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Asymptotic Behavior of a Network of Oscillators Coupled to Thermostats of Finite Energy

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Abstract. We study the asymptotic behavior of a finite network of oscillators (harmonic or anharmonic) coupled to a number of deterministic Lagrangian thermostats of *finite* energy. In particular, we consider a chain of oscillators interacting with two thermostats situated at the boundary of the chain. Under appropriate assumptions, we prove that the vector (p, q) of moments and coordinates of the oscillators in the network satisfies $(p, q)(t) \to (0, q_c)$ as $t \to \infty$, where q_c is a critical point of some effective potential, so that the oscillators just stop. Moreover, we argue that the energy transport in the system stops as well without reaching thermal equilibrium. This result is in contrast to the situation when the energies of the thermostats are *infinite*, studied for a similar system in [14] and subsequent works, where the convergence to a nontrivial limiting regime was established.

The proof is based on a method developed in [22], where it was observed that the thermostats produce some effective dissipation despite the Lagrangian nature of the system.

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1. INTRODUCTION

We study the asymptotic behavior of a finite network of oscillators (harmonic or anharmonic), where some of the oscillators are coupled to thermostats. Our principal example is a chain of $N \geqslant 1$ oscillators interacting with two thermostats situated at the boundary of the chain. One of the first rigorous results in this direction was obtained in [14], where the just mentioned case of chain was considered. The authors modelled the thermostats by the linear wave equations and the interaction between the thermostats and the chain was chosen to be linear as well. The initial conditions of the thermostats were assumed to be random and distributed according to the Gibbs measures of given temperatures $\mathcal{T}_L$ and $\mathcal{T}_R$. Under appropriate assumptions, the mixing property was established, stating that the asymptotic behavior of the chain is governed by a stationary measure μ, which is unique and absolutely continuous with respect to the Lebesgue measure. More precisely, the authors proved the total variation convergence of measures

$$\mathcal{D}(p, q)(t) \to \mu \quad \text{as} \quad t \to \infty, \tag{1.1}$$

where $\mathcal{D}\xi$ denotes the distribution of a random variable ξ and $(p, q)(t)$ is the vector of coordinates and moments of the oscillators in the chain at the time t. Moreover, in [15], the authors proved that the stationary measure μ has positive entropy production. These results were subsequently developed in [20, 5–9].

Because of the Gibbs distribution of initial conditions, the initial energies $\mathcal{E}_L(0), \mathcal{E}_R(0)$ of the thermostats in the model above are almost surely infinite. In the present work, we address the following question: What happens if the initial conditions are chosen in such a way that the energies $\mathcal{E}_L(0), \mathcal{E}_R(0)$ are finite (but probably, very large)? Namely, we consider a system similar to that investigated in [14], but assume the initial conditions to be deterministic and the total energy of the system to be finite. Under appropriate assumptions, we show that, as $t \to \pm\infty$, the oscillators in the chain just stop. That is, we prove the convergence

$$(p, q)(t) \to (0, q_c^{\pm}) \quad \text{and} \quad \frac{d^i}{dt^i}(p, q)(t) \to 0 \quad \forall i \geqslant 1 \quad \text{as} \quad t \to \pm\infty, \tag{1.2}$$

where the constants $q_c^\pm$ are critical points of some effective potential. The critical points $q_c^\pm$ may depend on the initial conditions, however, the effective potential is independent of them, so that the set of all critical points $\{q_c^\pm\}$ does not depend on the initial conditions.

We next study the asymptotic behavior of the energies $\mathcal{E}_L$ and $\mathcal{E}_R$ of the thermostats as $t \to \pm\infty$. For the case in which the chain consists of a unique oscillator ($N = 1$), we prove that the energies $\mathcal{E}_L$ and $\mathcal{E}_R$ converge to some constants $\mathcal{E}_L^\pm$ and $\mathcal{E}_R^\pm$ respectively, so that, the energy transport between the thermostats stops as well. Then, one could expect that $\mathcal{E}_L^\pm = \mathcal{E}_R^\pm$, so that the system reaches a kind of thermal equilibrium. We show, however, that this situation is not generic, but takes place only for a special set of initial conditions, of codimension one. Thus, the chain plays a role of an insulator rather than of a conductor. For longer chains, when $N > 1$, we can prove only that the sum $\mathcal{E}_L + \mathcal{E}_R$ and the time derivatives

$$\frac{d^i}{dt^i}\mathcal{E}_L, \frac{d^i}{dt^i}\mathcal{E}_R$$

converge as $t \to \pm\infty$, for any $i \geqslant 1$. We conjecture, however, that the same results as for $N = 1$ hold, probably under stronger assumptions for the function specifying the interaction between the thermostats and the chain, and for the initial conditions of the thermostats.

We use a method developed by Treschev in [22]. The main idea behind is that the motion of the oscillators in the chain cannot create parametric resonances in the thermostats: otherwise, the total energy of the system would not be finite. If the coupling of the thermostats with the chain is "sufficiently strong," this implies a serious restriction for the dynamics of the chain which leads to the convergence (1.2).

The convergence (1.2) generalizes results obtained in [22, 10] and [21], while the asymptotic behavior of the energies $\mathcal{E}_{L,R}$ is studied here for the first time. In [22], under appropriate assumptions, the convergence (1.2) was established for the system of one oscillator ($N = 1$) interacting with one thermostat. In [10], the result of [22] was generalized to an arbitrary network of N oscillators (not necessarily forming a chain) interacting with one thermostat, under the assumption that the oscillators are harmonic. In [21], the assumption of harmonicity was removed but instead it was assumed that the number M of the thermostats interacting with the system of oscillators is not less than the number of oscillators, that is,

$$M \geqslant N. \tag{1.3}$$

To establish the convergence (1.2), in the present paper we significantly relax condition (1.3) taking into account geometry of the network.[1] Namely, we prove (1.2) under the assumption that the oscillators interacting with the thermostats "control" the other oscillators from the network in an appropriate sense (see assumption A5 in Section 2.2). For example, our results apply to a chain of oscillators interacting through the boundary with at least one thermostat, as well as to a tree of oscillators, where the thermostats are coupled to the oscillators in the leafs of the tree. See Fig.1 for more examples. See also Section 2.4 for the comparison of our strategy with that used in [21].

When we were editing our manuscript, the paper [7] was published, where the convergence (1.1) was generalized to the case of networks under exactly the same assumption A5, which is called the controllability assumption in[7].

Let us note that, for systems of the type [14], where the energy is infinite and the coupling with the thermostats provides stochastic perturbation, an analog of assumption (1.3) is well-known. Namely, one assumes that each oscillator from the network is coupled with its own thermostat. This assumption significantly simplifies the investigation of the asymptotic dynamics and of the energy transport. Indeed, under this assumption, the mixing property (1.1) is well understood (see, e.g., [23, 17]) while, concerning energy transport, there is a number of recent developments (see [3, 4, 1, 11–13]). Relaxation of assumption (1.3) in this setting is an important and complicated problem, and paper [14] provides one of the first results in this direction.

Effects similar to the convergence (1.2) are also known in different infinite-dimensional Hamiltonian systems with finite total energy, mostly, in the context of nonlinear Hamiltonian PDEs.

[1] For simplicity, we restrict ourselves to the case when each thermostat is allowed to interact with a unique oscillator. In contrast, in [21], each thermostat is allowed to interact with several oscillators.

See the survey [18] and references therein. In particular, using a completely different method, in a similar but different setting in [19], it was proven that a single oscillator interacting with the thermostat enjoys the convergence (1.2).

The structure of the paper is as follows. In Section 2, we describe the model, state our main results and give an outline of their proof. In Section 3, we establish some technical lemmas in the spirit of [22, 10, 21], playing a central role in the sequel. In Section 4, we prove our main results. In Section 5, we study the asymptotic behavior of the energies of two thermostats which are coupled to a unique oscillator.

2. SETUP AND RESULTS

2.1. Setup

Let us take a finite undirected graph $\Gamma = (\mathcal{G}, \mathfrak{E})$, where $\mathcal{G}$ and $\mathfrak{E}$ stand for the sets of vertices and edges of Γ, correspondingly. We consider a system of one-dimensional oscillators enumerated by the vertices $j \in \mathcal{G}$ of the graph Γ. If the vertices $i, j \in \mathcal{G}$ are adjacent (we write $i \sim j$ or $(i,j) \in \mathfrak{E}$), we couple the corresponding oscillators by an interaction potential V_{ij}. The Lagrangian of the system has the form

$$\mathcal{L}^O(q,\dot q) = \sum_{j\in\mathcal{G}} \Big(\frac{\dot q_j^2}{2} - U_j(q_j)\Big) - \frac{1}{2}\sum_{i,j\in\mathcal{G}:\, i\sim j} V_{ij}(q_i - q_j),$$

where the dot stands for the derivative in time, $(q,\dot q) = (q_j, \dot q_j)_{j\in\mathcal{G}} \in \mathbb{R}^{2|\mathcal{G}|}$, U_j, V_{ij} are smooth real functions and $V_{ij}(q_i - q_j) = V_{ji}(q_j - q_i)$. We fix a set

$$\Lambda \subset \mathcal{G}$$

and couple each oscillator from Λ with its own thermostat. Each thermostat is modelled as a continuum collection of independent harmonic oscillators parametrized by their internal frequency ν. The thermostats are given by the Lagrangians

$$\mathcal{L}_m^T(\xi_m, \dot\xi_m) = \frac{1}{2}\int_{-\infty}^{\infty} \dot\xi_m^2(\nu) - \nu^2 \xi_m^2(\nu)\, d\nu, \qquad m \in \Lambda, \tag{2.1}$$

where $\xi_m(\nu,t), \dot\xi_m(\nu,t) \in \mathbb{R}$ stand for the coordinate and velocity of the oscillator which has the internal frequency ν.

Remark 2.1. It is more natural to assume that the Lagrangians have the form

$$\widetilde{\mathcal{L}}_m^T(\zeta_m, \dot\zeta_m) = \frac{1}{2}\int_{-\infty}^{\infty} \rho_m(\nu)\big(\dot\zeta_m^2(\nu) - \nu^2\zeta_m^2(\nu)\big)\, d\nu,$$

where the physical meaning of the function $\rho_m(\nu) > 0$ is the density of oscillators with the internal frequency ν. We consider the simplified Lagrangian (2.1), since it can be obtained from the Lagrangian $\widetilde{\mathcal{L}}_m^T$ by the transformation $\xi_m = \sqrt{\rho_m}\zeta_m$, $\dot\xi_m = \sqrt{\rho_m}\dot\zeta_m$.

