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Topological Complexity of Certain Classes of C^*-Algebras

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Abstract. We compute the topological complexity for some important classes of noncommutative C^*-algebras: AF algebras, AI algebras, and even Cuntz algebras.

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1. INTRODUCTION

Topological complexity was introduced in [9]. It is closely related to the Lyusternik–Shnirel'-man category and is very useful when describing configuration spaces of mechanical systems. By definition, the topological complexity of a space X is the Schwarz genus of the natural bundle $PX \to X \times X$, where PX is the space of paths in X. In [11], a noncommutative version of Farber's topological complexity for unital C^*-algebras was introduced, and it was shown that, for commutative C^*-algebras, it coincides with the usual ("commutative") topological complexity. Recall the definition of topological complexity. Throughout the paper, $\otimes$ denotes the minimal tensor product of C^*-algebras. Let $\alpha_0^A, \alpha_1^A \colon A \to A \otimes A$ be the $*$-homomorphisms $\alpha_0^A(a) = a \otimes 1$, $\alpha_1^A(a) = 1 \otimes a$, $a \in A$.

Definition 1. For a unital C^*-algebra A, the *topological complexity* $TC(A)$ is the smallest integer n (or ∞ if no such n exists) such that there exist unital C^*-algebras B_j, $j = 1, \dots, n$ and surjective $*$-homomorphisms $\beta_j \colon A \otimes A \to B_j$ for which $\oplus_{j=1}^n \beta_j$ is injective and the $*$-homomorphisms $\beta_j \circ \alpha_0^A, \beta_j \circ \alpha_1^A : A \to B_j$ are homotopic for every j.

2. CALCULATION OF TOPOLOGICAL COMPLEXITY. EXAMPLES

It is often too hard to calculate the topological complexity. For some noncommutative C^*-algebras, it was computed in [11]. In this section, we extend the list of noncommutative C^*-algebras with known topological complexity. Namely, we calculate it for the algebra $\mathbb{B}(H)$ of all bounded operators on a Hilbert space, for AF algebras, AI algebras, and for some NCCW-complexes.

The following lemma is useful for computing the topological complexity.

Lemma 2. *Let A, B be C^*-algebras with $TC(A) = 1$. If there exists a homotopy between $*$-homomorphisms $\gamma_0, \gamma_1 \colon B \to B \otimes B \otimes A$ defined by $\gamma_0(b) = b \otimes 1 \otimes 1$ and $\gamma_1(b) = 1 \otimes b \otimes 1$, then $TC(B \otimes A) = 1$.*

Proof. Let γ_t, $t \in [0, 1]$, be the homotopy mentioned in the lemma. Since $TC(A) = 1$, there exists a homotopy α_t between α_0^A and α_1^A. Define $\lambda_t \colon B \otimes A \to B \otimes B \otimes A \otimes A$ by

$$\lambda_t = \begin{cases} \alpha_0^B \otimes \alpha_{2t} & \text{for } 0 \leqslant t \leqslant 1/2; \\ \gamma_{2t-1} \otimes \mathrm{id} & \text{for } 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then λ_t is a homotopy connecting $\alpha_0^{B\otimes A}$ and $\alpha_1^{B\otimes A}$. This suffices to conclude that $TC(B \otimes A) = 1$.

Let $\mathbb{B}(H)$ ($\mathbb{K}(H)$) be the C^*-algebra of all bounded (compact, respectively) operators on a separable Hilbert space H.

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Proposition 3. $TC(\mathbb{B}(H)) = \infty$.

