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Manuilov Algebra, C^*-Hilbert Modules, and Kuiper Type Theorems[1]

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Abstract. We generalize (in two natural ways) the C^*-algebra generated by matrices of bounded operators in a separable Hilbert space H with a bounded number of nonzero elements in each row and each column, introduced recently by V. Manuilov. We consider the standard C^*-Hilbert module $H_{\mathcal{A}}$ instead of $H = H_{\mathbb{C}}$. Also we consider the algebras with finiteness conditions only on rows or only on columns. For related general linear groups, we prove the contractibility (Kuiper type theorems) and some other properties.

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INTRODUCTION

Recently V. Manuilov introduced the algebra $\mathbb{B}^f(H)$, which is the C^*-algebra generated by matrices of bounded operators in a separable Hilbert space H with a bounded number of nonzero elements in each row and each column [6]. He discovered very interesting properties of this algebra in the same paper, in particular, he proved the contractibility of the general linear group $GL(\mathbb{B}^f(H))$ (the group of invertible elements) of this algebra. The relation of the algebra to some previously known ones remains unclear (see the discussion in [6]). Then Manuilov applied the approach to the study of graphs in [7].

We introduce some natural generalizations of this algebra from operators on H to operators on the standard Hilbert module $H_{\mathcal{A}}$ over a unital C^*-algebra $\mathcal{A}$.

The study of contractibility of general groups of operators on $H_{\mathcal{A}}$ has a long and dramatic history (see [9, Chap. 7]). Here we have two operator algebras: the Banach algebra of all operators (bounded $\mathcal{A}$-homomorphisms) $\mathbb{B}(H_{\mathcal{A}})$ and the C^*-algebra $\mathbb{B}^\star(H_{\mathcal{A}})$ of adjointable operators. Respectively, we have two general linear groups: $GL(\mathbb{B}(H_{\mathcal{A}}))$ and $GL(\mathbb{B}^\star(H_{\mathcal{A}}))$. The contractibility of $GL(\mathbb{B}^\star(H_{\mathcal{A}}))$ when $\mathcal{A}$ is σ-unital was proved in [1] (see also the previous papers [10, 4, 16, 11]). The contractibility of $GL(\mathbb{B}(H_{\mathcal{A}}))$ was established only for some classes of algebras (see [17] and some examples and counterexamples in [14] and [15]). The proof of the case of $GL(\mathbb{B}(H_{\mathcal{A}}))$ in [16] contains a mistake (in contrast with the case of $GL(\mathbb{B}^\star(H_{\mathcal{A}}))$ in the same paper). An analog of the Dixmier–Douady theorem on strong contractibility was obtained for Hilbert modules in [17].

It is natural to consider two distinct generalizations of the Manuilov algebra to $H_{\mathcal{A}}$: the first one (strong) is more or less analogous to the case of H (Definition 3.1 below) and the other one (weak) supposes finitely many nonzero values $\varphi(da_ia_i^*d)$ for each pure state φ, an element $d \in \mathcal{A}$, and a_i running along a row of the matrix (and for columns in an adjoint manner) (see the precise form of this in Definition 3.3). Denote these algebras by $\mathbb{B}^f(H_{\mathcal{A}})$ and $W\mathbb{B}^f(H_{\mathcal{A}})$, respectively. We also consider the algebras with finiteness conditions only on rows or only on columns: $\mathbb{B}^f_L(H_{\mathcal{A}})$, $\mathbb{B}^f_C(H_{\mathcal{A}})$, $W\mathbb{B}^f_L(H_{\mathcal{A}})$, $W\mathbb{B}^f_C(H_{\mathcal{A}})$ (Definitions 3.1 and 3.3 below).

We prove here some properties of these algebras and the contractibility of the following groups of invertible elements:

$$GL(\mathbb{B}^f_C(H_{\mathcal{A}}) \cap \mathbb{B}^\star(H_{\mathcal{A}})) \quad \text{(Theorem 5.1)}, \quad GL(\mathbb{B}^f_L(H_{\mathcal{A}})) \quad \text{(Theorem 5.3)},$$

$$GL(\mathbb{B}^f(H_{\mathcal{A}})) \quad \text{(Theorem 5.5)}, \quad GL(W\mathbb{B}^f_C(H_{\mathcal{A}}) \cap \mathbb{B}^\star(H_{\mathcal{A}})) \quad \text{(Theorem 6.1)},$$

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$$GL(W\mathbb{B}^f_L(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A})) \quad \text{(Theorem 6.2)}, \quad GL(W\mathbb{B}^f(H_\mathcal{A})) \cap \mathbb{B}^\star(H_\mathcal{A}) \quad \text{(Theorem 6.3)}.$$

We also prove that the group $GL(W\mathbb{B}^f(H_\mathcal{A}))$ is contractible inside $GL(\mathbb{B}(H_\mathcal{A}))$ (Theorem 6.4).

As a corollary, we obtain a Kuiper type theorems for one-side algebras in the initial case of a Hilbert space: the contactibility of $GL(\mathbb{B}^f_C(H))$ (Theorem 5.2) and $GL(\mathbb{B}^f_L(H))$ (Theorem 5.4).

Some open questions and directions of research are indicated.

1. PRELIMINARIES

For the general theory of C^*-Hilbert modules, we refer to the monographs [5, 9] and the survey paper [8]. Some current trends in the field can be found in [3, 13].

We deal here only with the standard module $H_\mathcal{A} = \ell_2(\mathcal{A})$, and thus we skip the general definition.