The coupling between the system of oscillators and the thermostats is linear and given by the potentials

$$V_m^{\rm int}(q_m, \xi_m) = -q_m \int_{-\infty}^{\infty} \kappa_m(\nu)\xi_m(\nu)\, d\nu, \qquad m \in \Lambda, \tag{2.2}$$

where κ_m are real continuous functions. The Lagrangian of the total system has the form

$$\mathcal{L}(q,\xi,\dot q,\dot\xi) = \mathcal{L}^O(q,\dot q) + \sum_{m\in\Lambda} \mathcal{L}_m^T(\xi_m,\dot\xi_m) - \sum_{m\in\Lambda} V_m^{\rm int}(q_m,\xi_m),$$

where $(\xi, \dot{\xi}) = (\xi_m, \dot{\xi}_m)_{m\in\Lambda}$. Set $\delta_{j\Lambda} = 1$ if $j \in \Lambda$ and $\delta_{j\Lambda} = 0$, otherwise. Then the equations of motion take the form

$$\ddot{q}_j = -U_j'(q_j) + \sum_{i\in\mathcal{G}:\, i\sim j} V_{ij}'(q_i - q_j) + \delta_{j\Lambda} \int_{-\infty}^{\infty} \kappa_j(\nu)\xi_j(\nu)\, d\nu, \tag{2.3}$$

$$\ddot{\xi}_m(\nu) = -\nu^2\xi_m(\nu) + \kappa_m(\nu) q_m, \quad m \in \Lambda, \quad \nu \in \mathbb{R}, \quad j \in \mathcal{G}, \tag{2.4}$$

where $U_j'(x)$, $V_{ij}'(x)$ denote the derivatives of the functions U_j, V_{ij} in x. We fix some initial conditions

$$(q, \xi, \dot{q}, \dot{\xi})(0) = (q_0, \xi_0, \dot{q}_0, \dot{\xi}_0). \tag{2.5}$$

The total energy of the system has the form

$$\begin{aligned} E(q,\xi,\dot{q},\dot{\xi}) = &\sum_{j\in\mathcal{G}} \Big(\frac{\dot{q}_j^2}{2} + U_j(q_j)\Big) + \frac{1}{2}\sum_{i\sim j} V_{ij}(q_i - q_j) + \frac{1}{2}\sum_{m\in\Lambda}\int_{-\infty}^{\infty} \dot{\xi}_m^2 + \nu^2\xi_m^2\, d\nu \\ &- \sum_{m\in\Lambda} q_m \int_{-\infty}^{\infty} \kappa_m \xi_m\, d\nu. \end{aligned} \tag{2.6}$$

Remark 2.2. In the paper [14], the authors considered a system similar to (2.3)–(2.4), where the graph Γ was chosen as a chain $\{1, \ldots, N\}$ of length $N \geqslant 1$, and $\Lambda = \{1, N\}$. The dynamics of the thermostats was given by the wave equations $\partial_{tt}\varphi_m(x,t) = \Delta\varphi_m(x,t) + \alpha_m(x) q_m(t)$, where $x \in \mathbb{R}^d$. If $d = 1$, under the Fourier transform, the wave equations take the form

$$\ddot{\hat{\varphi}}_m(\nu, t) = -\nu^2\hat{\varphi}_m(\nu,t) + \hat{\alpha}_m(\nu) q_m(t),$$

and the total system takes the form (2.3)–(2.4). Note, however, that the effective potential arising in the formula (3.7) of [14] is different from our effective potential (2.8), introduced in the next subsection.

2.2. Assumptions

We impose on the system the following assumptions.

A1. *The potentials U_j, V_{ij} are smooth for every $i, j \in \mathcal{G}$. The second derivatives V_{ij}'' have only isolated zeros.*

Denote by L^n the space of measurable functions $f : \mathbb{R} \mapsto \mathbb{C}$ satisfying $\int_{-\infty}^{\infty} |f(x)|^n\, dx < \infty$.

A2. *The functions κ_m are C^2-smooth and $\kappa_m(\nu) = 0$ if and only if $\nu = 0$. Moreover, κ_m, κ_m'' belong to $L^1 \cap L^2$ and there exists an integer $r \geqslant 0$ such that $\nu^r\kappa$, $\nu^{r/2}\kappa' \in L^2$.*

Denote

$$K_m := \int_{-\infty}^{\infty} \frac{\kappa_m^2}{\nu^2}\, d\nu < \infty, \tag{2.7}$$

where the inequality $K_m < \infty$ follows from Assumption A2. Introduce the *effective potential*

$$V^{\text{eff}}(q) := \sum_{j\in\mathcal{G}} U_j(q_j) + \frac{1}{2}\sum_{i\sim j} V_{ij}(q_i - q_j) - \sum_{m\in\Lambda} \frac{K_m q_m^2}{2}. \tag{2.8}$$

A3. *The effective potential V^{eff} satisfies $\left|V^{\text{eff}}(q)\right| \to \infty$ as $|q| \to \infty$.*

The total energy (2.6) of the system can be written in the form

$$E(q,\xi,\dot{q},\dot{\xi}) = \sum_{j\in\mathcal{G}} \frac{\dot{q}_j^2}{2} + \sum_{m\in\Lambda}\int_{-\infty}^{\infty} \frac{\dot{\xi}_m^2}{2}\, d\nu + V^{\text{eff}}(q) + \sum_{m\in\Lambda}\int_{-\infty}^{\infty} \frac{\nu^2}{2}\Big(\xi_m - \frac{\kappa_m q_m}{\nu^2}\Big)^2 d\nu. \tag{2.9}$$

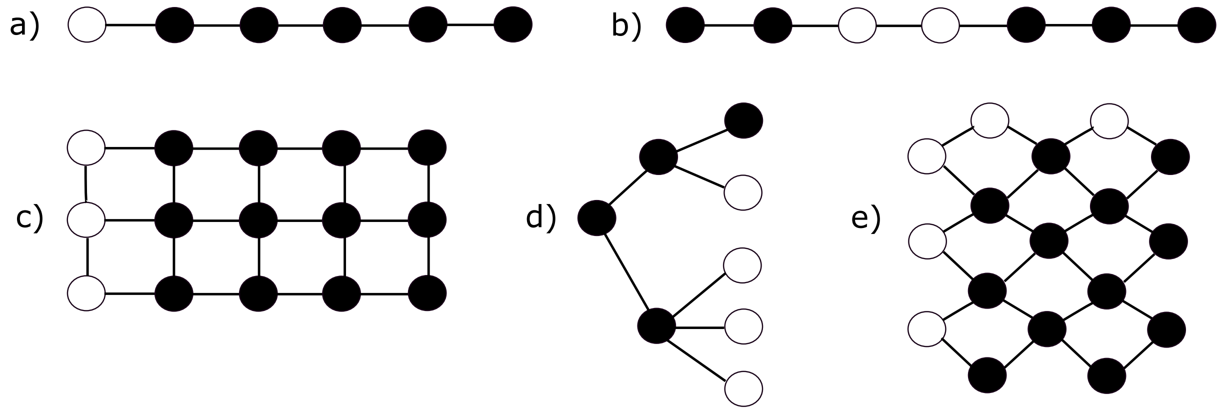

Fig. 1. Networks a)-e) satisfy Assumption A5 provided that the set Λ contains the white oscillators.

Due to Assumption A3, without loss of generality, we can assume $V^{\text{eff}} \geqslant 0$, so that $E \geqslant 0$. Denote by $\mathcal{E}_m$ the energy of the m-th thermostat,

$$\mathcal{E}_m(\xi, \dot{\xi}) := \frac{1}{2} \int_{-\infty}^{\infty} \dot{\xi}_m^2(\nu) + \nu^2 \xi_m^2(\nu)\, d\nu. \tag{2.10}$$

A4. *The initial conditions* (2.5) *satisfy* $\mathcal{E}_m(\xi_0, \dot{\xi}_0) < \infty$ *for any* $m \in \Lambda$.

Assumption A4 together with (2.7) implies that the initial total energy $E(q_0, \xi_0, \dot{q}_0, \dot{\xi}_0)$ is finite. Indeed, by the Cauchy-Bunyakovsky inequality, we have

$$\int_{-\infty}^{\infty} \kappa_m \xi_{0m}\, d\nu \leqslant \Big(\int_{-\infty}^{\infty} \frac{\kappa_m^2}{\nu^2}\, d\nu \Big)^{1/2} \Big(\int_{-\infty}^{\infty} \nu^2 \xi_{0m}^2\, d\nu \Big)^{1/2} < \infty,$$

so that, due to (2.6), we obtain $E(q_0, \xi_0, \dot{q}_0, \dot{\xi}_0) < \infty$.

To formulate the next assumption, we construct a set Λ_Γ by the following inductive procedure. We start by setting $\Lambda_\Gamma := \Lambda$ and consider the set Λ_Γ^1 of vertices $j \in \Lambda_\Gamma$ for which there exists a *unique* vertex $n(j) \in \mathcal{G} \setminus \Lambda_\Gamma$ adjacent to j (in particular, for all other $i \in \mathcal{G}$, $i \sim j$, we have $i \in \Lambda_\Gamma$). We add $n(j)$ to Λ_Γ, so that $\Lambda_\Gamma := \cup_{j \in \Lambda_\Gamma^1} n(j) \cup \Lambda_\Gamma$, and iterate the procedure. We finish when we get $\Lambda_\Gamma^1 = \emptyset$.

A5. *We have* $\Lambda_\Gamma = \mathcal{G}$.

See Fig. 1 for several examples of networks which satisfy Assumption A5.

Remark 2.3. We give Assumption A2 in the form above for simplicity of formulation, while in reality, we use the following weaker assumption.

The functions κ_m are continuous and $\kappa_m(\nu) = 0$ if and only if $\nu = 0$. Moreover,

$$\nu^l \kappa_m \in L^2 \quad \text{and} \quad \kappa_m, \nu^l \kappa_m^2 \in \hat{L}^1, \quad \text{for any} \quad -1 \leqslant l \leqslant r, \tag{2.11}$$

where $\hat{L}^1 = \mathcal{F}(L^1)$ stands for the Fourier transform of the space L^1. It is straightforward to check that Assumption A2 implies relation (2.11). To do this, one should use the fact that a C^2-smooth function g satisfying the inclusion $g, g'' \in L^1$, satisfies $\hat{g} \in L^1$, or, equivalently, $g \in \hat{L}^1$, where $\hat{g}$ is the Fourier transform of the function g. Indeed, this follows from the relations $\big|\hat{g}(\lambda)\big| \leqslant \|g\|_{L^1}$ and $\big|\lambda^2 \hat{g}(\lambda)\big| \leqslant \|g''\|_{L^1}$, which hold for any λ.