Proof. Assume that $TC(\mathbb{B}(H)) < \infty$. Let $p \in \mathbb{B}(H)$ be a rank one projection, $e_1, \dots, e_n, \dots$ an orthogonal basis of H, P_n the orthogonal projection onto $span\langle e_1, \dots, e_n\rangle$, $Q_N = P_N \otimes P_N$. Since $\oplus_j \beta_j$ is injective, there is a j such that $\beta_j(p \otimes p) \neq 0$. Let $\langle p \otimes p\rangle$ be the ideal generated by $p \otimes p$. Clearly, $\langle p \otimes p\rangle = \mathbb{K}(H) \otimes \mathbb{K}(H)$. Since this ideal is simple, $\beta_j|_{\mathbb{K}(H)\otimes\mathbb{K}(H)}$ is injective. Let us prove that $\beta_j|_{\mathbb{B}(H)\odot\mathbb{B}(H)}$ is injective, where $\odot$ is the algebraic tensor product. Let $0 \neq \omega \in \mathbb{B}(H) \odot \mathbb{B}(H)$. Then $\omega = \sum_i a_i \otimes b_i$ for some operators $a_i, b_i \in \mathbb{B}(H)$. Since $Q_n\omega Q_n \to \omega$ in the weak operator topology, we have $Q_N\omega Q_N \neq 0$ for some N. Obviously, $Q_N\omega Q_N \in \mathbb{K}(H)\otimes\mathbb{K}(H)$. So $\beta_j(Q_N\omega Q_N) \neq 0$, and therefore $\beta_j(\omega) \neq 0$. Since β_j is surjective and $\beta_j|_{\mathbb{B}(H)\odot\mathbb{B}(H)}$ is injective, B_j contains $\mathbb{B}(H) \odot \mathbb{B}(H)$ as a dense $*$-subalgebra; thus, $B_j \cong \mathbb{B}(H) \otimes_\alpha \mathbb{B}(H)$ for some tensor product $\otimes_\alpha$. However, $\beta_j \colon \mathbb{B}(H) \otimes \mathbb{B}(H) \to B_j \cong \mathbb{B}(H) \otimes_\alpha \mathbb{B}(H)$ is surjective and is equal to the identity mapping on the elementary tensors; hence $\otimes_\alpha = \otimes$, and β_j is an isomorphism.

Thus, $\gamma_0 = \beta_j \circ \alpha_0^{\mathbb{B}(H)}$ and $\gamma_1 = \beta_j \circ \alpha_1^{\mathbb{B}(H)}$ are homotopic. Consider the quotient $*$-homomorphism $\pi \colon \mathbb{B}(H) \otimes \mathbb{B}(H) \to \mathbb{Q}(H) \otimes \mathbb{B}(H)$, where $\mathbb{Q}(H) = \mathbb{B}(H)/\mathbb{K}(H)$. Then the $*$-homomorphisms $\pi \circ \gamma_0$ and $\pi \circ \gamma_1$ are also homotopic, where $\pi \circ \gamma_0(p) = 0$ and $\pi \circ \gamma_1(p) \neq 0$. This contradiction completes the proof.

One of the reasons for the validity of the condition $TC(\mathbb{B}(H)) = \infty$ is that the spectrum is not Hausdorff. Recall that $Prim(\mathbb{B}(H)) = A_2$, where A_2 is the two-point Alexandroff arrow, i.e., $A_2 = \{0, 1\}$ with the topology in which the open sets are $\{\{0\}, \{0, 1\}\}$. This seems similar to the case of $A = \mathbb{C}^2$ (i.e., a two-point Hausdorff space), for which the topological complexity is infinite because the space is not path-connected. Although, in some non-Hausdorff cases, the topological complexity feels a 'similar' Hausdorff space, this is not always the case, as is shown by the following example.

Example 4. Let $\mathbb{C}$ ($\mathbb{C}^2$) be the subalgebra of scalar (diagonal) matrices in the algebra M_2 of all 2×2 matrices. Write $A = \{f \in C[0, 2] \otimes M_2 : f(0) \in \mathbb{C}, f(t) \in \mathbb{C}^2 \text{ for } t \in (0, 1)\}$. It is easy to see that the spectrum of A is non-Hausdorff, more precisely, $Prim(A) = ([0, 2] \sqcup [0, 2])/\sim$, where $\sim$ is gluing the points in $\{0\} \cup (1, 2]$ from both segments, and the point $\{1\}$ is a "non-Hausdorff" point. Let $\phi_1(f)(s) = f(0)$, $f \in A$, $\phi_0 = id_A$. Then

$$\phi_t(f)(s) = \begin{cases} f(0) & \text{for } 0 \leqslant s \leqslant 2t, \\ f(s - 2t) & \text{for } 2t \leqslant s \leqslant 2, \end{cases}$$

is a homotopy connecting ϕ_0 and ϕ_1, and hence A is homotopy equivalent to $\mathbb{C}$. Since the topological complexity is homotopy invariant, we conclude that $TC(A) = 1$. It is unclear whether or not the topological complexity is infinite when a C^*-algebra is not homotopy equivalent to a C^*-algebra with a Hausdorff spectrum, even in the case of NCCW-complexes.

