1-matching has the form

$$\varepsilon\frac{\partial U^{(+)}(x^*, t, \varepsilon)}{\partial x} - \varepsilon\frac{\partial U^{(-)}(x^*, t, \varepsilon)}{\partial x} = 0, \quad t \in R. \tag{3.14}$$

We can readily see that

$$H(x^*, t) = \frac{\partial Q_0^{(+)}(0, x^*, t)}{\partial \xi} - \frac{\partial Q_0^{(-)}(0, x^*, t)}{\partial \xi},$$

where H was defined in Condition A3.

Using (3.11), we obtain

$$\frac{\partial \bar{u}_0^{(+)}(x^*, t)}{\partial x} + \frac{\partial Q_1^{(+)}(0, t; x^*)}{\partial \xi} - \left(\frac{\partial \bar{u}_0^{(-)}(x^*, t)}{\partial x} + \frac{\partial Q_1^{(-)}(0, t; x^*)}{\partial \xi}\right)$$
$$= \frac{\partial \varphi^+}{\partial x}(x^*, t) - \frac{\partial \varphi^-}{\partial x}(x^*, t) - \bar{u}_1^{(+)}(x^*, t)\frac{\partial \tilde{v}_\xi^+(0, x^*, t)}{\partial \tilde{v}^+(0, x^*, t)} + \bar{u}_1^{(-)}(x^*, t)\frac{\partial \tilde{v}_\xi^-(0, x^*, t)}{\partial \tilde{v}^-(0, x^*, t)}$$
$$- \frac{1}{\tilde{v}^+(0, x^*, t)}\int_0^{+\infty} p^+(\eta, x^*, t)\tilde{v}^+(\eta, x^*, t)r_1^+(\eta, t)d\eta$$
$$+ \frac{1}{\tilde{v}^-(0, x^*, t)}\int_0^{-\infty} p^-(\eta, x^*, t)\tilde{v}^-(\eta, x^*, t)r_1^-(\eta, t)d\eta$$
$$= \left[-\frac{1}{\tilde{v}^{\pm}(0, x^*, t)}\left(\int_0^{\pm\infty} p^{\pm}(\eta, x^*, t)\tilde{v}^{\pm}(\eta, x^*, t)\left(\frac{\partial A}{\partial x}(\tilde{u}^{(\pm)}, x^*, t)\frac{\partial Q_0^{(\pm)}(\partial\eta, t; x^*)}{\eta} + \frac{\partial F}{\partial x}(\tilde{u}^{(\pm)}, x^*, t, 0)\right)\eta d\eta\right.\right.$$

$$\begin{aligned} &+ \int_0^{\pm\infty} p^{\pm}(\eta, x^*, t)\tilde{v}^{\pm}(\eta, x^*, t)\frac{\partial F}{\partial \varepsilon}(\tilde{u}^{(\pm)}, x^*, t, 0)d\eta\Big)\Big]_-^+ \\ &+ \frac{\partial x^*}{\partial t}\left[\frac{1}{\tilde{v}^{\pm}(0, x^*, t)}\int_0^{\pm\infty} p^{\pm}(\eta, x^*, t)(\tilde{v}^{\pm}(\eta, x^*, t))^2 d\eta\right]_-^+ \\ &=: G_1(x^*, t) + \frac{\partial x^*}{\partial t}K(x^*, t). \end{aligned} \tag{3.15}$$

Here $[\quad]_-^+$ means the difference between the expressions containing the functions with symbol $+$ and $-$. We denote the expression in the first square brackets by $G_1(x^*, t)$ and in the second square brackets by $K(x^*, t)$. The function (G depends not only on x^*, but on its derivative $\frac{x^*}{t}$)

$$G_i(\varepsilon, t) = \frac{\partial \bar{u}_{i-1}^{(+)}(x^*, t)}{\partial x} + \frac{\partial Q_i^{(+)}(0, t; x^*)}{\partial \xi} - \left(\frac{\partial \bar{u}_{i-1}^{(-)}(x^*, t)}{\partial x} + \frac{\partial Q_i^{(-)}(0, t; x^*}{\partial \xi}\right), \quad i = 2, 3, \dots,$$

looking for a formal asymptotic expansion for $x^*(t, \varepsilon)$ in the form $x^*(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$, we obtain the following expansion:

$$\begin{aligned} &\varepsilon\frac{\partial U^{(+)}(x^*, t, \varepsilon)}{\partial x} - \varepsilon\frac{\partial U^{(-)}(x^*, t, \varepsilon)}{\partial x} \\ &\qquad = H(x_0, t) + \varepsilon\left(\frac{\partial H}{\partial x}\bigg|_{x=x_0} x_1(t) + G_1|_{x=x_0} + \frac{\partial x_0}{\partial t}\, K|_{x=x_0}\right) \\ &\qquad + \varepsilon^2\left[\frac{x_1^2(t)}{2}\,\frac{\partial^2 H}{\partial x^2}\bigg|_{x=x_0} + x_2(t)\,\frac{\partial H}{\partial x}\bigg|_{x=x_0} + x_1(t)\,\frac{\partial G_1}{\partial x}\bigg|_{x=x_0}\right. \\ &\qquad\qquad + \frac{\partial x_0}{\partial t}(t)\,\frac{\partial K}{\partial x}\bigg|_{x=x_0} x_1(t) + \frac{\partial x_1}{\partial t}K\bigg|_{x=x_0} + \left. G_2|_{\varepsilon=0}\right] + \cdots = 0. \end{aligned} \tag{3.16}$$