Definition 1.1. The standard Hilbert module is the right $\mathcal{A}$-module of sequences $a = (a_1, a_2, \dots)$, $a_i \in \mathcal{A}$, such that $\sum_i (a_i)^* a_i$ is norm-convergent in $\mathcal{A}$. It is equipped with the inner product $\langle a, b\rangle = \sum_i (a_i)^* b_i$, where $b = (b_1, b_2, \dots)$.

The left module is defined via the inner product $\sum_i a_i (b_i)^*$.

Definition 1.2. Denote the elements of the *standard base* of $H_\mathcal{A}$ by e_i, where e_i has $\mathbf{1}_\mathcal{A}$ at the i^{th} place and zeros at the remaining ones. The submodule L_m generated by $e_1, \dots, e_m$ will be called the "first base submodule." Denote by q_m the orthogonal projection $q_m : H_\mathcal{A} \to L_m$.

The main distinction of $H_\mathcal{A}$ from a Hilbert space is the following: a bounded $\mathcal{A}$-homomorphism need not admit an adjoint in the evident sense (see the discussion in [9, Sec. 2.1]). The same holds for $\mathcal{A}$-functionals. They can be described in the unital case as sequences $(\alpha_1, \alpha_2, \dots)$ such that partial sums of $\sum_i \alpha_i (\alpha_i)^*$ are uniformly bounded (see, e.g., [9, Proposition 2.5.5]).

Example 1.3. A typical example of an $\mathcal{A}$-functional on $H_\mathcal{A}$ without adjoint over $\mathcal{A} = C[0, 1]$ is defined by a sequence $(\gamma_1, \gamma_2, \dots)$ of functions of norm 1 with disjoint supports (see [9, Example 2.1.2]).

Definition 1.4. Denote the C^*-algebra of adjointable operators (i.e., bounded adjointable $\mathcal{A}$-homomorphisms) by $\mathbb{B}^\star(H_\mathcal{A})$ and the Banach algebra of all bounded operators by $\mathbb{B}(H_\mathcal{A})$. A *matrix* of an operator F is evidently formed by columns which are the components of $F(e_i)$ regarded as an element of $H_\mathcal{A}$.

The following statement is evident.

Theorem 1.5. *An operator from $\mathbb{B}(H_\mathcal{A})$ is adjointable if and only if each row of its matrix defines an adjointable functional.*

We will need the following known fact, for which we have not found a reference.

Lemma 1.6. *For any $a \in \mathcal{A}$, we have*

$$\|a\| \leqslant 2 \sup_{\varphi \text{ is a pure state}} |\varphi(a)|.$$

Proof. Since $a = \frac{1}{2}(a + a^*) + i \cdot \frac{1}{2i}(a - a^*)$, we have by [12, Th. 3.3.6], for some states φ_1 and φ_2,

$$\begin{aligned}\|a\| &\leqslant \frac{1}{2}\|a + a^*\| + \frac{1}{2}\|i(a - a^*)\| = \frac{1}{2}(\varphi_1(a + a^*) + \varphi_2(i(a - a^*)) \\ &\leqslant \frac{1}{2}(|\varphi_1(a) + \overline{\varphi_1(a)}| + |\varphi_2(a) - \overline{\varphi_2(a)}|) \leqslant 2 \sup_{\varphi \text{ is a state}} |\varphi(a)| \leqslant 2 \sup_{\varphi \text{ is a pure state}} |\varphi(a)|,\end{aligned}$$

because any state is approximated by convex combinations of pure states.

2. A GENERAL SCHEME FOR KUIPER TYPE THEOREMS

In this section, we summarize and slightly modify some argument known mostly from [16, 17] (see also [18] and [9, Chap. 7]) for proving of Kuiper type theorems for Hilbert modules.

We suppose $\mathcal{A}$ to be unital. The general argument based on the Atiyah theorem "on small balls" reduces a proof of contractibility of the general linear group of a Banach algebra $\mathcal{B}\subseteq\mathbb{B}(H_\mathcal{A})$ to the proof of the following fact: for any finite polyhedron P with vertexes $A_1,\dots,A_N$ and its inclusion $J: P\to GL(\mathcal{B})$, there is a homotopy of J to the constant mapping $P\to\mathbf{1}\in GL(\mathcal{B})$. Also, as a result of arbitrary small perturbation, we may assume that the columns of the matrix of A are of finite length:

$$\text{for any } i, \text{ there exists } j(i) \text{ such that } a^i_j=0 \text{ for } j>j(i). \tag{2.1}$$

Step 1. Find, for arbitrary small ε, a homotopy of the polyhedron P to another one, denoted P', such that there exists a decomposition $H_\mathcal{A}=H^0_\mathcal{A}\oplus H^1_\mathcal{A}\oplus H^2_\mathcal{A}$, where $H^0_\mathcal{A}$, $H^1_\mathcal{A}$, and $H^2_\mathcal{A}$ are some sums of the base modules E_i, with bases $\{e_{i_0(r)}\}$, $\{e_{i_1(r)}\}$, and $\{e_{i_2(r)}\}$, respectively (i.e., elements of these bases are some sequences of e_j with increasing indices), such that

$$i_0(1)=1,\qquad i_0(r+1)=\max\{j(i_1(1)),\dots,j(i_1(r))\}+1, \tag{2.2}$$

(in the notation of (2.1)), and

$$\|q_{m(r)}A'e_{i_1(r+1)}\|<\frac{\varepsilon}{2^r}, \tag{2.3}$$

for any element $A'\in P'$, where $q_{m(r)}$ is the projection on the first basic module $L_{m(r)}$, which contains $e_{i_0(1)},\ \dots\ ,e_{i_0(r+1)}$, $A'e_{i_1(1)},\ \dots\ ,A'e_{i_1(r)}$. In other words, $m(r)=i_0(r+1)$.