The homotopy theory of NCCW-complexes is incomplete; however, in some cases, one can find conditions implying that the topological complexity of an NCCW-complex is infinite. By an NCCW-complex we mean an algebra of the form $A = \{f \in C(X) \otimes M_n : f|_{X_j} \in A_j\}$, where X is a CW complex, $X_j \subset X$ are CW subcomplexes, and $A_j \subset M_n$ are $*$-subalgebras.

Proposition 5. *Let $A = \{f \in C(X) \otimes M_n : f|_{X_j} \in \mathbb{C}\}$, $n \geqslant 2$, for some CW complex X and at most two CW subcomplexes $X_j \subset X$ with $X_i \cap X_j = \emptyset$ for every pair i, j. Then $TC(A) = \infty$.*

Proof. Let $TC(A) < \infty$. Then there are surjections $\beta_j \colon A \otimes A \to B_j$. It can readily be seen that $\beta_j(f) = f|_{Y_j}$ for some $Y_j \subset X \times X$. Consider two points of different subcomplexes, $x \in X_k$, $y \in X_l$, $k \neq l$. Since $\oplus\beta_j$ is injective, there is a j such that $(x, y) \in Y_j \subset X \times X$. Then, by the definition of topological complexity, the $*$-homomorphisms $\varphi_x, \varphi_y : A \to \mathbb{C}$ defined by the relations $\varphi_x(f) = f(x), \varphi_y(f) = f(y)$ are homotopic. Since every epimorphism $\varphi \colon A \to A'$ is of the form $\varphi(f) = f|_Y$ for some $Y \subset X$, it is clear that $Hom(A, \mathbb{C})$ is homeomorphic to $\sqcup_j X_j \subset X$. Since φ_x and φ_y are in different path-connected components, they cannot be homotopic.

Corollary 6. *Let $A = \{f \in C(X) \otimes M_n : f(x_1), \dots, f(x_m) \in \mathbb{C}\}$, where X is a CW-complex and $x_1, \dots, x_m \in X$, $m \geqslant 2$. Then $TC(A) = \infty$.*

Proposition 7. *Suppose that A is a unital AF algebra. If A is UHF, then $TC(A) = 1$; otherwise $TC(A) = \infty$.*

Proof. If A is UHF, then $TC(A) = 1$ ([11], Proposition 3.6). Let A be not UHF. Let us prove that, in this case, $K_0(A)$ contains a copy of $\mathbb{Z}^2$. Suppose the contrary. Then, for every projection $p \in A$, there exist $x, y \in \mathbb{Z}$ such that $x[p] = y[1] \in K_0(A)$ (because, if not, then $x[p] \neq y[1]$ for some $x, y \in \mathbb{Z}$, and thus $[p]$ and $[1]$ generate a rank-2 free Abelian subgroup in $K_0(A)$). Let us construct a mapping $\varphi\colon K_0(A) \to \mathbb{Q}$. For a projection $p \in A$, set $\varphi([p]) = y/x$, where $x[p] = y[1]$ (we may assume that $x \neq 0$). Let us prove that φ is well defined. Assume that $x_p[p] = y_p[1]$ and $x'_p[p] = y'_p[1]$. Then $0 = x_p x'_p([p]-[p]) = (x'_p y_p - x_p y'_p)[1]$. Since the K-theory of an AF algebra has no torsion, we conclude that $(x'_p y_p - x_p y'_p) = 0$, and hence $y_p/x_p = y'_p/x'_p$. Since $x_p x_q([p]+[q]) = (x_p y_q + x_q y_p)[1]$, we have $\varphi([p]+[q]) = \frac{x_p y_q + x_q y_p}{x_p x_q} = \frac{y_p}{x_p} + \frac{y_q}{x_q} = \varphi([p]) + \varphi([q])$. Thus, φ is well defined on the formal differences $[p]-[q] \in K_0(A)$, and is a group homomorphism. It is monotone. Indeed, let $[p]-[q] \geqslant 0$, $x_p[p] = y_p[1]$, and $x_q[q] = y_q[1]$. Then $0 \leqslant x_p x_q([p]-[q]) = (x_q y_p - x_p y_q)[1]$ and $\varphi([p]-[q]) = y_p/x_p - y_q/x_q = \frac{x_q y_p - x_p y_q}{x_p x_q} \geqslant 0$. Finally, φ is injective. Let $[p]-[q] \neq 0$. Then $x_p x_q([p]-[q]) \neq 0$, since the K-theory of an AF algebra has no torsion. However, $x_p x_q([p]-[q]) = (x_q y_p - x_p y_q)[1]$. Thus, $(x_q y_p - x_p y_q) \neq 0$ and $\varphi([p]-[q]) \neq 0$.