The term at ε^0 is equal to zero by virtue of Condition A3. Equating the terms at ε^i to zero, we obtain the problems for $x_i(t)$, $i = 1, 2, \dots$:

$$x_i(t)\,\frac{\partial H}{\partial x}\bigg|_{x=x_0} = f_i(t), \tag{3.17}$$

where $f_i(t)$ are known functions. These problems are uniquely solvable by virtue of Condition A2 and have T-periodic solutions. Thus, the construction of the solution asymptotics, including the construction of the transition layer curve, can be continued to any number n.

4. EXISTENCE RESULTS

4.1. Existence Theorem

Let $D_n^{(-)}$ and $D_n^{(+)}$ be the domains

$$D_n^{(-)} := \left\{(x, t) \in R^2 : -1 \leqslant x \leqslant \sum_{i=0}^{n+1} x_i(t)\varepsilon^i,\ t \in R\right\},$$

$$D_n^{(+)} := \left\{(x, t) \in R^2 : \sum_{i=0}^{n+1} x_i(t)\varepsilon^i \leqslant x \leqslant 1,\ t \in R\right\},$$

and let $U_n^{(\pm)}$ be the partial sums of order n of the expansions (3.6), respectively, where ξ is replaced by $\left(x - \sum_{i=0}^{n+1} x_i(t)\varepsilon^i\right)/\varepsilon$. We introduce the notation

$$U_n(x,t,\varepsilon) := \begin{cases} U_n^{(-)}(x,t,\varepsilon) & \text{for} \quad (x,t) \in D_n^{(-)}, \\ U_n^{(+)}(x,t,\varepsilon) & \text{for} \quad (x,t) \in D_n^{(+)}. \end{cases}$$

Then we have the following existence theorem.

Theorem 4.1. *Suppose the assumptions A0-A4 are valid. Then, for sufficiently small ε, there exists a solution $u(x,t,\varepsilon)$ of* (1.1) *which satisfies*

$$|u(x,t,\varepsilon) - U_n(x,t,\varepsilon)| \leqslant c_n\varepsilon^{n+1} \quad \text{for} \quad (x,t) \in \overline{D},$$

where the positive constant c_n does not depend on ε.

4.2. Construction of Upper and Lower Solutions

The proof of this theorem is based on the technique of lower and upper solutions. For the convenience of the reader, we recall the definition of these functions.

Definition 4.1. We say that the functions $\alpha, \beta : \overline{D} \times \overline{I}_{\varepsilon_0} \to R$ have *smoothness property S*, if they are continuous, twice continuously differentiable in x, continuously differentiable in t and T-periodic in t. The functions α and β are called *ordered lower and upper solutions* of (1.1) for $\varepsilon \in I_{\varepsilon_0}$, if they have the smoothness property S and satisfy for $\varepsilon \in I_{\varepsilon_0}$ the following conditions:

$$1^\circ \qquad \alpha(x,t,\varepsilon) \leqslant \beta(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in \overline{D}, \tag{4.1}$$

$$2^\circ \qquad N_\varepsilon(\alpha) \geqslant 0 \geqslant N_\varepsilon(\beta) \quad \text{for } (x,t) \in D, \tag{4.2}$$

$$3^\circ \qquad \alpha(-1,t,\varepsilon) \leqslant u^{(-)}(t) \leqslant \beta(-1,t,\varepsilon), \tag{4.3}$$

$$\alpha(1,t,\varepsilon) \leqslant u^{(+)}(t) \leqslant \beta(1,t,\varepsilon) \quad \text{for} \quad t \in R.$$

In the case when there exists in D a smooth curve $x = \bar{x}(t), t \in R$, periodic in t and dividing D into two subregions D^+ and D^- such that α and β are continuous in $\overline{D}$, but have the smoothness property S only in D^+ and D^-, then α and β are called *ordered lower and upper solutions* of (1.1) for $\varepsilon \in I_{\varepsilon_0}$, if they satisfy the relations (4.1) and (4.2) in D^+ and D^-, the relation (4.3) and the inequalities

$$\frac{\partial \alpha}{\partial x}(\bar{x}(t)+0,t,\varepsilon) \geqslant \frac{\partial \alpha}{\partial x}(\bar{x}(t)-0,t,\varepsilon), \tag{4.4}$$

$$\frac{\partial \beta}{\partial x}(\bar{x}(t)+0,t,\varepsilon) \leqslant \frac{\partial \beta}{\partial x}(\bar{x}(t)-0,t,\varepsilon). \tag{4.5}$$

Remark 1. It is known (see, e.g., [2]) that the existence of ordered lower and upper solutions implies the existence of a solution $u(x,t,\varepsilon)$ of (1.1) satisfying

$$\alpha(x,t,\varepsilon) \leqslant u(x,t,\varepsilon) \leqslant \beta(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in D \quad \text{and} \quad \varepsilon \in I_{\varepsilon_0}.$$

In what follows, we construct lower and upper solutions with admissible jump of the derivative, defined by (4.4) and (4.5) . For this case, we have the same existence result (see, e.g., [2]).