We will take ε such that the ε-neighbourhood of P' lies in $GL(\mathcal{B})$.

Note that for some algebras under consideration below, this step will be unnecessary.

Step 2. Consider the linear homotopy

$$A'_t(e_l)=\begin{cases} A'(e_l)-t\cdot q_{m(r)}A'(e_l), & \text{if } l=i_1(r+1),\\ A'(e_l), & \text{otherwise,}\end{cases}\qquad t\in[0,1],\quad l=1,2,\dots \tag{2.4}$$

Denote the resulting polyhedron by P'' with elements A''. They satisfy

$$q_{m(r)}A''e_{i_1(r+1)}=0,\qquad r=1,2,\dots \tag{2.5}$$

for any r. The inequality (2.3) and the choice of ε imply that this linear homotopy is in $GL(\mathcal{B})$.

Step 3. Properties (2.2) and (2.5) imply that the elements $e_{i_0(r)}$ and $A'e_{i_0(u)}$ form an orthogonal system and generate an orthogonally complementable submodule (its complement $H^3_\mathcal{A}$ is equal to the orthogonal sum of complements to $A''e_{i_1(r+1)}$ in $L_{m(r+1)}\ominus L_{m(r)}$).

Thus, we can perform a generalized rotation,

$$\begin{aligned} R_{A''}(t)(e_{i_0(r)}) &= \cos t\cdot e_{i_0(r)}-\sin t\cdot A''e_{i_1(r)},\\ R_{A''}(t)(A''e_{i_1(r)}) &= \sin t\cdot e_{i_0(r)}+\cos t\cdot A''e_{i_1(r)},\\ R_{A''}(t)(x)=x,&\qquad \text{for } x\in H^3_\mathcal{A},\end{aligned} \tag{2.6}$$

$r=1,2,\dots$, $t\in[0,\pi/2]$. Evidently it is continuous in A''.

Then

$$R_{A''}(\pi/2)\cdot A''|_{H^1_\mathcal{A}}=\mathrm{Id}_{H^1_\mathcal{A}}. \tag{2.7}$$

For the argument used below, we need to find the matrix form for (2.6). Evidently, it consists of square blocks $R^r_{A''}(t)$ corresponding to r. The size of $R^r_{A''}(t)$ is $m(r)-m(r-1)$. Denote the components of $A''e_{i_1(r)}$ in $L_{m(r)}\ominus L_{m(r-1)+1}$ by $a_1,\dots,a_u$, $u=m(r)-m(r-1)-1$ (the other components vanish by (2.2) and (2.5)). Denote $b := a_1^* a_1 + a_2^* a_2 + \dots + a_u^* a_u$. Then the mentioned block $R^r_{A''}(t)$ has the form

$$\begin{pmatrix} \cos(t) & \sin(t)b^{-1}a_1^* & \dots & \sin(t)b^{-1}a_u^* \\ -\sin(t)a_1 & & & \\ \vdots & & \cos(t)\cdot \mathrm{pr}_a & \\ -\sin(t)a_u & & & \end{pmatrix} + \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & 1-\mathrm{pr}_a & \\ 0 & & & \end{pmatrix}, \tag{2.8}$$

where

$$\mathrm{pr}_a = \begin{pmatrix} a_1 b^{-1} a_1^* & \dots & a_1 b^{-1} a_u^* \\ \vdots & \vdots & \vdots \\ a_u b^{-1} a_1^* & \dots & a_u b^{-1} a_u^* \end{pmatrix}$$

is the projection on the submodule in $L_{m(r)} \ominus L_{m(r-1)+1}$ generated by $(a_1, \dots, a_u)$. Here the inclusion $A'' \in GL(\mathcal{B})$ implies the invertibility of $b = \langle A'' e_{i_1(r)}, A'' e_{i_1(r)} \rangle$.

The last two steps are very similar to Kuiper's.

Step 4. After this generalized rotation and re-scaling, we obtain the following matrices for all points of the corresponding polyhedron P''' with respect to the decomposition $H_\mathcal{A} = (H^1_\mathcal{A})^\perp \oplus H^1_\mathcal{A}$:

$$A''' = \begin{pmatrix} D''' & 0 \\ C''' & 1 \end{pmatrix}.$$

Evidently, here D''' is an isomorphism and the linear homotopy

$$A'''_t = \begin{pmatrix} D''' & 0 \\ t\cdot C''' & 1 \end{pmatrix} \tag{2.9}$$

consists of isomorphisms and connects $A''' = A'''_1$ and $A'''_0 =: A^D$, which is of the block-diagonal form $\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}$, $D = D'''$.

Step 5. Now decompose $H^1_\mathcal{A}$ in an orthogonal sum of countably many pairs of sub-modules H^0_i, H^1_i, $i = 1, 2, \dots$, generated by certain countable collections of basic elements e_s. Connect $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} DD^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ at each summand $H^0_i \oplus H^1_i$ with $\begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix}$ via the homotopy

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} D^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \qquad t \in [0, \pi/2].$$

If we extend this homotopy to be constant are equal to D on $(H^1_\mathcal{A})^\perp$, we obtain a path connecting

$$A^D = \mathrm{diag}(D, 1, 1, 1, \dots) \quad \text{and} \quad \mathrm{diag}(D, D^{-1}, D, D^{-1}, D, \dots) =: A^{DD},$$

with respect to the decomposition $H_\mathcal{A} = (H^1_\mathcal{A})^\perp \oplus (H^0_1 \oplus H^1_1) \oplus (H^0_2 \oplus H^1_2) \oplus \cdots$. Write down this $H_\mathcal{A}$ as $H_\mathcal{A} = ((H^1_\mathcal{A})^\perp \oplus H^0_1) \oplus (H^1_1 \oplus H^0_2) \oplus (H^1_2 \oplus H^0_3) \oplus \cdots$ and connect the restriction $\begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}$ of A^{DD} on each of these couples of subspaces with the identity by using the following homotopy:

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} D^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}, \qquad t \in [0, \pi/2].$$

Taking the direct sum, we obtain a homotopy $A^{DD} \sim \mathrm{Id}$. This completes the scheme.