By the Elliott theorem (Theorem IV.4.3 of [6]), A is $*$-isomorphic to a UHF algebra U with $K_0(U) = \varphi(K_0(A)) \subset \mathbb{Q}$, a contradiction.

Thus, in the non-UHF case, there is a projection $p \in A$ such that $N[p] \neq M[1]$ for every $N, M \in \mathbb{Z}$ (except for $N = M = 0$). Let $A = \overline{\cup_{n\in\mathbb{N}} A_n}$, where A_n are finite-dimensional C^*-algebras, and $\gamma_n\colon A_n \subset A_{n+1}$ and $\gamma_{n,k}\colon A_n \subset A_k$, where $\gamma_{n,k} = \gamma_{n+k-1} \circ \dots \circ \gamma_n$, $k > n$. Then $A_n = \oplus_{i=1}^{k_n} M_{[n,i]}$, where $[n,i]$ are some positive integers. If the sequence of integers k_n contains infinitely many ones, then A is an UHF algebra; thus, we may assume that $k_n > 1$ for any $n \in \mathbb{N}$. Since A is an AF algebra, $A \otimes A$ is also an AF algebra, and $A \otimes A = \overline{\cup_{n\in\mathbb{N}} A_n \otimes A_n}$. The quotients of AF algebras can be described in terms of Bratteli diagrams (see Chapter III of [6]). Every quotient is determined by a projection onto a subdiagram which is a complement to some hereditary directed diagram. Let Ω be a diagram for $A \otimes A$, and let Ω_J be the hereditary directed diagrams for the algebras $\ker \beta_J \subset A \otimes A$, $J = 1, \dots, K$. Then $B_J = \overline{\cup_{n\in\mathbb{N}} B_{J,n}}$, where $B_{J,n} = \oplus_{(i,j)\in\Omega^n\setminus\Omega^n_J} M_{[n,i][n,j]}$, and $\beta_J\colon A\otimes A \to B_J$ is the limit of the mappings $\beta_{J,n} : A_n \otimes A_n \to B_{J,n}$, which are the usual projections $\oplus_{(i,j)\in\Omega^n} M_{[n,i][n,j]} \to \oplus_{(i,j)\in\Omega^n\setminus\Omega^n_J} M_{[n,i][n,j]}$. Since Ω_J is directed and $\cap_{1\leqslant J\leqslant K} \ker \beta_J = 0$, it follows that $\cap \Omega_J = \emptyset$, and therefore $\cup_{1\leqslant J\leqslant K}(\Omega\setminus\Omega_J) = \Omega$. We may think that $p \in A_{n_0}$ for some n_0. Let $d_{[k,i]}$ be the dimension of the projection $\gamma_{n_0,k}(p)$ in the direct summand $M_{[k,i]} \subset A_k$. Since $N[p] \neq M[1]$ for every $N, M \in \mathbb{Z}$ except for $N = M = 0$, it follows that, for every $n > n_0$, there are i, j such that $d_{[n,i]}/[n,i] \neq d_{[n,j]}/[n,j]$. The projections $p \otimes 1$ and $1 \otimes p$ have the dimensions $d_{[n,i]}[n,j]$ and $d_{[n,j]}[n,i]$, respectively, in the direct summand $M_{[n,i][n,j]} \subset A_n \otimes A_n$. Since $\cap_{1\leqslant J\leqslant K} \Omega_J = \emptyset$, it follows that, for every n, there is a J_n for which $(i,j) \in \Omega^n\setminus\Omega^n_{J_n}$. Assume that $TC(A) = K < \infty$. Consider the sequence $\{J_n\}$, $n \in \mathbb{N}$. There is a $J \in \{1, 2, \dots, K\}$ which repeats infinitely in the sequence $\{J_n\}$. By assumption, the projections $p_0 = \beta_J \circ \alpha_0^A(p)$ and $p_1 = \beta_J \circ \alpha_1^A(p)$ are homotopic in B_J. This is equivalent to the fact that there is an $n_1 > n_0$ such that p_0 and p_1 are homotopic in $B_{J,n_1} \subset B_J$; however, this is impossible, because the dimensions of p_0 and p_1 in the direct summand $M_{[n,i][n,j]} \subset B_{J,n} = \oplus_{(i,j)\in\Omega^n\setminus\Omega^n_J} M_{[n,i][n,j]}$ are $d_{[n,i]}[n,j]$ and $d_{[n,j]}[n,i]$, respectively, and differ (here n is any integer greater than n_1). Thus, $TC(A) = \infty$.