In this section, we describe a method for constructing upper and lower solutions by a modification of the formal asymptotic expansion of the solution to (1.1). For this purpose, we introduce T-periodic functions x_β and x_α as the $(n+2)$-th partial sums of the asymptotic expansion of $x^*(t,\varepsilon)$ with a small shift in the last term

$$x_\beta(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \cdots + \varepsilon^{n+1}(x_{n+2}(t) - \delta),$$

$$x_\alpha(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \cdots + \varepsilon^{n+2}(x_{n+2}(t) + \delta),$$

where $\delta > 0$ is a small number independent of ε. The curves $x = x_\beta(t,\varepsilon)$ and $x = x_\alpha(t,\varepsilon)$ divide the domain $\overline{D}$ into two subdomains $\overline{D}_\beta^{(-)}$, $\overline{D}_\beta^{(+)}$ and $\overline{\mathcal{D}}_\alpha^{(-)}$, $\overline{D}_\alpha^{(+)}$, where

$$\overline{D}_\beta^{(-)} := \{(x,t) \in R^2 : -1 \leqslant x \leqslant x_\beta(t,\varepsilon),\ t \in R\},$$
$$\overline{D}_\beta^{(+)} := \{(x,t) \in R^2 : x_\beta(t,\varepsilon) \leqslant x \leqslant 1,\ t \in R\}.$$

The domains $\overline{D}_\alpha^{(\pm)}$ are defined similarly.

Now we can define an upper solution $\beta(x,t,\varepsilon) = \beta_n(x,t,\varepsilon)$ together with a lower solution $\alpha(x,t,\varepsilon) = \alpha_n(x,t,\varepsilon)$ for $\varepsilon \in I_{\varepsilon_0}$ in D by the expressions

$$\begin{aligned}\beta_n(x,t,\varepsilon) = \beta_n^{(\pm)}(x,t,\varepsilon) &= \bar{u}_0^{(\pm)}(x,t) + \varepsilon\bar{u}_1^{(\pm)}(x,t) + \cdots + \varepsilon^{n+2}\bar{u}_{n+2}^{(\pm)}(x,t)\\ &+ Q_0^{(\pm)}(\xi_\beta,t;x_\beta) + \varepsilon Q_1^{(\pm)}(\xi_\beta,t;x_\beta) + \cdots + \varepsilon^{n+2}Q_{(n+2)}^{(\pm)}(\xi_\beta,t;x_\beta)\\ &+ \varepsilon^{n+2}\Big(\gamma + q_\beta^{(\pm)}(\xi_\beta,t)\Big) + \Pi_{n\beta}^{(\pm)}(\tau,t).\end{aligned} \tag{4.6}$$

Here, $\gamma = \text{const} > 0$, $\xi_\beta = (x - x_\beta(t,\varepsilon))/\varepsilon$. The functions $Q_i^{(\pm)}(\xi_\beta,t;x_\beta)$, $i = 0,\dots,n+1$, are defined by replacing ξ by ξ_β and x^* by x_β in the terms of the asymptotics $Q_i^{(\pm)}(\xi,t;x^*)$.

The functions $q_\beta(\xi_\beta,t)$ are introduced into the interior layer part to compensate the changes produced by the modification of the $(n+2)$-th order term of the regular part of the asymptotic expansion and the interior layer curve expansion. The functions $q_\beta^{(\pm)}(\xi_\beta,t)$ are determined by the problems

$$\begin{aligned}&\frac{\partial^2 q_\beta^{(\pm)}(\xi,t;x_\beta)}{\partial\xi^2} - A(\tilde{u}_\beta^{(\pm)},x_\beta,t)\frac{\partial q_\beta^{(\pm)}(\xi,t;x_\beta)}{\partial\xi}\\ &\qquad = \left(\frac{\partial A}{\partial u}(\tilde{u}_\beta^{(\pm)},x_\beta,t)\frac{\partial Q_0^{(\pm)}(\xi,t;x_\beta)}{\partial\xi} + \frac{\partial F}{\partial u}(\tilde{u}_\beta^{(\pm)},x_\beta,t,0)\right)q_\beta^{(\pm)}(\xi,t;x_\beta) + r_\beta(\xi,t)\\ &q_\beta^{(\pm)}(0,t;x_\beta) + \gamma = 0,\\ &q_\beta^{(\pm)}(\pm\infty,t;x_\beta) = 0,\end{aligned} \tag{4.7}$$

where

$$\tilde{u}_\beta^{(\pm)} = \tilde{u}^{(\pm)}(\xi,x_\beta,t), \quad r_\beta(\xi,t) = \gamma\left(\frac{\partial F}{\partial u}(\tilde{u}^{(\pm)},x_\beta,t,0) - \frac{\partial F}{\partial u}(\varphi^{(\pm)},x_\beta,t,0)\right). \tag{4.8}$$

The functions $\Pi_{n\beta}^{(\pm)}(\tau,t)$ are sums with modified boundary functions and can be constructed by the scheme developed in [4].

Problems (4.7) can be investigated analogously as the problems (3.6) and have unique exponentially decaying solutions.