2.3. Main Results

Our main goal is to study the large-time asymptotic behavior of the system of oscillators. However, since the total system has infinite dimension, even its well-posedness is not immediate.

Theorem 2.4. ([21]) *Assume A1–A4. Then system (2.3)–(2.5) has a unique solution* $q(t), \xi(\nu, t)$ *and this solution is defined for all* $t \in \mathbb{R}$. *The energy computed on this solution is finite and does not depend on time,* $E(q, \xi, \dot{q}, \dot{\xi})(t) \equiv E(q_0, \xi_0, \dot{q}_0, \dot{\xi}_0) < \infty$. *The functions* $q_k(t)$, $\mathcal{E}_j(q, \xi, \dot{q}, \dot{\xi})(t)$ *are uniformly bounded and the functions* $q_k(t)$ *are uniformly Lipschitz.*

Theorem 2.4 is a particular case of a theorem from [21, Th. 3.1]; see also [22, Th. 1] for an earlier similar result. The uniform Lipschitz property of the functions q_k is not stated in the cited works, but it follows immediately from the uniform boundedness of the derivatives $\dot{q}_k$ which takes place because the energy (2.9) is constant.

Next we state our main results. Let $\mathcal{N}^{\text{eff}}$ be a set of critical points of the effective potential,

$$\mathcal{N}^{\text{eff}} = \{q \in \mathbb{R}^N : \partial_{q_j} V^{\text{eff}}(q) = 0 \ \forall j \in \mathcal{G}\}.$$

The set $\mathcal{N}^{\text{eff}}$ is closed. Denote by $\text{dist}(q, \mathcal{N}^{\text{eff}})$ the Euclidean distance in $\mathbb{R}^{|\mathcal{G}|}$ from the point q to the set $\mathcal{N}^{\text{eff}}$. Recall that the integer r is defined in Assumption A2.

Theorem 2.5. *Under Assumptions A1–A5, the solution* $q(t), \xi(\nu, t)$ *of system* (2.3) − −(2.5) *satisfies*

$$\text{dist}\left(q(t), \mathcal{N}^{\text{eff}}_{\pm}\right) \to 0 \quad \text{as} \quad t \to \pm\infty, \tag{2.12}$$

where $\mathcal{N}^{\text{eff}}_{+}, \mathcal{N}^{\text{eff}}_{-}$ *are some connected components of the set* $\mathcal{N}^{\text{eff}}$. *Moreover, the solution* $q(t)$ *is* C^{r+3}*-smooth and for its time derivatives* $q^{(l)}$, $1 \leqslant l \leqslant r + 3$, *we have*

$$q^{(l)}(t) \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{2.13}$$

The proof of Theorem 2.5 is given in Section 4.1. The choice of the connected components $\mathcal{N}^{\text{eff}}_{+}, \mathcal{N}^{\text{eff}}_{-} \subset \mathcal{N}^{\text{eff}}$ may depend on the initial conditions (2.5). Let us now additionally assume

A6. *The effective potential* V^{eff} *has only isolated critical points.*

Then any connected component of the set $\mathcal{N}^{\text{eff}}$ is a singleton. Consequently, Theorem 2.5 implies the convergence

$$q(t) \to q_c^{\pm} \quad \text{as} \quad t \to \pm\infty, \tag{2.14}$$

where the critical points $q_c^{+}, q_c^{-} \in \mathcal{N}^{\text{eff}}$ may depend on the initial conditions. If $q_c^{-} \neq q_c^{+}$, we observe a transition from one equilibrium at $t = -\infty$ to another one at $t = +\infty$.

As a corollary of Theorem 2.5, we obtain the following result specifying the limiting behavior as $t \to \pm\infty$ of the energies $\mathcal{E}_m$ of the thermostats.

Corollary 2.6. *Under Assumptions* $A1 - -A6$, *the energies of the thermostats* $\mathcal{E}_m(t)$, $m \in \Lambda$, *computed on the solution of equation* (2.3) − −(2.5) *satisfy*

$$\sum_{m \in \Lambda} \mathcal{E}_m(t) \to E(q_0, \xi_0, \dot{q}_0, \dot{\xi}_0) - V^{\text{eff}}(q_c^{\pm}) + \sum_{m \in \Lambda} \frac{K_m (q_c^{\pm})^2}{2} \quad \text{as} \quad t \to \pm\infty, \tag{2.15}$$

where $q_c^{\pm}$ *are the critical points of the effective potential given by* (2.14). *Moreover, the functions* $\mathcal{E}_m$ *are* C^{r+1}*-smooth and their derivatives satisfy*

$$\frac{d^l}{dt^l} \mathcal{E}_m(t) \to 0 \quad \text{as} \quad t \to \pm\infty \quad \forall 1 \leqslant l \leqslant r + 1. \tag{2.16}$$

Corollary 2.6 is established in Section 4.2 and it provokes the following questions. Is it true that the energies $\mathcal{E}_m(t)$ of the thermostats converge as $t \to \pm\infty$? If this is the case, let us assume that the graph $\mathcal{G}$ forms a chain $\{1, \ldots, N\}$ and only the first and the last oscillators are coupled to the thermostats, that is $\Lambda = \{1, N\}$. Assume also that the functions U_j, V_{ij}, κ_j are independent of i, j. Is it then true that the limits of the energies of the thermostats are equal, so that a kind of thermal equilibrium is achieved? We are able to give an answer to these questions only in a particular case. Namely, in Section 5, we consider a system of one oscillator interacting with two thermostats and prove that the energies of the thermostats converge. We show, however, that the set of initial conditions for which the limits coincide is of codimension one, so this situation is not generic.

Although we do not have a proof, we believe that the same situation takes place for longer chains as well, at least, in the case when the functions κ_m together with the initial conditions $\xi_0, \dot{\xi}_0$ decay sufficiently fast at infinity. In the case when the chain consists of a unique oscillator, we are able to establish the convergence of the energies because of the existence of a special change of variables (see (5.2)). The latter transforms our system to a system consisting of one oscillator interacting with one thermostat, and of another isolated thermostat, so that the convergence of energies immediately follows from Corollary 2.6. In the general case, however, an analogous transformation does not exist and the main ingredient we lack to prove the convergence of the energies $\mathcal{E}_m$ is an appropriate estimate for the rate of convergence (2.12).

2.4. Outline of the Proof of Theorem 2.5

The proof of the theorem relies on a method invented by Treschev in [22] and subsequently developed in [10, 21]. To explain its main idea, let us study equation (2.4), where the function q_m is viewed as an external force. Assume that $q_m(t) = \sin(\lambda t)$ or $q_m(t) = \cos(\lambda t)$ for some $\lambda \neq 0$. Then in equation (2.4) with $\nu = \lambda$, the parametric resonance will occur, since $\kappa_m(\lambda) \neq 0$ by Assumption A2. However, this contradicts energy conservation, so that such a situation is impossible. This suggests that the function q_m cannot have components oscillating with any frequency $\lambda \neq 0$.

To make these considerations rigorous, following [22], we introduce the notion of singular support $\operatorname{sing\,supp}\hat{x}$ of a distribution $\hat{x}$, where $\hat{x} = \mathcal{F}(x)$ is the Fourier transform of a bounded uniformly continuous real function x. We say that $\lambda \in \operatorname{sing\,supp}\hat{x}$ if, roughly speaking, the function x has a component oscillating with the frequency λ; see Definition 3.3. In particular, if $\operatorname{sing\,supp}\hat{x} = \emptyset$, we have $x(t) \to 0$ as $t \to \pm\infty$. The conservation of energy together with the relation $\kappa_m(\nu) \neq 0$ for $\nu \neq 0$ implies that $\operatorname{sing\,supp}\hat{q}_m \subset \{0\}$ for $m \in \Lambda$, and it follows that $\operatorname{sing\,supp}\big(\lambda^l \hat{q}_m(\lambda)\big) = \emptyset$ for any $l \geqslant 1$, since $\lambda^l \hat{q}_m(\lambda)$ vanishes at $\lambda = 0$. Since $\mathcal{F}\big(q_m^{(l)}\big)(\lambda) = (i\lambda)^l \hat{q}_m(\lambda)$, we obtain

$$\operatorname{sing\,supp}\mathcal{F}\big(q_m^{(l)}\big) = \emptyset,$$

so that

$$q_m^{(l)}(t) \to 0 \quad \text{as} \quad t \to \pm\infty, \quad \text{for any} \quad l \geqslant 1 \tag{2.17}$$

and $m \in \Lambda$. We then use the Duhamel formula to express the function $\varphi_m(t) := \int_{-\infty}^{\infty} \kappa_m(\nu)\xi_m(\nu, t)\, d\nu$ from equation (2.4) through the function $q_m(t)$. Using that $\operatorname{sing\,supp}\hat{q}_m \subset \{0\}$, we prove that

$$\varphi_m(t) = K_m q_m(t) + \vartheta_m(t), \tag{2.18}$$

where the constant K_m is given by (2.7) and the function $\vartheta_m(t) \to 0$ as $t \to \pm\infty$.

Assume first that $\Lambda = \mathcal{G}$, so that each oscillator from the network is coupled with a thermostat and, consequently, (2.17) holds for all $m \in \mathcal{G}$. Then, inserting (2.18) into equation (2.3), letting $t \to \pm\infty$ and using (2.17), we obtain the desired convergence (2.12); a similar argument was used in the papers [22] and [21]. In difference with these works, we do not have the relation $\Lambda = \mathcal{G}$, so that an additional argument is needed to establish (2.17) for $m \notin \Lambda$. By construction of the set Λ_Γ, due to Assumption A5, there exists a $j \in \Lambda$ for which there is exactly one $n \in \mathcal{G} \setminus \Lambda$ adjacent to j. In particular, for any $i \in \mathcal{G}$ adjacent to j, $i \neq n$, we have $i \in \Lambda$, so for such an i, the convergence (2.17) takes place. Then we differentiate in time both sides of the j-th equation from (2.3) and find that all the terms of the resulting equation vanish as $t \to \pm\infty$, except the term $V_{nj}''(q_n - q_j)\dot{q}_n$. Thus, the latter term vanishes as well and we prove that this implies the convergence $\dot{q}_n(t) \to 0$ as $t \to \pm\infty$. Using the uniform continuity of $\dot{q}_n$, we then establish the convergence (2.17) for $m = n$. Then we replace the set Λ by $\Lambda \cup \{n\}$ and repeat the procedure. Finally, we obtain (2.17) for all $m \in \Lambda_\Gamma$, that is, for all $m \in \mathcal{G}$ since, due to Assumption A5, $\Lambda_\Gamma = \mathcal{G}$. Arguing as above, we conclude that this implies the convergence (2.12).