The ideology is very clear: to compute the topological complexity of a nonsimple C^*-algebra A, we must have a good understanding of the quotients of $A \otimes A$. In the case of a simple C^*-algebra A, we must have effective methods for checking when $*$-homomorphisms α_0^A and α_1^A are homotopic. The homotopy theory of $*$-homomorphisms between AH algebras is vaguely known, even in the case of simple algebras. However, in the case of simple AI algebras, the topological complexity can be computed (see Sec. 3.2 of [13] or Chap. 2 of [16] for an information about AI algebras). Recall Theorem 3.10 of [1].

Theorem 8. *Let A, B be simple unital AI algebras. Two unital $*$-homomorphisms $\alpha, \beta : A \to B$ are homotopic if and only if $\alpha_* = \beta_* \colon K_0(A) \to K_0(B)$.*

We have an easy corollary to this important theorem.

Corollary 9. *Let A be a simple unital AI algebra. Then $TC(A) = 1$ if and only if there is an injective homomorphism $K_0(A) \subset \mathbb{Q}$. Otherwise $TC(A) = \infty$.*

Proof. It is an easy exercise to show that, for a torsion-free countable Abelian group G, the homomorphisms $\gamma_0, \gamma_1 \colon G \to G \otimes G$, $\gamma_0(g) = g \otimes 1$, $\gamma_1(g) = 1 \otimes g$ coincide if and only if G does not contain a copy of a rank-two free Abelian group $\mathbb{Z}^2$, which holds in turn if and only if there is an injective homomorphism $G \subset \mathbb{Q}$.

It is of interest that the form of conditions for $TC(A) = 1$ in the case of simple AI algebras coincides with that in the case of AF algebras. It can readily be seen that a unital AF algebra A is UHF if and only if $K_0(A)$ embeds in $\mathbb{Q}$. Recall that every AF algebra is also an AI algebra.

Example 10. Let A_F be a Fibonacci AF algebra, i.e., $A = \overline{\cup_{n\in\mathbb{N}} A_n}$, where $A_n = M_{\varphi_n} \oplus M_{\varphi_{n+1}}$, with connective $*$-homomorphisms $\alpha_n \colon A_n \to A_{n+1}$ defined by $\alpha_n(a \oplus b) = b \oplus (a \oplus b)$. This algebra is the simplest example of a simple non-UHF unital AF algebra (see Example III.2.6 of [6]). By Proposition 7, $TC(A_F) = \infty$.

Example 11. An interesting example of a simple AI algebra is the Goodearl algebra (see Example 3.1.7 of [13] for more details). Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $[0, 1]$. Write $A_n = C[0, 1] \otimes M_{2^n}$ and define $\alpha_n \colon A_n \to A_{n+1}$ by $\alpha_n(f)(x) = diag(f(x), f(x_n))$. Let $A = \lim A_n$ be the inductive limit of the system. Then A is a simple unital AI algebra with $K_0(A) = \mathbb{Z}[\frac{1}{2}]$, because $K_0(A_n) = \mathbb{Z}$, and $(\alpha_n)_* \colon \mathbb{Z} \to \mathbb{Z}$ is given by $(\alpha_n)_*(x) = 2x$. Thus $TC(A) = 1$.

3. REDUCING THE TOPOLOGICAL COMPLEXITY BY TENSORING BY MATRICES

In what follows, all CW complexes are finite with a base point. By $H^*(X)$ ($\tilde{K}^*(X)$) we denote the cohomology (the reduced K-theory) groups of a CW complex X.