To verify that the introduced functions α_n and β_n are upper and lower solutions, we substitute the expressions for α_n and β_n into the operator $N_\varepsilon(u)$ defined in (1.1). We obtain

$$N_\varepsilon(\beta_n(x,t,\varepsilon)) = -\varepsilon^{n+2}\gamma\frac{\partial F}{\partial u}(\varphi^{(\pm)},x_0,t,0) + O(\varepsilon^{n+3}). \tag{4.9}$$

For the lower solution, we have similar result. The proof of the ordering $(\alpha_n \leqslant \beta_n)$ and of the inequality on the ends of the interval $(-1;1)$ can be done according to the scheme from paper [4].

We summarize the results of our construction of upper and lower solutions in the following lemma.

Lemma 2.1. *The functions* $\beta_n(x,t,\varepsilon)$ *and* $\alpha_n(x,t,\varepsilon)$ *satisfy Definition* 2.1, *and therefore they are upper and lower solutions of problem* (1.1). *Moreover, they obey the following relations*:

$$\beta_n(x,t,\varepsilon) - \alpha_n(x,t,\varepsilon) = O(\varepsilon^{n+1}) \quad \textit{for} \quad x \in [0,1],\ t \in R, \tag{4.10}$$

$$\beta_n(x,t,\varepsilon) - U_n(x,t,\varepsilon) = O(\varepsilon^{n+1}) \quad \textit{for} \quad x \in [0,1],\ t \in R, \tag{4.11}$$

$$\frac{\partial \alpha_n}{\partial x} = \frac{\partial U_n}{\partial x} + O(\varepsilon^{n-1}) \quad \textit{for} \quad x \in (0,1) \setminus \{x_\alpha(t,\varepsilon),\ \sum_{i=0}^{n+1} x_i(t)\varepsilon^i\}, t \in R, \tag{4.12}$$

$$\frac{\partial \beta_n}{\partial x} = \frac{\partial U_n}{\partial x} + O(\varepsilon^{n-1}) \quad \textit{for} \quad x \in (0,1) \setminus \{x_\beta(t,\varepsilon),\ \sum_{i=0}^{n+1} x_i(t)\varepsilon^i\}, t \in R. \tag{4.13}$$

Therefore, we see that problem (1.1) has a solution that satisfies (see Remark 1) the relations

$$\alpha_n(x,t,\varepsilon) \leqslant u(x,t,\varepsilon) \leqslant \beta_n(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in D \quad \text{and} \quad \varepsilon \in I_{\varepsilon_0}.$$

The statements of Theorem 4.1 follow from the estimates (4.10), (4.11) of Lemma 2.1, taking into account that $U_n - \beta_{n+1} = O(\varepsilon^{n+1})$, $U_n - \alpha_{n+1} = O(\varepsilon^{n+1})$.

The statements of Theorem 4.1 follow from the estimates (4.10), (4.11) of Lemma 2.1.

In order to investigate stability (see Section 5), we need estimates for the lower and upper solutions.

Lemma 2.2. *The functions* $\beta_n(x,t,\varepsilon)$ *and* $\alpha_n(x,t,\varepsilon)$ *satisfy the following relations:*

$$\begin{aligned} \frac{\partial \alpha_n}{\partial x} &= \frac{\partial u}{\partial x} + O(\varepsilon^{n-1}), \quad x \in (0,1) \setminus \{x_\alpha(t,\varepsilon)\}, \quad t \in R, \\ \frac{\partial \beta_n}{\partial x} &= \frac{\partial u}{\partial x} + O(\varepsilon^{n-1}), \quad x \in (0,1) \setminus \{x_\beta(t,\varepsilon)\}, \quad t \in R, \end{aligned} \tag{4.14}$$

where $u = u(x,t,\varepsilon)$ *is the periodic interior layer solution of problem* (1.1), *stated in Theorem* 4.1.

Proof. The proof of Lemma 2.2 is based on the estimate for the difference $z_n(x,t,\varepsilon) \equiv u(x,t,\varepsilon) - U_n(x,t,\varepsilon)$; estimate (4.14) then trivially follows from estimate (4.12). Indeed,

$$\frac{\partial \alpha_n}{\partial x} - \frac{\partial u}{\partial x} = \frac{\partial \alpha_n}{\partial x} - \frac{\partial U_n}{\partial x} + \frac{\partial U_n}{\partial x} - \frac{\partial u}{\partial x} = \frac{z_n}{x} + O(\varepsilon^{n-1}).$$

A similar estimate is valid for the upper solution.

The proof is carried out, following the lines of the proof of a similar assertion presented in [10].