To follow the strategy, outlined above, we need to develop some technical tools used in the papers [22, 10, 21]. This is done in the next section.

3. PRELIMINARY RESULTS

In this section, we develop the method used in the papers [22, 10, 21] and obtain technical results playing a key role in the proof of Theorem 2.5.

3.1. Functional Spaces

We denote by $\mathcal{F}$ the Fourier transform and by $\mathcal{F}^{-1}$ its inverse. We agree to the convention

$$\hat\psi(\lambda) = \mathcal{F}(\psi)(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t}\psi(t)\,dt \quad \text{and} \quad \mathcal{F}^{-1}(\hat\psi)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda t}\hat\psi(\lambda)\,d\lambda, \tag{3.1}$$

for $\psi \in L^1$. We also denote the Fourier transform by a "hat." Let us recall that, for any $\varphi, \psi \in L^1$, we have

$$\mathcal{F}(\varphi\psi) = \frac{1}{2\pi}\hat\varphi * \hat\psi \quad \text{and} \quad \mathcal{F}^{-1}(\hat\varphi\hat\psi) = \varphi * \psi,$$

where $*$ stands for the convolution.

We denote by $\mathcal{C}_b$ the space of *uniformly* continuous bounded functions $\psi : \mathbb{R} \mapsto \mathbb{C}$. This is a Banach space with the standard norm $\|\psi\|_\infty = \sup_{t\in\mathbb{R}} |\psi(t)|$. Let $\mathcal{C}_0^+, \mathcal{C}_0^-$ be subspaces of $\mathcal{C}_b$ consisting of functions $\psi \in \mathcal{C}_b$ satisfying $\psi(t) \to 0$ as $t \to \infty$ or $t \to -\infty$, correspondingly. Set $\mathcal{C}_0 := \mathcal{C}_0^+ \cap \mathcal{C}_0^-$, so that, for $\psi \in \mathcal{C}_0$, we have $\psi(t) \to 0$ as $t \to \pm\infty$. We will also use the Fourier transforms $\hat{\mathcal{C}}_b, \hat{\mathcal{C}}_0^\pm$, and $\hat{\mathcal{C}}_0$ of the spaces $\mathcal{C}_b, \mathcal{C}_0^\pm$ and $\mathcal{C}_0$, understood in the sense of distributions.

Let $\mathcal{M}$ be the space of complex Radon measures of bounded variation on $\mathbb{R}$. This is a Banach space with the norm

$$\|\mu\|_{\mathcal{M}} := \sup_{\psi\in\mathcal{C}_b:\, \|\psi\|_\infty=1} \langle \mu, \psi\rangle, \quad \text{where} \quad \langle \mu, \psi\rangle := \int_{-\infty}^{\infty} \psi(\tau)\,\mu(d\tau).$$

The Fourier transform $\hat{\mathcal{M}}$ of the space $\mathcal{M}$ is defined by

$$\hat\mu(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t}\,\mu(dt), \tag{3.2}$$

and it is known that $\hat{\mathcal{M}} \subset \mathcal{C}_b$ (see, e.g., [2]). In particular, we have $\hat L^1 := \mathcal{F}(L^1) \subset \hat{\mathcal{M}}$, since any measure $\mu \in \mathcal{M}$ which is absolute continuous with respect to the Lebesgue measure has the form $\mu(dt) = \tilde\mu(t)\,dt$ with $\tilde\mu \in L^1$. By $\mathcal{F}^{-1}(\hat\mu)$, we write the inverse Fourier transform of $\hat\mu \in \hat{\mathcal{M}}$, which is defined as the action of the operator $\mathcal{F}^{-1}$ from (3.1) on the space $\mathcal{C}_b$ and is understood in the sense of distributions. Thus, $\mathcal{F}^{-1}(\hat\mu)$ is a distribution $\tilde\mu$ which can be viewed as a density of the measure μ, so that, informally, $\mu(dt) = \tilde\mu(t)\,dt$. We identify μ with $\tilde\mu$ and, abusing notation sometimes, we write $\mu(t)\,dt$ instead of $\mu(dt)$. For $\hat\psi \in \hat{\mathcal{C}}_b$ and $\hat\mu \in \hat{\mathcal{M}}$, we define the pairing

$$\langle \hat\mu, \hat\psi\rangle := 2\pi\langle \mu, \bar\psi\rangle, \tag{3.3}$$

where $\psi = \mathcal{F}^{-1}(\hat\psi)$ and $\mu = \mathcal{F}^{-1}(\hat\mu)$. In particular, when $\mu(dt) = \widetilde{\mu}(t)\,dt$ and the functions $\widetilde{\mu}, \psi$ belong to the Schwartz class, we have[2]

$$\langle \hat\mu, \hat\psi\rangle = \int \hat\mu(\nu)\overline{\hat\psi(\nu)}\,d\nu = 2\pi \int \mu(\nu)\overline{\psi(\nu)}\,d\nu.$$

---

[2]Usually one defines the Fourier transform of measures via (3.2) with the exponent $e^{-i\lambda t}$ replaced by $e^{i\lambda t}$. The sign "-" in the definition is more convenient for us since we want to have the same formula for the Fourier transform of spaces $\mathcal{M}$ and $\mathcal{C}_b$. The price we pay is the conjugation arising in the formula (3.3).

Lemma 3.1. *Let $\hat{\mu}, \hat{\mu}_1, \hat{\mu}_2 \in \hat{\mathcal{M}}$, $\hat{\psi} \in \hat{\mathcal{C}}_b$ and $\hat{\phi} \in \hat{\mathcal{C}}_0$. Then $\hat{\mu}_1\hat{\mu}_2 \in \hat{\mathcal{M}}$, $\hat{\mu}\hat{\psi} \in \hat{\mathcal{C}}_b$ and $\hat{\mu}\hat{\phi} \in \hat{\mathcal{C}}_0$.*

The proof of the lemma can be obtained via the Fourier transform in a standard way. We do not present it but address the reader to [22] (Section 8.1 and Lemma 8.2). In this connection (and for further needs), we note that we have the following relations:

$$\mathcal{F}^{-1}(\hat{\psi}\hat{\mu}) = \psi * \mu = \int_{-\infty}^{\infty} \psi(t-s)\,\mu(ds) \quad \text{and} \quad \mathcal{F}^{-1}(\hat{\mu}_1\hat{\mu}_2) = \mu_1 * \mu_2,$$

where the convolution $\mu_1 * \mu_2 \in \mathcal{M}$ is defined by the formula

$$\langle \mu_1 * \mu_2, \psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(t+s)\,\mu_1(dt)\mu_2(ds), \quad \text{where} \quad \psi \in \mathcal{C}_b.$$

For a more detailed discussion of the spaces $\mathcal{C}_b, \mathcal{C}_0, \mathcal{M}$ and their Fourier transforms, see [16].

Lemma 3.2. *Let $\hat{\mu} \in \hat{\mathcal{M}}$, $a \in \mathbb{R}$ and $\hat{\mu}(a) = 0$. Then, for any $\varepsilon > 0$, there exists a $\hat{\mu}^\varepsilon \in \hat{\mathcal{M}}$ satisfying $\operatorname{supp}\hat{\mu}^\varepsilon \subset (a-\varepsilon, a+\varepsilon)$ and $a \notin \operatorname{supp}(\hat{\mu} - \hat{\mu}^\varepsilon)$, so that $\|\mu^\varepsilon\|_{\mathcal{M}} \to 0$ as $\varepsilon \to 0$, where $\mu^\varepsilon := \mathcal{F}^{-1}(\hat{\mu}^\varepsilon)$.*

Proof. Without loss of generality, we assume that $a = 0$. Let $\hat{\chi}$ be a smooth real function satisfying $\operatorname{supp}\hat{\chi} \subset [-1,1]$ and $\hat{\chi}(\nu) = 1$ for any $|\nu| \leqslant 1/2$. Set

$$\hat{\chi}^\varepsilon(\nu) := \hat{\chi}(\nu/\varepsilon) \quad \text{and} \quad \hat{\mu}^\varepsilon := \hat{\chi}^\varepsilon\hat{\mu}.$$

Then it remains to show that $\|\mu^\varepsilon\|_{\mathcal{M}} \to 0$ as $\varepsilon \to 0$. Since $\mu^\varepsilon = \mu * \chi^\varepsilon$, where $\chi^\varepsilon = \mathcal{F}^{-1}(\hat{\chi}^\varepsilon)$, we have

$$\|\mu^\varepsilon\|_{\mathcal{M}} = \sup_{\varphi\in\mathcal{C}_b:\,\|\varphi\|_\infty=1} \big|\langle \mu * \chi^\varepsilon, \varphi\rangle\big| = \sup_{\varphi\in\mathcal{C}_b:\,\|\varphi\|_\infty=1} \big|\langle \mu, \varphi * \widetilde{\chi}^\varepsilon\rangle\big|,$$

where $\widetilde{\chi}^\varepsilon(t) := \chi^\varepsilon(-t)$. Set $b := (\varphi * \widetilde{\chi}^\varepsilon)(0)$. Since $\hat{\mu}(0) = 0$, we have $\langle \mu, b\rangle = b\hat{\mu}(0) = 0$. Then, denoting by $\mathbb{I}_A$ the indicator function of a set A, we find

$$\begin{aligned}\langle \mu, \varphi * \widetilde{\chi}^\varepsilon\rangle &= \langle \mu, \varphi * \widetilde{\chi}^\varepsilon - b\rangle = \big\langle \mu, \mathbb{I}_{\{|t|\leqslant\varepsilon^{-1/2}\}}\big(\varphi * \widetilde{\chi}^\varepsilon - b\big)\big\rangle + \big\langle \mu, \mathbb{I}_{\{|t|>\varepsilon^{-1/2}\}}\big(\varphi * \widetilde{\chi}^\varepsilon - b\big)\big\rangle \\ &=: I_1^\varepsilon + I_2^\varepsilon.\end{aligned}$$