We need to verify the continuity of all above homotopies with respect to the operator argument (i.e., in A, A', ..., A^{DD}). Because of the formulas, this is evident for all steps, except maybe the above (generalized) rotations. But they are decomposed in a direct orthogonal sum of rotations in some finite-dimensional modules of the form $L_{m(r)} \ominus L_{m(r-1)+1}$, which are contunuous uniformly in r, because the distance between $R^r_{A''}(t)$ and $R^r_{A''_0}(t)$ can be estimated in terms of the distance between $A'' e_{i_1(r)}$ and $A''_0 e_{i_1(r)}$. This implies the continuity.

3. MANUILOV ALGEBRAS ON C^*-HILBERT MODULES

Passing from the separable Hilbert space ℓ_2 to the standard C^*-Hilbert module $\ell_2(\mathcal{A}) = H_\mathcal{A}$ over a C^*-algebra $\mathcal{A}$, we can give several distinct definitions.

It is natural to restrict ourselves to the unital case, because only in this case an operator from $\mathbb{B}(H_\mathcal{A})$ has a matrix $\|a_{ij}\|$, $a_{ij} \in \mathcal{A}$, with respect to the standard base $\{e_i\}$. In the general case, a_{ij} are left multipliers, are they do not form a C^*-algebra.

Definition 3.1. Denote by $\mathbb{B}_L^{(k)}(H_\mathcal{A})$ the set of operators in $\mathbb{B}(H_\mathcal{A})$ having no more than k nonzero elements in each row of their matrices, and by $\mathbb{B}_C^{(k)}(H_\mathcal{A})$ the set of operators in $\mathbb{B}(H_\mathcal{A})$ having no more than k nonzero elements in each column of their matrices. Let us put $\mathbb{B}^{(k)}(H_\mathcal{A}) = \mathbb{B}_L^{(k)}(H_\mathcal{A}) \cap \mathbb{B}_C^{(k)}(H_\mathcal{A})$. Write

$$\mathbb{B}_L^{\infty}(H_\mathcal{A}) := \bigcup_k \mathbb{B}_L^{(k)}(H_\mathcal{A}), \quad \mathbb{B}_C^{\infty}(H_\mathcal{A}) := \bigcup_k \mathbb{B}_C^{(k)}(H_\mathcal{A}), \quad \mathbb{B}^{\infty}(H_\mathcal{A}) := \bigcup_k \mathbb{B}^{(k)}(H_\mathcal{A}).$$

Definition 3.2. For a positive functional $\varphi : \mathcal{A} \to \mathbb{C}$, denote by φ_a the positive functional $\varphi_a(b) = \varphi(aba^*)$, where $a \in \mathcal{A}$.

Definition 3.3. Denote by $W\mathbb{B}_L^{(k)}(H_\mathcal{A}) \to \mathbb{B}(H_\mathcal{A})$ (weakly having no more than k nonzero elements in a row) the set of all operators from $\mathbb{B}(H_\mathcal{A})$ such that, for any pure state φ on $\mathcal{A}$ and any $d \in \mathcal{A}$, there is no more than k elements in any row of the matrix of the operator, say $a_i^{j_1}, \dots a_i^{j_k}$ in the i^{th} row, with the property $\varphi_d(a_i^{j_s}(a_i^{j_s})^*) \neq 0$.

Similarly, denote by $W\mathbb{B}_C^{(k)}(H_\mathcal{A}) \to \mathbb{B}(H_\mathcal{A})$ the set of all operators from $\mathbb{B}(H_\mathcal{A})$ such that, for any pure state φ on $\mathcal{A}$ and any element $d \in \mathcal{A}$, there is no more than k elements in any column of the matrix of the operator, say $a_{j_1}^i, \dots, a_{j_k}^i$ in the i^{th} column, with the property $\varphi_d((a_i^{j_s})^* a_i^{j_s}) \neq 0$.

Write $W\mathbb{B}^{(k)}(H_\mathcal{A}) := W\mathbb{B}_L^{(k)}(H_\mathcal{A}) \cap W\mathbb{B}_C^{(k)}(H_\mathcal{A})$ and

$$W\mathbb{B}_L^{\infty}(H_\mathcal{A}) := \bigcup_k W\mathbb{B}_L^{(k)}(H_\mathcal{A}), \quad W\mathbb{B}_C^{\infty}(H_\mathcal{A}) := \bigcup_k W\mathbb{B}_C^{(k)}(H_\mathcal{A}), \quad W\mathbb{B}^{\infty}(H_\mathcal{A}) := \bigcup_k W\mathbb{B}^{(k)}(H_\mathcal{A}).$$

Remark 3.4. Note that, for any positive functional φ, if $\varphi(aa^*) = 0$, then $\varphi(ba^*) = 0$ and $\varphi(ab) = 0$ for any $b \in \mathcal{A}$, because

$$|\varphi(yx^*)|^2 \leqslant \varphi(xx^*)\varphi(yy^*) \tag{3.1}$$

(see [2, Subsec. 2.1]).