The function $z_n(x,t,\varepsilon)$ satisfies the equation

$$\begin{aligned} &\varepsilon^2 \left(\frac{\partial^2 z_n}{\partial x^2} - \frac{\partial z_n}{\partial t} \right) - \varepsilon \Big[A(u,x,t)\,\frac{\partial u}{\partial x} - A(U_n,x,t)\,\frac{\partial U_n}{\partial x} \Big] \\ &\quad - [F(u,x,t,\varepsilon) - F(U_n,x,t,\varepsilon)] = \varepsilon^{n+1}\psi(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in D \end{aligned} \tag{4.15}$$

with zero boundary conditions, where $|\psi(x,t,\varepsilon)| \leqslant c_1$. From Theorem 2.1, we obtain

$$z_n(x,t,\varepsilon) \equiv u(x,t,\varepsilon) - U_n(x,t,\varepsilon) \leqslant c\varepsilon^{n+1}, \tag{4.16}$$

and therefore,

$$|r_1| := |F(u,x,t,\varepsilon) - F(U_n,x,t,\varepsilon)| \leqslant c\varepsilon^{n+1}.$$

The second term of equation (4.15) can be represented in the form

$$A(u,x,t)\,\frac{\partial u}{\partial x} - A(U_n,x,t)\,\frac{\partial U_n}{\partial x} = \frac{\partial}{\partial x}\int_{U_n}^{u} A(s,x,t)ds - \int_{U_n}^{u} A_x(s,x,t)ds.$$

From (4.16), it follows that

$$|r_2| := \left|\int_{U_n}^{u} A_x(s,x,t)ds\right| \leqslant c\varepsilon^{n+1}.$$

We can rewrite equation (4.15) in the following form

$$\begin{aligned}\frac{\partial^2 z_n}{\partial x^2} - \frac{\partial z_n}{\partial t} - kz_n &= -kz_n + \frac{1}{\varepsilon}\frac{\partial}{\partial x}\int_{U_n}^{u} A(s,x,t)ds \\ &+ \frac{1}{\varepsilon^2}\Big[r_1(x,t,\varepsilon) - \varepsilon r_2(x,t,\varepsilon) + \varepsilon^{n+1}\psi(x,t,\varepsilon)\Big] \quad \text{for} \quad (x,t)\in D,\end{aligned} \tag{4.17}$$

where $k > 0$ is an arbitrary constant value. Now we can define

$$r(x,t,\varepsilon) := \frac{1}{\varepsilon^2}\Big[r_1(x,t,\varepsilon) - \varepsilon r_2(x,t,\varepsilon) + \varepsilon^{n+1}\psi(x,t,\varepsilon)\Big].$$

From the estimates above, we obtain

$$|r(x,t,\varepsilon)| \leqslant c\varepsilon^{n-1}.$$

Using the Green function for the parabolic operator of the left hand side of (4.17), we get the following representation for z_n (see, for example, [11]).

$$\begin{aligned}z_n = \int_{-1}^{1} G(x,t,\xi,t_0)z_n(\xi,t_0)d\xi - \int_{t_0}^{t} d\tau \int_{-1}^{1} G(x,t,\xi,\tau) \\ \left(-kz_n(\xi,\tau) + r(\xi,\tau,\varepsilon) + \frac{1}{\varepsilon}\frac{\partial}{\partial \xi}\int_{U_n(\xi,\tau,\varepsilon)}^{u(\xi,\tau,\varepsilon)} A(s,\xi,\tau)ds\right) d\xi.\end{aligned} \tag{4.18}$$

Using integration by parts and the boundary conditions for G, we can transform the last term in (4.18) as follows

$$\begin{aligned}&\int_{t_0}^{t} d\tau \int_{-1}^{1} G(x,t,\xi,\tau)\frac{1}{\varepsilon}\frac{\partial}{\partial \xi}\int_{U_n(\xi,\tau,\varepsilon)}^{u(\xi,\tau,\varepsilon)} A(s,\xi,\tau)dsd\xi \\ &\qquad = -\int_{t_0}^{t} d\tau \int_{-1}^{1} G_\xi(x,t,\xi,\tau)\frac{1}{\varepsilon}\int_{U_n(\xi,\tau,\varepsilon)}^{u(\xi,\tau,\varepsilon)} A(s,\xi,\tau)dsd\xi.\end{aligned} \tag{4.19}$$

Using (4.19) and (4.18) we obtain following representation for the derivative $\frac{\partial z_n}{\partial x}$:

$$\begin{aligned}\frac{\partial z_n}{\partial x} = \int_{-1}^{1} G_x(x,t,\xi,t_0)z_n(\xi,t_0)d\xi - \int_{t_0}^{t} d\tau \int_{-1}^{1} G_x(x,t,\xi,\tau) \\ \left(-kz_n(\xi,\tau) + r(\xi,\tau,\varepsilon)\right)d\xi + \int_{t_0}^{t} d\tau \int_{-1}^{1} G_{\xi x}(x,t,\xi,\tau)\frac{1}{\varepsilon}\int_{U_n(\xi,\tau,\varepsilon)}^{u(\xi,\tau,\varepsilon)} A(s,\xi,\tau)dsd\xi,\end{aligned} \tag{4.20}$$

$$\left|\int_{-1}^{1} G_x(x,t,\xi,t_0)d\xi\right| \leqslant C, \quad \left|\int_{t_0}^{t} d\tau \int_{-1}^{1} G_x(x,t,\xi,\tau)d\xi\right| \leqslant C.$$

We see that the first and second terms of the representation (4.20) have the estimates $O(\varepsilon^{n+1})$ and $O(\varepsilon^{n-1})$, respectively. From the estimates for $G_{\xi x}(x,t,\xi,\tau)$ [12], it also follows in the standard way that the last term in the representation (4.20) can be estimated by

$$\frac{C}{\varepsilon}\int_{U_n(\xi,\tau,\varepsilon)}^{u(\xi,\tau,\varepsilon)}\Big|A(s,\xi,\tau)\Big|ds = O(\varepsilon^n).$$

Using these estimates, we finally, get from (4.20)

$$\frac{\partial z_n}{\partial x}(x,t,\varepsilon) = O(\varepsilon^{n-1}) \quad \text{for} \quad (x,t)\in D. \tag{4.21}$$

This completes the proof of Lemma 2.2.