Let us estimate the term I_1^ε. Using that $\widetilde{\chi}^\varepsilon(t) = \chi^\varepsilon(-t) = \varepsilon\chi(-\varepsilon t)$, we obtain

$$\begin{aligned}|I_1^\varepsilon| &\leqslant \|\mu\|_{\mathcal{M}}\,\big\|\mathbb{I}_{\{|t|\leqslant\varepsilon^{-1/2}\}}\big(\varphi * \widetilde{\chi}^\varepsilon - b\big)\big\|_\infty = \|\mu\|_{\mathcal{M}} \sup_{|t|\leqslant\varepsilon^{-1/2}} \Big|\int_{-\infty}^{\infty} \varphi(y)(\widetilde{\chi}^\varepsilon(t-y) - \widetilde{\chi}^\varepsilon(-y))\,dy\Big| \\ &\leqslant \|\mu\|_{\mathcal{M}} \sup_{|t|\leqslant\varepsilon^{-1/2}} \varepsilon\int_{-\infty}^{\infty} |\chi(\varepsilon(y-t)) - \chi(\varepsilon y)|\,dy = \|\mu\|_{\mathcal{M}} \sup_{|t|\leqslant\varepsilon^{-1/2}} \int_{-\infty}^{\infty} |\chi(y-\varepsilon t) - \chi(y)|\,dy,\end{aligned}$$

so that $I_1^\varepsilon \to 0$ as $\varepsilon \to 0$, uniformly in $\varphi \in \mathcal{C}_b$ satisfying $\|\varphi\|_\infty = 1$. To estimate the term I_2^ε, we recall that, for any $\mu \in \mathcal{M}$, there exist nonnegative $\mu_1, \dots, \mu_4 \in \mathcal{M}$ satisfying $\mu_1 - \mu_2 + i\mu_3 - i\mu_4 = \mu$, see, e.g., [2]. Then,

$$|I_2^\varepsilon| \leqslant \sum_{j=1}^4 \langle \mu_j, \mathbb{I}_{\{|t|>\varepsilon^{-1/2}\}}\rangle \|\varphi * \widetilde{\chi}^\varepsilon - b\|_\infty \leqslant 2\|\varphi * \widetilde{\chi}^\varepsilon\|_\infty \sum_{j=1}^4 \langle \mu_j, \mathbb{I}_{\{|t|>\varepsilon^{-1/2}\}}\rangle.$$

We have

$$\|\varphi * \widetilde{\chi}^\varepsilon\|_\infty \leqslant \|\varphi\|_\infty \|\widetilde{\chi}^\varepsilon\|_{L^1} = \|\chi\|_{L^1}.$$

Since for any $\alpha \in \mathcal{M}$, we have $\langle \alpha, \mathbb{I}_{\{|t|>\varepsilon^{-1/2}\}}\rangle \to 0$ as $\varepsilon \to 0$, we obtain $I_2^\varepsilon \to 0$ as $\varepsilon \to 0$, uniformly in $\varphi \in \mathcal{C}_b$ satisfying $\|\varphi\|_\infty = 1$.

3.2. Singular Support

Here we discuss the notion of *singular support*, first introduced in [22] which is central for our paper.

Definition 3.3. ([22]) Let $\hat{\phi} \in \hat{\mathcal{C}}_b$, $\lambda \in \mathbb{R}$ and $\star \in \{+,-\}$. We say that $\lambda \in \operatorname{sing\,supp}^\star \hat{\phi}$ if, for any interval $I \subset \mathbb{R}$ containing the point λ, there exists a $\hat{\mu} \in \hat{\mathcal{M}}$, $\operatorname{supp} \hat{\mu} \subset I$, such that $\hat{\mu}\hat{\phi} \notin \hat{\mathcal{C}}_0^\star$. We set $\operatorname{sing\,supp} \hat{\phi} := \operatorname{sing\,supp}^+ \hat{\phi} \cup \operatorname{sing\,supp}^- \hat{\phi}$.

For example, if $\hat{\phi}(\lambda) = \delta(\lambda - \lambda_0)$, where $\lambda_0 \in \mathbb{R}$ and δ is the Dirac delta-function, we have $\operatorname{sing\,supp} \hat{\phi} = \{\lambda_0\}$. On the other hand, for $\hat{\varphi} \in \hat{\mathcal{C}}_0$, we have $\operatorname{sing\,supp} \hat{\varphi} = \emptyset$ since, by Lemma 3.1, $\hat{\mu}\hat{\varphi} \in \hat{\mathcal{C}}_0$, for any $\hat{\mu} \in \hat{\mathcal{M}}$.

Lemma 3.4. *Let $\hat{\phi} \in \hat{\mathcal{C}}_b$ and $\hat{\mu} \in \hat{\mathcal{M}}$.*

(1) *Assume that $\hat{\mu}\hat{\phi} \in \hat{\mathcal{C}}_0$. Then* $\operatorname{sing\,supp} \hat{\phi} \subset \{\nu \in \mathbb{R} : \hat{\mu}(\nu) = 0\}$.

(2) *Assume that* $\operatorname{sing\,supp} \hat{\phi} = \emptyset$ *or* $\operatorname{sing\,supp} \hat{\phi} \subset \{a_1, \dots, a_n\}$ *for $a_1, \dots, a_n \in \mathbb{R}$, $n \geqslant 1$, and $\hat{\mu}(a_1) = \cdots = \hat{\mu}(a_n) = 0$. Then $\hat{\mu}\hat{\phi} \in \hat{\mathcal{C}}_0$. Moreover, the equality* $\operatorname{sing\,supp} \hat{\phi} = \emptyset$ *implies $\hat{\phi} \in \hat{\mathcal{C}}_0$.*

Proof. Item 1 is proven in Lemma 8.9 of [22]. If $\operatorname{sing\,supp} \hat{\phi} = \emptyset$, then Lemma 8.6 from [22] implies that $\hat{\mu}\hat{\phi} \in \hat{\mathcal{C}}_0$. Choosing $\hat{\mu} = 1$, we get $\hat{\phi} \in \hat{\mathcal{C}}_0$.

It remains to study the case when $\operatorname{sing\,supp} \hat{\phi} \subset \{a_1, \dots, a_n\}$. We apply Lemma 3.2 to $\hat{\mu}$ at the point a_1 and construct $\hat{\mu}_1^\varepsilon \in \hat{\mathcal{M}}$ satisfying $\|\mu_1^\varepsilon\|_{\mathcal{M}} \to 0$ as $\varepsilon \to 0$, $a_1 \notin \operatorname{supp}(\hat{\mu} - \hat{\mu}_1^\varepsilon)$ and $a_2, \dots, a_n \notin \operatorname{supp} \hat{\mu}_1^\varepsilon$, for ε sufficiently small. Next we apply Lemma 3.2 to $\hat{\mu} - \hat{\mu}_1^\varepsilon$ at the point a_2 and obtain $\hat{\mu}_2^\varepsilon \in \hat{\mathcal{M}}$ satisfying $\|\mu_2^\varepsilon\|_{\mathcal{M}} \to 0$ as $\varepsilon \to 0$, $a_1, a_2 \notin \operatorname{supp}(\hat{\mu} - \hat{\mu}_1^\varepsilon - \hat{\mu}_2^\varepsilon)$ and $a_3, \dots, a_n \notin \operatorname{supp} \hat{\mu}_2^\varepsilon$, for ε sufficiently small. Iterating the procedure, we construct $\hat{\mu}_3^\varepsilon, \dots, \hat{\mu}_n^\varepsilon \in \hat{\mathcal{M}}$ satisfying

$$\|\mu_j^\varepsilon\|_{\mathcal{M}} \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{3.4}$$

and

$$a_1, \dots, a_n \notin \operatorname{supp}(\hat{\mu} - \hat{\mu}_1^\varepsilon - \cdots - \hat{\mu}_n^\varepsilon). \tag{3.5}$$

Lemma 8.6 from [22] states that relation (3.5), together with the assumption $\operatorname{sing\,supp} \hat{\phi} \subset \{a_1, \dots, a_n\}$, ensures that $(\hat{\mu} - \hat{\mu}_1^\varepsilon - \cdots - \hat{\mu}_n^\varepsilon)\hat{\phi} \in \hat{\mathcal{C}}_0$. On the other hand, in view of (3.4), we have $\|\mathcal{F}^{-1}(\hat{\mu}_j^\varepsilon \hat{\phi})\|_\infty \leqslant \|\mu_j^\varepsilon\|_{\mathcal{M}} \|\phi\|_\infty \to 0$ as $\varepsilon \to 0$. This implies the desired inclusion $\hat{\mu}\hat{\phi} \in \hat{\mathcal{C}}_0$.

Lemma 3.5. *Assume that a C^l-smooth, $l \geqslant 1$, real function ϕ satisfies* $\operatorname{sing\,supp} \hat{\phi} \subset \{0\}$ *and its derivatives $\phi^{(k)}$, $0 \leqslant k \leqslant l$, belong to the space $\mathcal{C}_b$. Then $\phi^{(k)} \in \mathcal{C}_0$ for any $1 \leqslant k \leqslant l$.*

In particular, the assumption $\operatorname{sing\,supp} \hat{\phi} \subset \{0\}$ is satisfied if $\phi \in \mathcal{C}_0$, since in this case we have $\operatorname{sing\,supp} \hat{\phi} = \emptyset$.

Proof. Note that $\mathcal{F}\big(\phi^{(k)}\big)(\nu) = (i\nu)^k \hat{\phi}(\nu)$. We first claim that, for any $\hat{\mu} \in \hat{\mathcal{M}}$, with compact support, $(i\nu)^k \hat{\mu} \in \hat{\mathcal{M}}$. Indeed, take a smooth function $\hat{\chi}$ with compact support satisfying $\hat{\chi}(\nu) = 1$ for all $\nu \in \operatorname{supp} \hat{\mu}$. Then $(i\nu)^k \hat{\chi} \in \hat{\mathcal{M}}$, since $(i\nu)^k \hat{\chi}$ is a Schwartz function and Schwartz functions belong to the space $\hat{\mathcal{M}}$. Then, $(i\nu)^k \hat{\mu} = ((i\nu)^k \hat{\chi})\hat{\mu} \in \hat{\mathcal{M}}$, by Lemma 3.1.