Lemma 3.5. *The above sets* $\mathbb{B}_L^{\infty}(H_\mathcal{A})$, $\mathbb{B}_C^{\infty}(H_\mathcal{A})$, $\mathbb{B}^{\infty}(H_\mathcal{A})$, $W\mathbb{B}_L^{\infty}(H_\mathcal{A})$, $W\mathbb{B}_C^{\infty}(H_\mathcal{A})$, *and* $W\mathbb{B}^{\infty}(H_\mathcal{A})$ *are subalgebras of* $\mathbb{B}(H_\mathcal{A})$. *More precisely,*

$$W\mathbb{B}_L^{(k)}(H_\mathcal{A}) \cdot W\mathbb{B}_L^{(m)}(H_\mathcal{A}) \subseteq W\mathbb{B}_L^{(k+m)}(H_\mathcal{A}),$$

$$W\mathbb{B}_C^{(k)}(H_\mathcal{A}) \cdot W\mathbb{B}_C^{(m)}(H_\mathcal{A}) \subseteq W\mathbb{B}_C^{(k+m)}(H_\mathcal{A}),$$

and the same for $\mathbb{B}^{(k)}$.

Proof. Everything is obvious except the above inclusions for $W\mathbb{B}^{(k)}$ (for $\mathbb{B}^{(k)}$ they are obvious and quite similar to the complex case [6]).

Suppose, $A \in W\mathbb{B}_L^{(k)}(H_\mathcal{A})$, $B \in W\mathbb{B}_L^{(m)}(H_\mathcal{A})$, $d \in \mathcal{A}$, and φ is a pure state on $\mathcal{A}$. Consider the i^{th} row of A. Suppose, $\varphi_d(a_i^{j_s}(a_i^{j_s})^*) = 0$ for all elements of the i^{th} row, except maybe $a_i^{j_1}, \dots, a_i^{j_k}$.

Then for each $s = 1, \dots, k$, we can find such elements $b_{j_s}^{r(s,1)}, \dots, b_{j_s}^{r(s,m)}$ of the row number j_s of B that $\varphi_{da_i^{j_s}}(b_{j_s}^t (b_{j_s}^t)^*) = \varphi(da_i^{j_s} b_{j_s}^t (b_{j_s}^t)^* (a_i^{j_s})^* d^*) = 0$ if $t \notin \{r(s,1), \dots, r(s,m)\}$. Then $\varphi_d((AB)_i^r((AB)_i^r)^*) = 0$ if $r \neq r(s,u)$, $s = 1, \dots, k$, $u = 1, \dots, m$. Indeed, in all nonzero summands of

$$\varphi_d((AB)_i^r((AB)_i^r)^*) = \varphi\left(\sum_v da_i^v b_v^r \left(\sum_v a_i^v b_v^r\right)^* d^*\right) = \sum_{v,w} \varphi(da_i^v b_v^r (b_w^r)^*, (a_i^w)^* d^*)$$

one has u and v among $j_1, \dots, j_k$ by Remark 3.4. Thus,

$$\varphi_d((AB)_i^r((AB)_i^r)^*) = \sum_{s,z=1}^{k} \varphi(da_i^{j_s} b_{j_s}^r (b_{j_z}^r)^* (a_i^{j_z})^* d^*).$$

By (3.1),

$$|\varphi_d((AB)_i^r((AB)_i^r)^*)| \leqslant \sum_{s,z=1}^{k} \left(\varphi(da_i^{j_s} b_{j_s}^r (b_{j_s}^r)^* (a_i^{j_s})^* d^*)\varphi(da_i^{j_z} b_{j_z}^r (b_{j_z}^r)^* (a_i^{j_z})^* d^*)\right)^{1/2} = 0$$

for our choice of r. So the i^{th} row of AB can have entries with $\varphi_d((AB)_i^r((AB)_i^r)^*) \neq 0$ only for $r = r(s,u)$, $s = 1, \dots, k$, $u = 1, \dots, m$.

Similarly for $W\mathbb{B}_C^{(k)}(H_\mathcal{A})$, but starting from B. More precisely, suppose, that $A \in W\mathbb{B}_C^{(k)}(H_\mathcal{A})$, $B \in W\mathbb{B}_C^{(m)}(H_\mathcal{A})$, $d \in \mathcal{A}$, and φ is a positive functional on $\mathcal{A}$. Consider the i^{th} column of B. Suppose, $\varphi_d((b_{j_s}^i)^* b_{j_s}^i) = \varphi(d(b_{j_s}^i)^* b_{j_s}^i d^*) = 0$ for all elements of the i^{th} column, except maybe $b_{j_1}^i, \dots, b_{j_m}^i$. Then for each $s = 1, \dots, m$, we can find such elements $a_{r(s,1)}^{j_s}, \dots, a_{r(s,k)}^{j_s}$ of the column number j_s of A that $\varphi_{d(b_{j_s}^i)^*}((a_t^{j_s})^* a_t^{j_s}) = \varphi_d((b_{j_s}^i)^* (a_t^{j_s})^* a_t^{j_s} b_{j_s}^i) = 0$ if $t \notin \{r(s,1), \dots, r(s,k)\}$. Then $\varphi_d(((AB)_r^i)^*(AB)_r^i) = 0$ if $r \neq r(s,u)$, $s = 1, \dots, m$, $u = 1, \dots, k$. Indeed, in all nonzero summands of

$$\varphi_d(((AB)_r^i)^*(AB)_r^i) = \varphi_d\left(\left(\sum_v a_r^v b_v^i\right)^* \sum_v a_r^v b_v^i\right) = \sum_{v,w} \varphi(d(b_w^i)^* (a_r^w)^* a_r^v b_v^i d^*),$$

one has u and v among $j_1, \dots, j_k$ by Remark 3.4. Thus,

$$\varphi_d(((AB)_r^i)^*(AB)_r^i) = \sum_{s,z=1}^{k} \varphi(d(b_{j_z}^i)^* (a_r^{j_z})^* a_r^{j_s} b_{j_s}^i d^*).$$