Remark 2. Taking into account the relation $\frac{\partial u}{\partial x} - \frac{\partial \alpha_{n+1}}{\partial x} = O(\varepsilon^n)$, $\quad \frac{\partial \alpha_n}{\partial x} - \frac{\partial \alpha_{n+1}}{\partial x} = O(\varepsilon^n)$, we arrived the following estimate

$$\frac{\partial u}{\partial x} - \frac{\partial \alpha_n}{\partial x} = O(\varepsilon^n).$$

5. STABILITY RESULTS

In this section, we investigate the asymptotic stability (in the sense of Lyapunov) of the periodic solution $u(x,t,\varepsilon)$. Periodic solutions of problem (1.1) can be regarded as the solutions of the following initial boundary value problem on the semi-infinite time interval:

$$\begin{aligned} &N_\varepsilon(v) := \varepsilon^2\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t}\right) - \varepsilon A(v,x,t)\frac{\partial v}{\partial x} - F(v,x,t,\varepsilon) = 0,\\ &(x,t)\in R^2 : -1 < x < 1, \quad 0 < t < \infty,\\ &v(-1,t,\varepsilon) = u^{(-)}(t), \quad v(1,t,\varepsilon) = u^{(+)}(t), \quad 0 < t < \infty,\\ &v(x,0,\varepsilon) = v^0(x,\varepsilon), \quad x\in[0,1]. \end{aligned} \tag{5.1}$$

Obviously, if $v^0(x,\varepsilon) = u(x,0,\varepsilon)$, where $u(x,t,\varepsilon)$ is a solution to problem (1.1), then problem (5.1) has the solution $v(x,t,\varepsilon) = u(x,t,\varepsilon)$.

In order to investigate its asymptotical stability, we use the asymptotic method of differential inequalities, which is called the method of attenuated barriers (see [13, 14]). We will look for the upper and lower solutions to the problem. The problem of the Lyapunov stability of this solution is solved in the following way: $\alpha(x,t,\varepsilon) = u(x,t,\varepsilon) + e^{-\lambda(\varepsilon)t}(\alpha_n(x,t,\varepsilon) - u(x,t,\varepsilon))$, $\beta(x,t,\varepsilon) = u(x,t,\varepsilon) + e^{-\lambda(\varepsilon)t}(\beta_n(x,t,\varepsilon) - u(x,t,\varepsilon))$, where $\lambda(\varepsilon) > 0$ will be chosen later. Obviously $\alpha < \beta$, and to check the classical theorems of differential inequalities for parabolic systems [11], it is sufficient to show that $N_\varepsilon\beta < 0$, $N_\varepsilon\alpha > 0$. Substituting the above expressions for the functions α and β and taking into account that u is a solution to (1.1), it is not difficult to obtain the required inequalities. For example, the expression for $N_\varepsilon\beta$ is transformed (for brevity, in the following formulas, we omit all the arguments of the functions F, A, F_u, A_u except for the first one) into

$$\begin{aligned} N_\varepsilon\beta = e^{-\lambda t}\Bigg\{&\left[\varepsilon^2\left(-\frac{\partial\beta_n}{\partial t} + \frac{\partial^2\beta_n}{\partial x^2}\right) - \varepsilon A(\beta_n)\frac{\partial\beta_n}{\partial x}\right) - F(\beta_n)\Bigg] + \left[\varepsilon^2\left(-\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2}\right) - \varepsilon A(u)\frac{\partial u}{\partial x}\right) - F(u)\Bigg]\\ &+\varepsilon\frac{\partial\beta_n}{\partial x}[A(\beta_n) - A(u) - A_u^*\cdot(\beta_n - u)] + \varepsilon A_u^*\cdot(\beta_n - u)\frac{\partial(\beta_n - u)}{\partial x}[1 - e^{-\lambda t}] + [F(\beta_n) - F(u) - F_u^*\cdot(\beta_n - u)]\\ &+\varepsilon^2\lambda(\beta_n - u)\Bigg\}. \end{aligned}$$

Here the symbol "*" to the right of the function means that their value is taken for the argument $u(x,t,\varepsilon) + \theta e^{-\lambda(\varepsilon)t}(\alpha_n(x,t,\varepsilon) - u(x,t,\varepsilon)), 0 < \theta < 1$. By using $\varepsilon^2(-\frac{\partial\beta_n}{\partial t} + \frac{\partial^2\beta_n}{\partial x^2} - \varepsilon A(\beta_n)\frac{\partial\beta_n}{\partial x}) - F(\beta_n) = -\gamma\bar{F}_u\varepsilon^{n+1} + O(\varepsilon^{n+2})$, where $\gamma > 0$, $\beta_n - u = O(\varepsilon^{n+1})$ (see Theorem 4.1), $A(\beta_n) - A(u) - A_u^*(\beta_n - u) = O(\varepsilon^{2n+2})$, $F(\beta_n) - F(u) - F_u^*(\beta_n - u) = O(\varepsilon^{2n+2})$ and $\frac{\partial(\beta_n - u)}{\partial x} = O(\varepsilon^n)$ (see Lemma 2.2) and choosing $\lambda(\varepsilon) = \lambda_0$ sufficiently small and γ sufficiently large, we obtain $N_\varepsilon\beta < 0$ for $n \geqslant 0$. Similarly, we can check the inequality $N_\varepsilon\alpha > 0$. Thus, we can formulate the following theorem.