Since $(i\nu)^k \hat{\mu}(\nu)$ vanishes at the point $\nu = 0$, Lemma 3.4 (2) implies $(i\nu)^k \hat{\mu}\hat{\phi} \in \hat{\mathcal{C}}_0$. Thus, by the definition of singular support, we find $\operatorname{sing\,supp}\big((i\nu)^k \hat{\phi}\big) = \emptyset$. Then, applying Lemma 3.4(2) once more, we find $(i\nu)^k \hat{\phi} \in \hat{\mathcal{C}}_0$ and consequently, $\phi^{(k)} \in \mathcal{C}_0$.

3.3. Main Lemma

In this section, we establish the following proposition, which plays a key role in the proof of Theorem 2.5. Let $(q(t),\xi(t))$ be the solution of equation (2.3)-(2.5). Set

$$\varphi_m(t) := \int_{-\infty}^{\infty} \kappa_m(\lambda)\xi_m(\lambda,t)\,d\lambda.$$

Recall that the numbers r and K_m are defined in Assumption A2 and (2.7).

Proposition 3.6. *Under Assumptions A1–A4, for any* $m\in\Lambda$,

(1) $\operatorname{sing\,supp}\hat{q}_m \subset \{0\}$;
(2) *the function* φ_m *is* C^{r+1}*-smooth and all its derivatives belong to the space* $\mathcal{C}_b$*; the functions* q_j, $j\in\mathcal{G}$, *are* C^{r+3}*-smooth and their derivatives belong to* $\mathcal{C}_b$ *as well*;
(3) $\varphi_m - K_m q_m = \vartheta_m$, *where* $\vartheta_m\in\mathcal{C}_0$*; the function* ϑ_m *is* C^{r+1}*-smooth and its derivatives belong to the space* $\mathcal{C}_0$.

Proposition 3.6 is a corollary of a result obtained in Theorem 3.2 of [21], which we formulate below, in Theorem 3.7. See also Theorem 2 from [22] and Theorem 2.2 from [10] for similar results. Set $\hat{w}_m(\nu) := \dfrac{2\pi\kappa_m^2(\nu)}{i\nu}$ and $w_m := \mathcal{F}^{-1}(\hat{w}_m)$. By (2.11), $\kappa_m^2/\nu \in \hat{L}^1$, so that $\hat{w}_m\in\hat{L}^1$ and $w_m\in L^1$. Denote

$$w_{\diamond m}^{\pm}(\tau) := \frac{1}{2}\big(\mathbb{I}_{[0,\pm\infty)}(\tau)(w_m(\tau)-w_m(-\tau))\big), \tag{3.6}$$

where $\mathbb{I}_{[0,-\infty)} := -\mathbb{I}_{(-\infty,0]}$. Using the formula (see, e.g., [24]),

$$\mathcal{F}(\mathbb{I}_{[0,\pm\infty)})(\lambda) = \pm\pi\delta(\lambda) + \frac{1}{i}v.p.\frac{1}{\lambda},$$

where $v.p.$ means the principal value, we find

$$\hat{w}_{\diamond m}^{\pm}(\nu) = \mathcal{F}(w_{\diamond m}^{\pm})(\nu) = \pm\frac{1}{4}\big(\hat{w}_m(\nu)-\hat{w}_m(-\nu)\big) + \frac{1}{2\pi i}v.p.\int_{-\infty}^{\infty}\frac{\lambda\hat{w}_m(\lambda)}{\nu^2-\lambda^2}\,d\lambda. \tag{3.7}$$

Since $w_m\in L^1$, we find $w_{\diamond m}^{\pm}\in L^1$ and $\hat{w}_{\diamond m}^{\pm}\in\hat{L}^1$.

Theorem 3.7. ([21]) *Under Assumptions A1–A4 for any* $m\in\Lambda$,

(1) $\hat{w}_m\hat{q}_m \in L^1\subset\hat{\mathcal{C}}_0$;
(2) $\hat{\varphi}_m - \hat{w}_{\diamond m}^{\pm}\hat{q}_m \in \hat{\mathcal{C}}_0^{\pm}$.

Despite that Theorem 3.7 is proven in [21], for the sake of consistency as well as for the further need, we give its proof below.

Proof of Theorem 3.7. Item 1. Using the Duhamel formula, from equation (2.3), we find

$$\xi_m(\nu,t) = \xi_m^0(\nu,t) + \xi_m^1(\nu,t), \tag{3.8}$$

where

$$\xi_m{}^0(\nu,t) := \Re\Big(\frac{1}{i\nu}\xi_{0m}e^{i\nu t}\Big), \quad \xi_{0m} := \dot{\xi}_{0m} + i\nu\xi_{0m},$$

and

$$\xi_m^1(\nu,t) := \frac{\kappa_m}{2i\nu}\int_0^t\big(e^{i\nu(t-\tau)}-e^{i\nu(\tau-t)}\big)q_m(\tau)\,d\tau = \Re\Big(\frac{\kappa_m}{i\nu}\hat{q}_m^t e^{i\nu t}\Big); \tag{3.9}$$

here $q_m^t := q_m \mathbb{I}_{[0,t]}$ if $t \geqslant 0$, $q_m^t := -q_m \mathbb{I}_{[t,0]}$ if $t < 0$, and $\hat{q}_m^t := \mathcal{F}(q_m^t)$. By a direct computation, from equation (3.8), we obtain

$$\mathcal{E}_m(t) = \|\kappa_m \hat{q}_m^t + \xi_{0m}\|_{L^2}^2. \tag{3.10}$$

By Assumption A4, we have $\|\xi_{0m}\|_{L^2}^2 = \mathcal{E}_m(0) < \infty$ and, by Theorem 2.4, the function $\mathcal{E}_m(t)$ is bounded. Then

$$\|\kappa_m \hat{q}_m^t\|_{L^2} < C, \tag{3.11}$$

where the constant C does not depend on the time t. Set

$$q_m^{\pm} := q_m \mathbb{I}_{[0,\pm\infty)}, \quad \text{where} \quad \mathbb{I}_{[0,-\infty)} := -\mathbb{I}_{(-\infty,0]},$$

and let $\hat{q}_m^{\pm} := \mathcal{F}(q_m^{\pm})$.

Lemma 3.8. *The function $\kappa_m \hat{q}_m^{\pm}$ belongs to L^2 and the functions $\kappa_m \hat{q}_m^t$ weakly converge in L^2 to $\kappa_m \hat{q}_m^{\pm}$ as $t \to \pm\infty$.*

Proof. We first claim that the functions $\kappa_m \hat{q}_m^t$ considered as functionals on the space $\hat{\mathcal{M}}$ converge to $\kappa_m \hat{q}_m^+$ as $t \to +\infty$. Indeed, due to (3.3), for any $\hat{\mu} \in \hat{\mathcal{M}}$, we have

$$\langle \hat{\mu}, \kappa_m \hat{q}_m^t \rangle = 2\pi \langle \mu, \overline{\mathcal{F}^{-1}(\kappa_m \hat{q}_m^t)} \rangle = 2\pi \langle \mu * \mathcal{F}^{-1}(\kappa_m), q_m^t \rangle.$$

Since, by (2.11), we have $\kappa_m \in \hat{L}^1 \subset \hat{\mathcal{M}}$, we obtain $\mu * \mathcal{F}^{-1}(\kappa_m) \in \mathcal{M}$ (see Lemma 3.1). Then it is straightforward to check that $2\pi\langle \mu * \mathcal{F}^{-1}(\kappa_m), q_m^t \rangle$ converges to

$$2\pi \langle \mu * \mathcal{F}^{-1}(\kappa_m), q_m^+ \rangle = \langle \hat{\mu}, \kappa_m \hat{q}_m^+ \rangle.$$

Due to the obtained convergence, the norm of the functional $\kappa_m \hat{q}_m^+$, considered as a functional on $\hat{\mathcal{M}} \cap L^2$, is bounded by the constant from (3.11). Since the space $\hat{\mathcal{M}} \cap L^2$ is dense in L^2, we can uniquely extend $\kappa_m \hat{q}_m^+$ to a continuous functional on L^2 with the same norm. Identifying the obtained functional with an element of L^2, we obtain the desired result. The case $t \to -\infty$ can be studied similarly.

Since $\hat{q}_m = \hat{q}_m^+ - \hat{q}_m^-$, Lemma 3.8 implies the inclusion $\kappa_m \hat{q}_m \in L^2$. Since, by (2.11), we have $\kappa_m/\nu \in L^2$, we find $\hat{w}_m \hat{q}_m = \dfrac{2\pi\kappa_m}{i\nu} \kappa_m \hat{q}_m \in L^1$.

Item 2. Due to (3.8), we have $\varphi_m = \varphi_m^0 + \varphi_m^1$, where $\varphi_m^j := \int_{-\infty}^{\infty} \kappa_m \xi_m^j \, d\nu$, $j = 0, 1$. Then

$$\varphi_m^0 = \Re \int_{-\infty}^{\infty} \frac{\kappa_m}{i\nu} \xi_{0m} e^{i\nu t} \, d\nu = \Re \mathcal{F}^{-1}\Big(\frac{2\pi\kappa_m}{i\nu} \xi_{0m}\Big). \tag{3.12}$$

Since by (2.11) and Assumption A4, we have $\kappa_m/\nu, \xi_{0m} \in L^2$, we obtain $\dfrac{2\pi\kappa_m}{i\nu}\xi_{0m} \in L^1$, so

$$\varphi_m^0 \in \mathcal{C}_0. \tag{3.13}$$

Next,

$$\varphi_m^1 = \Re \int_{-\infty}^{\infty} \frac{\kappa_m^2}{i\nu} \hat{q}_m^t e^{i\nu t} \, d\nu =: \Re \mathcal{A}(t). \tag{3.14}$$

We have

$$\mathcal{A}(t) = \mathcal{F}^{-1}\big(\hat{w}_m \hat{q}_m^t\big)(t) = (w_m * q_m^t)(t) = \mathcal{A}_1^{\pm}(t) - \mathcal{A}_0^{\pm}(t), \tag{3.15}$$

where

$$\mathcal{A}_1^{\pm}(t) := \int_{-\infty}^{\infty} q_m(t-\tau) \mathbb{I}_{[0,\pm\infty)}(\tau) w_m(\tau) \, d\tau$$

and

$$\mathcal{A}_0^+(t) := \int_t^\infty q_m(t-\tau)w_m(\tau)\,d\tau, \quad \mathcal{A}_0^-(t) := \int_{-\infty}^t q_m(t-\tau)w_m(\tau)\,d\tau.$$

Since q_m is real and $\hat{w}_m$ is purely imaginary, we find

$$\Re\mathcal{A}_1^\pm = \frac{1}{2}\big(\mathbb{I}_{[0,\pm\infty)}(\tau)(w_m(\tau)-w_m(-\tau))\big) * q_m = \mathcal{F}^{-1}(\hat{w}_{\diamond m}^\pm \hat{q}_m).$$

Moreover, since $w_m \in L^1$, we have $\mathcal{A}_0^\pm(t) \to 0$ as $t \to \pm\infty$. Consequently,

$$\varphi_m^1 = \Re\mathcal{A}_1^\pm(t) - \Re\mathcal{A}_0^\pm(t) = \mathcal{F}^{-1}(\hat{w}_{\diamond m}^\pm \hat{q}_m) + \vartheta_m^\pm,$$

where $\vartheta_m^\pm \in \mathcal{C}_0^\pm$. Thus, in view of (3.13), we have $\varphi_m = \mathcal{F}^{-1}(\hat{w}_{\diamond m}^\pm \hat{q}_m) + \widetilde{\vartheta}_m^\pm$, where $\widetilde{\vartheta}_m^\pm \in \mathcal{C}_0^\pm$. □

Now we deduce Proposition 3.6 from Theorem 3.7.