By (3.1),

$$|\varphi_d(((AB)_r^i)^*(AB)_r^i)| \leqslant \sum_{s,z=1}^{k} (\varphi(d(b_{j_s}^i)^* (a_r^{j_s})^* a_r^{j_s} b_{j_s}^i d^*)\varphi(d(b_{j_z}^i)^* (a_r^{j_z})^* a_r^{j_z} b_{j_z}^i d^*))^{1/2} = 0$$

for our choice of r. Now we can complete the proof as in the previous case.

Definition 3.6. Denote the closure in $\mathbb{B}(H_\mathcal{A})$ of each of the above subalgebras, respectively, by

$$\mathbb{B}_L^f(H_\mathcal{A}) := \overline{\mathbb{B}_L^\infty(H_\mathcal{A})}, \quad \mathbb{B}_C^f(H_\mathcal{A}) := \overline{\mathbb{B}_C^\infty(H_\mathcal{A})}, \quad \mathbb{B}^f(H_\mathcal{A}) := \overline{\mathbb{B}^\infty(H_\mathcal{A})},$$

$$W\mathbb{B}_L^f(H_\mathcal{A}) := \overline{W\mathbb{B}_L^\infty(H_\mathcal{A})}, \quad W\mathbb{B}_C^f(H_\mathcal{A}) := \overline{W\mathbb{B}_C^\infty(H_\mathcal{A})}, \quad W\mathbb{B}^f(H_\mathcal{A}) := \overline{W\mathbb{B}^\infty(H_\mathcal{A})}.$$

So, these closures are Banach subalgebras of $\mathbb{B}(H_\mathcal{A})$.

4. RELATION TO ADJOINTABLE OPERATORS

The following statement shows that the relationship between one-side algebras is complicated in the Hilbert module case.

Theorem 4.1. *The algebras* $\mathbb{B}_L^f(H_\mathcal{A})$ *and* $\mathbb{B}^f(H_\mathcal{A})$ *consist of adjointable operators, i.e., are subalgebras of the* C^**-algebra* $\mathbb{B}^\star(H_\mathcal{A})$*.*

Moreover, $\mathbb{B}^f(H_\mathcal{A})$ *is an involutive subalgebra, hence, a* C^**-algebra.*

The algebras $\mathbb{B}_R^f(H_\mathcal{A})$*,* $W\mathbb{B}_L^f(H_\mathcal{A})$*,* $W\mathbb{B}_R^f(H_\mathcal{A})$*, and* $W\mathbb{B}^f(H_\mathcal{A})$ *generally contain nonadjointable operators.*

Proof. Any element of $\mathbb{B}_L^{(k)}(H_\mathcal{A})$ is adjointable by Theorem 1.5. This implies the first statement.

Hence, we can consider an adjoint of an element from $\mathbb{B}^{(k)}(H_\mathcal{A})$. Evidently, it will be in $\mathbb{B}^{(k)}(H_\mathcal{A})$. This gives the second statement.

To prove the third statement, consider an operator with the first row $(\gamma_1, \gamma_2, \dots)$ from Example 1.3, and zeros in other places. This operator is not adjointable by Theorem 1.5. On the other hand, it belongs to $\mathbb{B}_C^{(1)}(H_\mathcal{A})$, $W\mathbb{B}_C^{(1)}(H_\mathcal{A})$, and $W\mathbb{B}_L^{(1)}(H_\mathcal{A})$.

Remark 4.2. If we would consider not only pure states, the argument with the example in the proof does not work. Thus, generally we obtain some other algebras if we will remove the word 'pure' in the above definitions. Their relationship with adjointable operators is an open question.

If $(a_1, a_2, \dots)$ is a column of a matrix and $(a_1^*, a_2^*, \dots)$ is the corresponding row of the matrix of the adjoint operator, then the condition $\varphi_d((a_i)^* a_i) = 0$ is the same as $\varphi_d((a_i)^*(a_i^*)^*) = 0$, i.e., two conditions of Definition 3.3 come to each other under taking the adjoint matrix. Hence, the proof of the previous theorem implies

$$(\mathbb{B}_L^{(k)}(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A}))^\star = (\mathbb{B}_C^{(k)}(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A})),$$
$$(W\mathbb{B}_L^{(k)}(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A}))^\star = (W\mathbb{B}_C^{(k)}(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A})).$$

Using the first statement of Theorem 4.1, we arrive to

Theorem 4.3. *One has the following identities*:

$$(\mathbb{B}_L^f(H_\mathcal{A}))^\star = (\mathbb{B}_C^f(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A})),$$
$$(W\mathbb{B}_L^f(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A}))^\star = (W\mathbb{B}_C^f(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A})).$$

5. KUIPER TYPE THEOREMS FOR THE STRONG CASE

Theorem 5.1. *The group* $GL(\mathbb{B}_C^f(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A}))$ *is contractible.*

Proof. As in the scheme in Section 2, if we have a polyhedron P with vertices $A_1, \dots, A_N$, $A_i \in GL(\mathbb{B}_C^{(k)}(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A}))$. Elements of a row of the matrix of an adjointable operator must tend to zero. Thus, we can find $e_{i_0(r)}$, $r = 1, 2, \dots$, with the properties as in Step 1 of Section 2. The homotopy of Step 2 will not increase the number of nonzero elements and remain operators in $\mathbb{B}_C^{(k)}(H_\mathcal{A})$. Since $A'' e_{i_0(r)}$ has no more than k nonzero entries, the block (2.8) will have no more than $k+1$ nonzero entries in each column. Then the matrix of the entire rotation will have this property and the resulting operators will be in $\mathbb{B}_C^{(k(k+1))}(H_\mathcal{A})$. Thus, Step 3 preserves our subspaces. Finally, Steps 4 and 5 keep operators in $\mathbb{B}_C^{(k')}(H_\mathcal{A})$, where $k' = 4(2k(k+1))^4$.