Theorem 5.1. *Suppose that the assumptions A0-A4 on satisfied. Then for sufficiently small* ε*, the periodic solution of problem* (1.1) *is asymptotically stable with domain of attraction at least* $\alpha_0(x,0,\varepsilon) \leqslant u \leqslant \beta_0(x,0,\varepsilon)$*. As a result, this solution is locally unique.*

Note that the width of the stability region, according to Lemma 2.1, $|\beta_0(x,0,\varepsilon) - \alpha_0(x,0,\varepsilon)| = O(\varepsilon)$.

6. LINEAR ADVECTION CASE

We consider the following important class of the systems (1.1):

$$\begin{aligned} &N_\varepsilon(u) := \varepsilon^2 \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\right) - \varepsilon a(x,t)\,\frac{\partial u}{\partial x} - (u - \varphi(x,t))(u^2-1) = 0, \\ &|\varphi(x,t)| < 1 \quad (x,t) \in D := \{(x,t) \in R^2 : -1 < x < 1, t \in R\}, \\ &u(-1,t,\varepsilon) = -1, \quad u(1,t,\varepsilon) = 1 \quad \text{for} \quad t \in R, \\ &u(x,t,\varepsilon) = u(x,t+T,\varepsilon) \quad \text{for} \quad t \in R, \quad -1 \leqslant x \leqslant 1. \end{aligned} \tag{6.1}$$

An similar problem was investigated in the paper [9] (see Example 3.2), in which the case was considered when the advection coefficient is positive and depends only on time. In this paper, it was proved only the existence of a solution with a transition layer. The applied approach does not allow us to investigate the stability of a solution and to construct its asymptotic behavior.

Here $\varphi^{(\pm)}(x,t) = \pm 1$, $\varphi^{(0)}(x,t) = \varphi(x,t)$. The separatrix that joins two saddles $S1(-1,0)$ and $S3(1,0)$ for $x = x_0(t)$ on the phase plane of the adjoint system

$$\frac{\partial \tilde v}{\partial \xi} = a(x,t)\tilde v + (u - \varphi(x,t))(u^2-1), \quad \frac{\partial \tilde u}{\partial \xi} = \tilde v, \quad -\infty < \xi < \infty, \tag{6.2}$$

is constructed in the form of the parabola $\tilde v = \lambda(x,t)(\tilde u^2 - 1)$. By performing the straightforward substitution into the first equation in system (6.2), we obtain:

$$(2\lambda^2 - 1)\tilde u = a(x,t)\lambda - \varphi(x,t). \tag{6.3}$$

If we require that both sides of (6.3) be identically equal to zero, then a separatrix in the form of the parabola exists. Hence, taking into account the fact that we consider the upper half-plane, we obtain the value $\lambda(x,t) = -1/\sqrt{2}$, and the corresponding equation for the transition location curve in the zero approximation has the form $w(x_0,t) := a(x_0,t) + \sqrt{2}\varphi(x_0,t) = 0$.

Note that the equation for the function $\tilde u(\xi,x,t)$ which is obtained from system (6.2) cannot be integrated by quadratures for fixed x. The equation for the separatrix that joins two saddles $S1(-1,0)$ and $S3(1,0)$ for $x = x_0(t)$ can also be written as

$$\frac{d\tilde v}{d\tilde u} = f(\tilde u,\tilde v) := a(x_0,t) + \frac{(\tilde u - \varphi(x_0,t))(\tilde u^2-1)}{\tilde v}, \quad \tilde v(-1) = 0. \tag{6.4}$$

Here and further, the functions f, f_1, f_3 have only two arguments, because x and t are fixed and treated as parameters.

For fixed x, we have the following equation for the separatrix issuing from the saddle $S1(-1,0)$:

$$\frac{d\tilde v_1}{d\tilde u} = f_1(\tilde u,\tilde v_1) := a(x,t) + \frac{(\tilde u - \varphi(x,t))(\tilde u^2-1)}{\tilde v_1}, \quad \tilde v_1(-1) = 0, \tag{6.5}$$

and for the separatrix entering the saddle $S3(1,0)$:

$$\frac{d\tilde v_3}{d\tilde u} = f_3(\tilde u,\tilde v_3) := a(x,t) + \frac{(\tilde u - \varphi(x,t))(\tilde u^2-1)}{\tilde v_3}, \quad \tilde v_1(1) = 0. \tag{6.6}$$