Proof of Proposition 3.6. Item 1. Due to Theorem 3.7 (1), we have $\hat{w}_m\hat{q}_m \in L^1 \subset \hat{\mathcal{C}}_0$. Then Lemma 3.4(1) implies that $\operatorname{sing\,supp}\hat{q}_m \subset \{\nu : \hat{w}_m(\nu)=0\}$. Since, due to assumption A2, $\{\nu : \hat{w}_m(\nu)=0\}=\{0\}$, we obtain the desired result.

Item 2. The result follows in a standard way from equations of motion (2.3)–(2.4). Namely, due to formulas (3.12)–(3.15), we have $\varphi_m = \varphi_m^0 + \varphi_m^1$, where $\varphi_m^0 \in \mathcal{C}_0 \subset \mathcal{C}_b$ and $\varphi_m^1(t) = \Re(w_m * q_m^t)(t)$. Using the inclusion $w_m \in L^1$, it can be checked that $\varphi_m^1 \in \mathcal{C}_b$, so that $\varphi_m \in \mathcal{C}_b$.

For the time derivatives of the function φ_m, we have $\varphi_m^{(l)} = (\varphi_m^0)^{(l)} + (\varphi_m^1)^{(l)}$. By (3.12), we obtain $(\varphi_m^0)^{(l)} = \Re\mathcal{F}^{-1}(2\pi(i\nu)^{l-1}\kappa_m\xi_{0m})$. Since by Assumptions A2 and A4, $\nu^r\kappa_m, \xi_{0m}$ belongs to L^2, we find $(\varphi_m^0)^{(l)} \in \mathcal{F}^{-1}(L^1) \subset \mathcal{C}_0 \subset \mathcal{C}_b$, for any $l \leqslant r+1$.

Let us now study the derivatives $(\varphi_m^1)^{(l)}$. Due to (3.9), we have

$$\dot{\varphi}_m^1(t) = \Re\int_{-\infty}^\infty \kappa_m^2(\nu)e^{i\nu t}\int_0^t e^{-i\nu\tau}q_m(\tau)\,d\tau d\nu = \Re\mathcal{F}^{-1}(2\pi\kappa_m^2\hat{q}_m^t)(t) = \Re\big(\mathcal{F}^{-1}(\kappa_m^2) * q_m^t\big)(t).$$

Since by (2.11), we have $\mathcal{F}^{-1}(\kappa_m^2) \in L^1$, we find $\dot{\varphi}_m^1 \in \mathcal{C}_b$ and, consequently, $\dot{\varphi}_m \in \mathcal{C}_b$. Thus, the r.h.s. of equation (2.3) is differentiable in time and its derivative is from the space $\mathcal{C}_b$. Then the l.h.s. also is, so that the functions q_j are differentiable at least three times and $\dddot{q}_j \in \mathcal{C}_b$, for any $j \in \mathcal{G}$. Next we compute $\ddot{\varphi}_m^1$. Arguing as above, we find

$$\ddot{\varphi}_m^1(t) = \big(\mathcal{F}^{-1}((i\nu)\kappa_m^2) * q_m^t\big)(t) + q_m(t)\int_{-\infty}^\infty \kappa_m^2(\nu)\,d\nu.$$

Then, using that $\mathcal{F}^{-1}((i\nu)\kappa_m^2) \in L^1$, we obtain $\ddot{\varphi}_m^1 \in \mathcal{C}_b$, so that $\ddot{\varphi}_m \in \mathcal{C}_b$. Differentiating equation (2.3) twice, we then find $q_j^{(4)} \in \mathcal{C}_b$. Continuing the procedure in the same way, we arrive at the desired result.

Item 3. Set $\hat{\delta}_m^\pm(\nu) := \hat{w}_{\diamond m}^\pm(0) - \hat{w}_{\diamond m}^\pm(\nu)$. Since by (3.7), we have $\hat{w}_{\diamond m}^\pm(0) = K_m$, Theorem 3.7 (2) implies

$$\hat{\varphi}_m - K_m\hat{q}_m + \hat{\delta}_m^\pm\hat{q}_m \in \hat{\mathcal{C}}_0^\pm.$$

Since $\hat{\delta}_m^\pm(0)=0$, Lemma 3.4 (2) together with the first item of the present proposition implies that $\hat{\delta}_m^\pm\hat{q}_m \in \hat{\mathcal{C}}_0$, so that $\vartheta_m := \varphi_m - K_m q_m \in \mathcal{C}_0^+ \cap \mathcal{C}_0^- = \mathcal{C}_0$.

Due to item 2 of the proposition, the functions ϑ_m are C^{r+1}-smooth and their derivatives belong to the space $\mathcal{C}_b$. Since $\vartheta_m \in \mathcal{C}_0$, we have $\operatorname{sing\,supp}\hat{\vartheta}_m = \emptyset$, so the fact that the derivatives of ϑ_m belong to the space $\mathcal{C}_0$ follows from Lemma 3.5.

4. PROOFS OF MAIN RESULTS

4.1. Proof of Theorem 2.5

Step 1. Proposition 3.6 (3) implies that equation (2.3) can be written in the form

$$\ddot{q}_j = -U_j'(q_j) + \sum_{i\in\mathcal{G}:\, i\sim j} V_{ij}'(q_i - q_j) + \delta_{j\Lambda}(K_j q_j + \vartheta_j), \quad j \in \mathcal{G}. \tag{4.1}$$

Due to Proposition 3.6 (1), we have $\operatorname{sing\,supp} \hat{q}_j \subset \{0\}$ for any $j \in \Lambda$. Then, Lemma 3.5 implies that all time-derivatives of q_j belong to the space $\mathcal{C}_0$, that is

$$q_j^{(l)}(t) \to 0 \quad \text{as} \quad t \to \pm\infty \quad \text{for any} \quad 1 \leqslant l \leqslant r+3 \quad \text{and} \quad j \in \Lambda, \tag{4.2}$$

where we recall that the functions q_j are C^{r+3}-smooth and their derivatives belong to the space $\mathcal{C}_b$ for any $j \in \mathcal{G}$, due to Proposition 3.6 (2).

The goal of this step is to show that convergence (4.2) holds for any $j \in \mathcal{G}$. If $\Lambda = \mathcal{G}$, we obtain this automatically, so we pass directly to Step 2. Below we assume that $\Lambda \neq \mathcal{G}$. Due to Proposition 3.6 (2), we can differentiate equation (4.1) in time, obtaining

$$\dddot{q}_j = -U_j''(q_j)\dot{q}_j + \sum_{i\in\mathcal{G}:\, i\sim j} V_{ij}''(q_i - q_j)(\dot{q}_i - \dot{q}_j) + \delta_{j\Lambda}(K_j \dot{q}_j + \dot{\vartheta}_j). \tag{4.3}$$

In view of convergence (4.2), for $j \in \Lambda$, the l.h.s. of equation (4.3) goes to zero as $t \to \pm\infty$, as well as all the terms from the r.h.s., which are multiplied by $\dot{q}_l$ with $l \in \Lambda$, and as well as the function $\dot{\vartheta}_j$. Then, the sum of remaining terms has to vanish as well, that is

$$\sum_{\substack{i\,\in\,\mathcal{G}\setminus\Lambda\,:\\ i\,\sim\, j}} V_{ij}''(q_i - q_j)\dot{q}_i \to 0 \quad \text{as} \quad t \to \pm\infty \quad \text{for any} \quad j \in \Lambda. \tag{4.4}$$

By construction of the set Λ_Γ and in view of Assumption A5, there exists a $k \in \Lambda$ for which there is a unique $n \in \mathcal{G} \setminus \Lambda$ satisfying $n \sim k$. Then, the convergence (4.4) implies

$$V_{nk}''(q_n - q_k)\dot{q}_n(t) \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{4.5}$$

In Step 3 of the proof, we will show that convergence (4.5) implies

$$\dot{q}_n(t) \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{4.6}$$

Now let us assume that (4.6) takes place, so that $\dot{q}_n \in \mathcal{C}_0$ and, consequently, $\operatorname{sing\,supp} \mathcal{F}(\dot{q}_n) = \emptyset$. Then, due to Lemma 3.5, we have $q_n^{(l)} \in \mathcal{C}_0$ for any $1 \leqslant l \leqslant r+3$, so that $q_n^{(l)}(t) \to 0$ as $t \to \pm\infty$, and we have the convergence (4.2) with the set Λ replaced by the set $\Lambda \cup \{n\}$. Next we repeat the argument above with Λ replaced by $\Lambda \cup \{n\}$, and iterate the procedure. Finally, we obtain the convergence (4.2) with the set Λ replaced by Λ_Γ. Since, by Assumption A5, we have $\Lambda_\Gamma = \mathcal{G}$, the convergence (4.2) holds for any $j \in \mathcal{G}$.