For C^*-algebras A and B, by $[A, B]$ we denote the set of homotopy classes of $*$-homomorphisms if either A or B is nonunital and the set of homotopy classes of unital $*$-homomorphisms if both A and B are unital. Consider the bifunctor $kk(Y, X) = \lim_n [C(X), C(Y) \otimes M_n]$ introduced in [4] with the mappings $\alpha_n : [C(X), C(Y) \otimes M_n] \to [C(X), C(Y) \otimes M_{n+1}]$ defined by $\alpha_n(\beta)(f)(y) = \beta(f)(y) \oplus f(x_0)$, where x_0 is the base point of X, $f \in C(X)$, $\beta \in [C(X), C(Y) \otimes M_n]$. Write $C_0(X) = C(X\backslash\{x_0\})$. As is known (see [4]), $kk(Y, X) \cong [C_0(X) \otimes \mathbb{K}, C(Y) \otimes \mathbb{K}]$. This isomorphism gives rise to a homomorphism $\gamma \colon kk(Y, X) \to KK(C_0(X), C_0(Y))$.

Theorem 12 (Theorem 3.3 of [8])**.** *Let X be an n-dimensional CW complex and Y an m-connected CW-complex. Let $n + 3 \geqslant m$ and $H^n(X)$ be a finite group. Then $\gamma : kk(Y, X) \to KK(C_0(X), C_0(Y)]$ is an isomorphism.*

Theorem 13 (Theorem 6.6.4 of [4])**.** *Let $m > (3 dim Y)/2$. Then the following equality holds $kk(Y, X) = [C(X), C(Y) \otimes M_m]$.*

Combining these two theorems, we show that tensoring by matrix algebras can reduce topological complexity.

Corollary 14. *Let X be a two-dimensional finite CW complex such that $\tilde{K}^*(X) = 0$, and let $H^2(X)$ be a finite group. Then $TC(C(X) \otimes M_7) = 1$.*

Proof. By Lemma 2, it suffices to prove that the homomorphisms $\alpha, \beta \colon C(X) \to C(X^2) \otimes M_7$, $\alpha(f)(x, y) = f(x) \otimes 1$, $\beta(f)(x, y) = f(y) \otimes 1$ are homotopic. By Theorem 12 with $n = 2, m = 0$ and by Theorem 13, we have $[C(X), C(X^2) \otimes M_7] = KK(C_0(X), C_0(X^2))$. Since $\tilde{K}^*(X) = 0$, we see, by the Künneth theorem, that $KK(C_0(X), C_0(X^2)) = Hom(\tilde{K}^*(X), \tilde{K}^*(X^2)) = 0$. Thus, each of the two unital *-homomorphisms $C(X) \to C(X^2) \otimes M_7$ are homotopic.

Example 15. Conditions of Corollary 14 hold for the classifying space BH of the Higman group $H = \langle a,b,c,d | bab^{-1} = a^2, cbc^{-1} = b^2, dcd^{-1} = c^2, ada^{-1} = d^2\rangle$. The space BH is just the wedge sum of 4 circles with 4 two-dimensional discs attached by the mappings defined by the given relations. As was shown in ([7], Sec. 4), $\tilde{H}^n(BH) = 0$ for $n \geqslant 0$ and $\tilde{K}^*(BH) = 0$. Corollary 14 implies that $TC(BH \otimes M_7) = 1$, while $TC(BH) > 1$, since BH is not contractible.

4. TOPOLOGICAL COMPLEXITY OF EVEN CUNTZ ALGEBRAS

Recall that two $*$-homomorphisms $\alpha, \beta\colon A \to B$ are strongly asymptotically unitarily equivalent if there is a continuous unitary path $t \mapsto u_t \in U(B)$, $t \in [0,\infty)$, such that $u(0) = 1$ and $\lim_{t\to\infty} \|\alpha(a) - u_t\beta(a)u_t^*\| = 0$ for any $a \in A$. Strongly asymptotically unitarily equivalent $*$-homomorphisms are obviously homotopic.

Recall that a C^*-algebra A is said to be K_1-*injective* if the canonical mapping $U(A)/U(A)_0 \to K_1(A)$ is injective (here $U(A)_0$ is the path-connected component of 1_A in the unitary group $U(A)$ of A). A C^*-algebra D is said to be *strongly self-absorbing* if there are unitaries $u_n \in D \otimes D$ and a $*$-isomorphism $\beta\colon D \to D \otimes D$ such that $\beta(d) - u_n(1 \otimes d)u_n^* \to 0$ for every d (see [5] for more details).

The homotopy theory of $*$-homomorphisms between simple nuclear C^*-algebras is well studied in the literature (better than in the nonsimple case).