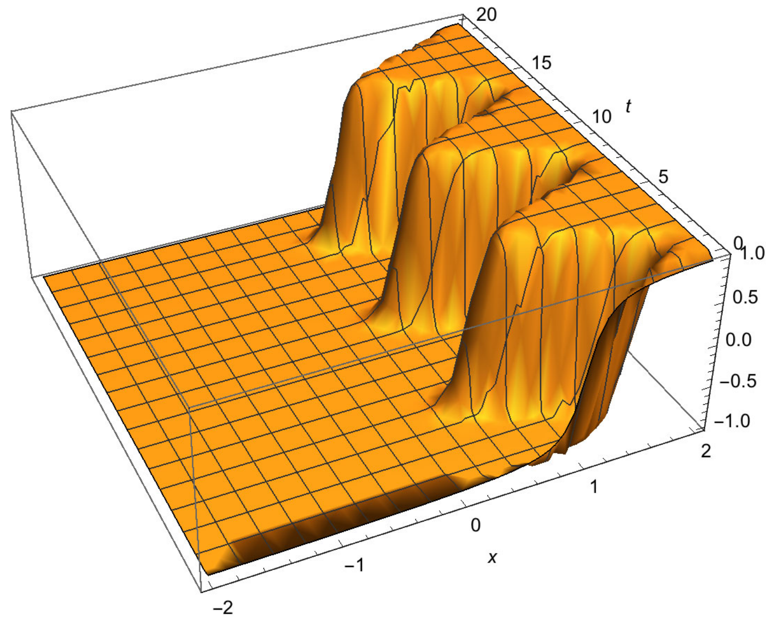

Fig. 1. The numerical solution to the initial boundary value problem corresponding to (6.1).

It can be shown, that tasks (6.5), (6.6) have solutions which are continuously dependent on the papramete x. As the separatrices in the form of parabola, issuing from S1 and S3 intersect the line $u = \phi(x,t)$, the separatrices $v_1(u)$ and $v_3(u)$ intersect it too for any x from a small neighborhood of x_0. It is clear that the condition **(A2)** can be formulated for x lying in this neighborhood. Therefore, the condition **(A2)** is satisfied.

To derive the sufficiency requirement for the Condition **(A3)** to be true, we use that according to [15], the following representation for the function $\frac{\partial H}{\partial x}$ in explicit form is valid:

$$\left.\frac{\partial H}{\partial x}\right|_{x=x_0(t)} = -\frac{1}{\tilde{v}(0,x_0,t)}\int_{-\infty}^{+\infty} f_x(\tilde{v}(\eta,x_0,t),\tilde{u}(\eta,x_0,t),x_0,t) \times \exp(-a(x_0,t)\eta)\tilde{v}(\eta,x_0,t)d\eta, \tag{6.7}$$

where

$$f_x(\tilde{v}(\eta,x_0,t),\tilde{u}(\eta,x_0,t),x_0,t) = \tilde{v}\left(a_x - \frac{\varphi_x}{\lambda}\right). \tag{6.8}$$

It can be shown that the solution $\tilde{u}$ of the problem (6.4) looks like this:

$$\tilde{u}(\xi,x_0,t) = \frac{C\exp(-2\lambda\xi)-1}{C\exp(-2\lambda\xi)+1}, \quad C = \frac{1+\varphi(x_0,t)}{1-\varphi(x_0,t)}. \tag{6.9}$$

From (6.9), we obtain

$$\xi = -\ln\left[\frac{1+\tilde{u}}{1-\tilde{u}}\,\frac{1-\varphi(x_0,t)}{1+\varphi(x_0,t)}\right]^{\frac{1}{2\lambda}}. \tag{6.10}$$

Substituting (6.8), (6.10) in (6.7), we see that $\frac{\partial H}{\partial x}$ has the form:

$$\left.\frac{\partial H}{\partial x}\right|_{x=x_0(t)} = \left[\frac{1-\varphi}{1+\varphi}\right]^{\varphi}\frac{\lambda}{\tilde{v}(0,x_0,t)}\left[a_x(x_0,t)+\sqrt{2}\varphi_x(x_0,t)\right]I, \tag{6.11}$$

where $I := \int_{-1}^{+1}(1+u)^{1+\varphi}(1-u)^{1-\varphi}du > 0$.

From (6.11), we conclude that the function $H(x,t) = v_3(\varphi(x,t)) - v_1(\varphi(x,t))$ has the positive derivative with respect to x at $x_0(t)$ for any real t, namely, if the inequality

$$w_x(x_0,t) = a_x(x_0,t) + \sqrt{2}\varphi_x(x_0,t) < 0$$

takes place, then the Condition **(A3)** is satisfied. It provides the existence of a step-like contrast structure with the transition from S1 to S3. Consequently, Theorems 4.1 and 5.1 for the problem (6.1) are satisfied when the above-listed conditions for the function $w(x,t)$ are fulfilled.

In Fig. 2, the result of the calculations in the "Wolfram Mathematica" application is depicted. Here is the numerical solution to the initial boundary value problem corresponding to (6.1) on the semi-infinite time interval. The problem is considered for $x \in (-2;2)$, $a(x,t) = \sin(t) - x + 2$, $\varphi(x,t) = -\frac{x}{2^{0.5}}$, $u(x,0) = \frac{2\tan^{-1}\left(\frac{x-1}{\epsilon}\right)}{\pi}$ and $\varepsilon = 0.1$. This figure shows that the solution to the problem (4.1) is asymptotically stable. For the zero-th term of the interior layer asymptotic for the problem (6.1), we have the expression $x_0(t) = 1 + \sin(t)/2$, which is in agreement with Fig. 2.

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