Step 2. Next we let $t \to \pm\infty$ in equation (4.1). Due to the convergence (4.2) which holds for any $j \in \mathcal{G}$, the functions $\ddot{q}_j$ vanish as $t \to \pm\infty$, as well as the functions ϑ_j. Then, $q(t)$ approaches a set consisting of points $q \in \mathbb{R}^{|\mathcal{G}|}$ satisfying

$$0 = -U_j'(q_j) + \sum_{i\in\mathcal{G}:\, i\sim j} V_{ij}'(q_i - q_j) + \delta_{j\Lambda} K_j q_j, \quad j \in \mathcal{G}.$$

That is, we have $\operatorname{dist}(q(t), \mathcal{N}^{\rm eff}) \to 0$ as $t \to \pm\infty$, where $\mathcal{N}^{\rm eff}$ is the set of critical points of the effective potential (2.8). Clearly, the latter convergence implies the desired convergence (2.12).

Step 3. Finally, we deduce the convergence (4.6) from (4.5). Assume that (4.6) is false as $t \to +\infty$; the case $t \to -\infty$ can be studied similarly. Then there is an $M > 0$ such that there exists an arbitrarily large time t_0 satisfying $|\dot q_n(t_0)| > M$. For definiteness, we assume that $\dot q_n(t_0)$ is positive, so that $\dot q_n(t_0) > M$. Due to the uniform continuity of $\dot q_n$, there exists $\delta > 0$ independent of t_0, such that

$$\dot q_n(t) > M/2 \quad \text{for any} \quad t \in U_\delta(t_0), \tag{4.7}$$

where $U_\delta(t_0)$ denotes a δ-neighbourhood of t_0. Since $\dot q_k(t) \to 0$ as $t \to \infty$, the inequality (4.7) implies that, for sufficiently large t_0, we have

$$\frac{d}{dt}(q_n - q_k)(t) > M/4 \quad \text{for any} \quad t \in U_\delta(t_0). \tag{4.8}$$

Take $\varepsilon > 0$ and choose t_0 to be so large that $\big|V''_{nk}(q_n - q_k)\dot q_n\big|(t_0) < \varepsilon M/2$. Then, by (4.7), we have

$$\big|V''_{nk}(q_n - q_k)\big|(t) < \varepsilon \quad \text{for all} \quad t \in U_\delta(t_0).$$

Letting ε go to zero, we see that this contradicts to (4.8). Indeed, by Assumption A1, the function V''_{nk} has only finite number of zeros on bounded sets, while the function $(q_n - q_k)(t)$ is bounded. □

4.2. Proof of Corollary 2.6

Applying Proposition 3.6 (3) to formula (2.6), we obtain

$$\begin{aligned} E(q_0, \xi_0, \dot q_0, \dot\xi_0) &= E(q, \xi, \dot q, \dot\xi)(t) \\ &= \sum_{j \in \mathcal{G}} \frac{\dot q_j(t)^2}{2} + \sum_{m \in \Lambda} \mathcal{E}_m(q, \xi, \dot q, \dot \xi)(t) + V^{\rm eff}(q(t)) - \sum_{m \in \Lambda} \frac{K_m q_m(t)^2}{2} + \vartheta(t), \end{aligned}$$

where the function ϑ belongs to the space $\mathcal{C}_0$. Then, using (2.13) and (2.14), we get the desired convergence (2.15).

It is straightforward to check that, due to equation (2.4), $\dot{\mathcal{E}}_m = q_m \int_{-\infty}^{\infty} \kappa_m \dot\xi_m \, d\nu$. Then, using Proposition 3.6 (3), we find

$$\dot{\mathcal{E}}_m = K_m q_m \dot q_m + \dot\vartheta_m,$$

where the function ϑ_m is C^{r+1}-smooth and belongs to the space $\mathcal{C}_0$ together with all its derivatives. Then (2.16) follows from (2.13). □

5. ENERGY TRANSPORT IN THE SYSTEM OF ONE OSCILLATOR COUPLED TO TWO THERMOSTATS

In this section, we consider a system of one oscillator interacting with two thermostats, given by the equation

$$\ddot q = -U'(q) + \int_{-\infty}^{\infty} \kappa \xi_1 \, d\nu + \int_{-\infty}^{\infty} \kappa \xi_2 \, d\nu, \quad \ddot\xi_m(\nu) = -\nu^2 \xi_m(\nu) + \kappa(\nu) q, \quad m = 1, 2, \tag{5.1}$$

where $q(0) = q_0, \dot q(0) = \dot q_0$, $\xi_m(0) = \xi_{0m}$, $\dot\xi_m(0) = \dot\xi_{0m}$. Our goal is to study the asymptotic behavior of the energies $\mathcal{E}_1$, $\mathcal{E}_2$ of the thermostats, which are defined as in (2.10). We assume that the functions U and κ satisfy assumptions A1–A3 and A6 while the initial conditions fulfil Assumption A4 with $m = 1, 2$. Formally, the system (5.1) does not satisfy assumptions of Section

2.1, since the oscillator interacts with *two* thermostats. However, the theory developed in the paper can be applied to this system as well. Indeed, to see this, it suffices to change the variables

$$\zeta := \frac{\xi_1 + \xi_2}{\sqrt{2}}, \quad \eta := \frac{\xi_1 - \xi_2}{\sqrt{2}}. \tag{5.2}$$

Then equation (5.1) takes the form

$$\ddot{q} = -U'(q) + \sqrt{2}\int_{-\infty}^{\infty} \kappa\zeta\, d\nu, \quad \ddot{\zeta}(\nu) = -\nu^2\zeta(\nu) + \sqrt{2}\kappa(\nu)q, \quad \ddot{\eta}(\nu) = -\nu^2\eta(\nu). \tag{5.3}$$

Equation (5.3) describes a system which consists of an oscillator interacting with a thermostat that satisfies assumptions of Section 2.1, and of another independent thermostat.

Proposition 5.1. *The energies* $\mathcal{E}_1(t), \mathcal{E}_2(t)$ *converge as* $t \to \pm\infty$.

Proof. Set

$$\xi_m := \dot{\xi}_m + i\nu\xi_m, \quad \zeta := \dot{\zeta} + i\nu\zeta, \quad \eta := \dot{\eta} + i\nu\eta,$$

where $m = 1, 2$. The equations of motion of the thermostats from (5.3) written in the variables η, ζ take the form

$$\dot{\zeta} = i\nu\zeta + \sqrt{2}\kappa q, \quad \dot{\eta} = i\nu\eta. \tag{5.4}$$

The energy $\mathcal{E}_1$ has the form

$$\mathcal{E}_1 = \frac{1}{2}\|\xi_1\|_{L^2}^2 = \frac{1}{4}\|\zeta + \eta\|_{L^2}^2 = \frac{1}{4}\big(\|\zeta\|_{L_2}^2 + \|\eta\|_{L_2}^2 + 2\Re\langle\zeta, \eta\rangle_{L_2}\big), \tag{5.5}$$

where $\|\cdot\|_{L^2}$ and $\langle\cdot\,,\cdot\rangle_{L^2}$ denote the standard norm and scalar product in the space L^2. Similarly,

$$\mathcal{E}_2 = \frac{1}{2}\|\xi_2\|_{L^2}^2 = \frac{1}{4}\|\zeta - \eta\|_{L^2}^2 = \frac{1}{4}\big(\|\zeta\|_{L_2}^2 + \|\eta\|_{L_2}^2 - 2\Re\langle\zeta, \eta\rangle_{L_2}\big). \tag{5.6}$$

Applying Corollary 2.6 to the system given by the first two equations of (5.3), we find that the energy of the thermostat ζ, given by $\|\zeta\|_{L_2}^2$, converges as $t \to \pm\infty$. Moreover, the second equation from (5.4) implies that the norm $\|\eta\|_{L^2}$ is independent of time. Thus, in view of (5.5) and (5.6), to finish the proof of the proposition, it suffices to show that the scalar product $\langle\zeta, \eta\rangle_{L_2}$ converges as $t \to \pm\infty$.

Set $\eta_0 = \eta(0)$ and $\zeta_0 = \zeta(0)$. Recall that $\zeta_0, \eta_0 \in L^2$, due to Assumption A4. In view of (5.4), we have

$$\eta = e^{i\nu t}\eta_0, \quad \zeta = e^{i\nu t}\zeta_0 + \sqrt{2}\kappa\int_0^t e^{i\nu(t-\tau)}q(\tau)\, d\tau = e^{i\nu t}(\zeta_0 + \sqrt{2}\kappa\hat{q}^t),$$

where $q^t := q\mathbb{I}_{[0,t]}$ if $t \geqslant 0$, $q^t := -q\mathbb{I}_{[t,0]}$ if $t < 0$ and $\hat{q}^t = \mathcal{F}(q^t)$. Set $q^\pm := q\mathbb{I}_{[0,\pm\infty)}$. In Lemma 3.8, it is shown that $\kappa\hat{q}^\pm \in L^2$ and $\kappa\hat{q}^t \to \kappa\hat{q}^\pm$ weakly in L^2 as $t \to \pm\infty$. Then

$$\langle\eta, \zeta\rangle_{L_2} = \langle\eta_0, \zeta_0 + \sqrt{2}\kappa\hat{q}^t\rangle_{L_2} \to \langle\eta_0, \zeta_0 + \sqrt{2}\kappa\hat{q}^\pm\rangle_{L_2} \quad \text{as} \quad t \to \pm\infty. \tag{5.7}$$

Now we study whether we generically have

$$\lim_{t\to\star\infty} \mathcal{E}_1(t) = \lim_{t\to\star\infty} \mathcal{E}_2(t), \tag{5.8}$$

where $\star = +$ or $\star = -$. Equations (5.5)–(5.6) imply that the equality (5.8) takes place if and only if we have $\lim_{t\to\star\infty} \Re\langle\eta, \zeta\rangle = 0$. In turn, due to (5.7), the latter convergence takes place if and only if

$$\Re\langle\eta_0, \zeta_0 + \sqrt{2}\kappa\hat{q}^\star\rangle_{L_2} = 0. \tag{5.9}$$

Obviously, the function $\kappa\hat{q}^\star$ depends on the initial conditions $q_0, \dot{q}_0$, so that the equality

$$\zeta_0 + \sqrt{2}\kappa\hat{q}^\star \equiv 0$$

is generically false. Moreover, the function $\zeta_0 + \sqrt{2}\kappa\hat{q}^\star$ does not depend on the initial conditions η_0. Consequently, equation (5.9) holds only for a set of the initial conditions $q_0, \dot{q}_0, \zeta_0, \eta_0$ of codimension one, so that (5.8) is not generic. For example, (5.8) takes place if $\eta_0 = 0$, that is when we have $(\xi_{01}, \dot{\xi}_{01}) = (\xi_{02}, \dot{\xi}_{02})$, so that the thermostats have the same initial conditions.

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