Theorem 5.2. *The group* $GL(\mathbb{B}_C^f(H))$ *is contractible.*

Proof. Take $\mathcal{A} = \mathbb{C}$ in Theorem 5.1 and note that bounded operators in a Hilbert space are always ajointable, i.e.,

$$GL(\mathbb{B}_C^f(H_\mathbb{C}) \cap \mathbb{B}^\star(H_\mathbb{C})) = GL(\mathbb{B}_C^f(H_\mathbb{C})).$$

Theorem 5.3. *The group $GL(\mathbb{B}^f_L(H_\mathcal{A}))$ is contractible.*

Proof. The statement follows immediately from Theorem 5.1 and the first equivalence in Theorem 4.3.

Taking $\mathcal{A} = \mathbb{C}$, we obtain the following statement.

Theorem 5.4. *The group $GL(\mathbb{B}^f_L(H))$ is contractible.*

Theorem 5.5. *The group $GL(\mathbb{B}^f(H_\mathcal{A}))$ is contractible.*

Proof. The proof repeats the proof of Theorem 5.1, because

$$GL(\mathbb{B}^f(H_\mathcal{A})) \to GL(\mathbb{B}^f_C(H_\mathcal{A}) \cap \mathbb{B}^*(H_\mathcal{A})).$$

We only need to observe that the ingredients of the construction of the homotopy (rotations, diagonal scaling and linear decreasing of a corner) lie not only in $GL(\mathbb{B}^f_C(H_\mathcal{A}))$, but in $GL(\mathbb{B}^f(H_\mathcal{A}))$. Once again, the explicit form (2.8) shows this.

6. KUIPER TYPE THEOREMS FOR THE WEAK CASE

Theorem 6.1. *The group $GL(W\mathbb{B}^f_C(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A}))$ is contractible.*

Proof. We repeat the proof of Theorem 5.1. We need only to verify that (2.8) is in $W\mathbb{B}^{(k+1)}_C(H_\mathcal{A})$ under the assumption that the operator A'' is in $W\mathbb{B}^{(k)}_C(H_\mathcal{A})$. Indeed, for any $d \in \mathcal{A}$ and any pure state φ, we have $\varphi_d((a_i)^*a_i) \neq 0$ no more than for k elements (this gives the desired property for the first column) and

$$\varphi_d((a_i b^{-1} a_j^*)^* a_i b^{-1} a_j^*) = \varphi_d(a_j b^{-1} (a_i)^* a_i b^{-1} a_j^*) = \varphi_{d a_j b^{-1}}((a_i)^* a_i) \neq 0$$

no more than for k values of i and fixed j (this gives the property for the j^{th} column of pr_a).

Theorem 6.2. *The group $GL(W\mathbb{B}^f_L(H_\mathcal{A}) \cap \mathbb{B}^\star(H_\mathcal{A}))$ is contractible.*

Proof. This follows immediately from Theorems 6.1 and 4.3.

In the same way as in Theorem 6.1, one can prove the following statement.

Theorem 6.3. *The group $GL(W\mathbb{B}^f(H_\mathcal{A})) \cap \mathbb{B}^\star(H_\mathcal{A})$ is contractible.*

Proof. Once again we need additionally to verify only that (2.8) defines an element from $W\mathbb{B}^{(k+1)}_L(H_\mathcal{A})$ under the assumption that the operator A'' is in $W\mathbb{B}^{(k)}(H_\mathcal{A})$ (only $W\mathbb{B}^{(k)}_L(H_\mathcal{A})$ is insufficient in this place). For this purpose we observe that, for any $d \in \mathcal{A}$,

$$\varphi_d(b^{-1}(a_i)^* a_i b^{-1}) = \varphi_{db^{-1}}((a_i)^* a_i), \quad \varphi_d(a_j b^{-1} a_i^* a_i b^{-1} (a_j)^*) = \varphi_{d a_j b^{-1}}((a_i)^* a_i),$$

then use $A'' \in W\mathbb{B}^{(k)}_C(H_\mathcal{A})$, and argue similarly to the proof of Theorem 6.1.

For operators without adjoint, we can prove only the following.

Theorem 6.4. *The group $GL(W\mathbb{B}^f(H_\mathcal{A}))$ is contractible inside $GL(\mathbb{B}(H_\mathcal{A}))$.*

Remark 6.5. The formulation means that the contracting homotopy can be valued not only in $GL(W\mathbb{B}^f(H_\mathcal{A}))$, but in the wider space $GL(\mathbb{B}(H_\mathcal{A}))$ (see the second paragraph of Section 2).

This theorem makes sense because the question about the contractibility of $GL(\mathbb{B}(H_\mathcal{A}))$ for general $\mathcal{A}$ is open, as was explained above.

Proof. Consider an operator A in $W\mathbb{B}^{(k)}(H_{\mathcal{A}})$. The next argument is based on the following two observations.

Observation 1. Suppose, $\mathbf{a}_i = (a_{i1}, a_{i2}, \dots)$ is a row of the matrix of A and $\varepsilon > 0$ is arbitrary small. Then, for sufficiently large n and any element of $H_{\mathcal{A}}$ of the form

$$x = \Big(0, \dots, 0, 0, \underbrace{\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}}_{n \text{ positions}}, 0, 0, \dots\Big) \tag{6.1}$$

we have $\|\mathbf{a}_i(x)\| < \varepsilon$, where we consider $\mathbf{a}_i$ as an $\mathcal{A}$-functional $H_{\mathcal{A}} \to \mathcal{A}$.

Indeed, take $n > \left(\frac{2k\|A\|}{\varepsilon}\right)^2$. Then by Lemma 1.6 and Remark 3.4, we have

$$\begin{aligned} \|\mathbf{a}_i(x)\| &\leqslant 2 \sup_{\varphi \text{ is a pure state}} |\varphi(\mathbf{a}_i(x))| = 2\frac{1}{\sqrt{n}} \sup_{\varphi \text{ is a pure state}} \sum_j |\varphi(a_{ij})| \\ &\leqslant 2\frac{k}{\sqrt{n}} \sup_k \|a_{ik}\| \leqslant \frac{2k\|A\|}{\sqrt{n}} < \varepsilon. \end{aligned}$$

Observation 2. Consider an increasing sequence n_i and corresponding projections 'on diagonal'

$$p_i = \begin{pmatrix} \frac{1}{n_i} & \frac{1}{n_i} & \cdots & \frac{1}{n_i} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n_i} & \frac{1}{n_i} & \cdots & \frac{1}{n_i} \end{pmatrix}.$$

According to Proposition 2.3 from [6], we have $\oplus p_i \in \mathbb{B}^f(H)$. Now we add some one-dimensional zero projections to the direct sum:

$$p = 0 \oplus p_{i_1} \oplus 0 \oplus p_{i_2} \oplus \cdots .$$

Evidently, we still have $p \in \mathbb{B}^f(H)$. Now consider the elements e'_s to be base elements, corresponding to zero summands above, and elements f_s of the form (6.1). Thus, p_{i_s} is the projection of n_{i_s}-dimensional space on its diagonal, generated by f_s. More formally:

$$e'_s := e_t, \quad \text{where} \quad t = s + n_{i_1} + \cdots + n_{i_{s-1}},$$

$f_s = (f_s^1, f_s^2, \dots)$, where

$$f_s^t := \begin{cases} 0, & \text{if } t \leqslant s + n_{i_1} + \cdots + n_{i_{s-1}} \text{ or } t \geqslant s + 1 + n_{i_1} + \cdots + n_{i_s}, \\ \frac{1}{\sqrt{n_{i_s}}}, & \text{if } s + n_{i_1} + \cdots + n_{i_{s-1}} < t < s + 1 + n_{i_1} + \cdots + n_{i_s}. \end{cases}$$

Then the rotation, which takes each f_s to e'_s, e'_s to $-f_s$, and is constant on the orthogonal complement to all f_s and e'_s, is an element of $\mathbb{B}^\star(H_{\mathcal{A}})$.

More explicitly, the matrix of the restriction of the rotation $R(\alpha)$ onto the span of all e'_s and f_s is composed of blocks of the form:

$$\begin{pmatrix} \cos(\alpha) & \frac{\sin(\alpha)}{\sqrt{n_i}} & \frac{\sin(\alpha)}{\sqrt{n_i}} & \cdots & \frac{\sin(\alpha)}{\sqrt{n_i}} \\ -\frac{\sin(\alpha)}{\sqrt{n_i}} & \frac{\cos(\alpha)}{n_i} & \frac{\cos(\alpha)}{n_i} & \cdots & \frac{\cos(\alpha)}{n_i} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{\sin(\alpha)}{\sqrt{n_i}} & \frac{\cos(\alpha)}{n_i} & \frac{\cos(\alpha)}{n_i} & \cdots & \frac{\cos(\alpha)}{n_i} \end{pmatrix}, \qquad \alpha \in [0, \pi/2]. \tag{6.2}$$

We need this explicit form for a negative observation in Remark 6.6 below.

Now we reduce the situation to the scheme of Section 2.

We can suppose (after an arbitrary small perturbation, if necessary) that all vertices A_i (hence, all elements) of P are in $W\mathbb{B}^{(k)}(H_\mathcal{A})$. Now, applying Observation 1, and taking a common estimation for all $\|A\|$ (from a compact set), we will find inductively n_{i_s} (in the notation of Observation 2) to keep the following estimations:

$$\|\langle Af_s, e_k\rangle\| < \frac{\varepsilon}{k2^k}, \qquad A \in P, \quad k = 1, 2, \dots \tag{6.3}$$

After that, we perform a rotation as in Observation 2. Then after performing the homotopy $A\circ R(\alpha)$, $\alpha \in [0, \pi/2]$, we have for $A' = A\,R(\pi/2)$, and the conditions of Step 2 in Section 2 is fulfilled.

Then we argue following the next steps in Section 2 without additional considerations, because we do not need to control the number of nonzero entries. Also, the "additional" homotopy at the beginning does not depend on A, hence, it is continuous.

Remark 6.6. It is very unlikely that this (Kuiper) approach will allow to prove the contractibility of $GL(W\mathbb{B}^f(H_\mathcal{A}))$ in itself, because the first rows of blocks (6.2) form an operator, which is not in $W\mathbb{B}^f(H_\mathcal{A})$ (cf. [6, Proposition 2.1]).

One can also remark that, at the next step, we also have no control on (2.8) because $(a_1, \dots, a_u)$ is now a mixture of $n_i \to \infty$ columns.

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