Theorem 16 (Theorem 2.2 of [5]). *Let A be a C^*-algebra, and let D be a strongly self-absorbing K_1-injective C^*-algebra. Then any two unital *-homomorphisms $\alpha, \beta\colon D \to D \otimes A$ are strongly asymptotically unitarily equivalent* (*and hence homotopic*).

Corollary 17. *Let D be a strongly self-absorbing C^*-algebra. Then $TC(D) = 1$.*

Proof. Take $A = D$ in Theorem 16.

Corollary 17 applies to many C^*-algebras, for example, to the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, to the Jiang–Su algebra $\mathcal{Z}$, and to the UHF algebras of infinite type (i.e., UHF algebras with supernatural numbers of the form $n = p_1^\infty p_2^\infty \ldots$, where $p_1, p_2, \ldots$ are primes). The Cuntz algebras $\mathcal{O}_n$ with $n \neq 2, \infty$ are not self-absorbing (i.e., $\mathcal{O}_n \not\cong \mathcal{O}_n \otimes \mathcal{O}_n$), and we can compute its topological complexity only for even n. Since the Cuntz algebras are simple, we have either $TC(\mathcal{O}_n) = 1$ or $TC(\mathcal{O}_n) = \infty$.

Proposition 18. *Let n be an odd positive integer. Then $TC(\mathcal{O}_{n+1}) = 1$.*

Proof. As is well known, $K_*(\mathcal{O}_{n+1}) = (\mathbb{Z}_n, 0)$. Write $A = \mathcal{O}_{n+1}$, $B = \mathcal{O}_{n+1} \otimes \mathcal{O}_{n+1}$, $\alpha = \alpha_0^A$, and $\beta = \alpha_1^A$. Define the flip automorphism $\sigma\colon B \to B$ by the formula $\sigma(a \otimes b) = b \otimes a$ for $a, b \in A$. Since A is simple, it suffices to prove that α and β are homotopic. By the Künneth theorem (see [15]), $K_*(B) = (\mathbb{Z}_n, \mathbb{Z}_n)$. For the unital C^*-algebra D, define the cone as follows: $CD = \{f \in C[0,1] \otimes D : f(0) = 0, f(1) \in \mathbb{C}1_D\}$. For a unital $*$-homomorphism $\gamma\colon A \to B$, define $C\gamma\colon CA \to CB \subset M(SB)$ by $C\gamma(f)(t) = \gamma(f(t))$, where $M(SB)$ stands for the multiplier algebra of $SB = C_0(0,1) \otimes B$ (for details, see [2]). Choose a unital absorbing homomorphism j (in our case, every unital homomorphism is automatically absorbing by Proposition 2.8 of [2], and therefore we can set $j = \alpha$). Let $\chi(\gamma) = \langle C\gamma, Cj\rangle$ be a pair of quasi-homomorphisms. By Corollary 3.10 of [2], α is homotopic to β if and only if $\chi(\alpha) = \chi(\beta) \in KK(CA, SB)$. We have $\chi(\alpha) = \langle C\alpha, C\alpha\rangle = 0$, and thus it suffices to prove that $\chi(\beta) = \langle C\beta, C\alpha\rangle = 0$. Write $\omega = \chi(\beta)$.

Consider he canonical exact sequence

$$0 \to SA \to CA \to \mathbb{C} \to 0.$$

The following six-term exact diagram holds:

$$\begin{array}{ccccc} K_1(A) & \longrightarrow & K_0(CA) & \longrightarrow & K_0(\mathbb{C}) \\ \uparrow{\scriptstyle i_*} & & & & {\scriptstyle i_*}\downarrow \\ K_1(\mathbb{C}) & \longleftarrow & K_1(CA) & \longleftarrow & K_0(A) \end{array}$$

As is known, the vertical homomorphisms are induced by the inclusion $i\colon \mathbb{C} \hookrightarrow A$ (see [12]) in the case of unital C^*-algebra A. Since $[1] \in K_0(A)$ is a generator of $\mathbb{Z}_n$, it follows that i_* is an epimorphism. Thus, $K_*(CA) = (\mathbb{Z}, 0)$. By the UCT theorem (see [14]), $0 \to 0 \to KK(CA, SB) \to \mathbb{Z}_n \to 0$. Therefore, $KK(CA, SB) = \mathbb{Z}_n$. We claim that $[\sigma] = 1 \in KK(B, B)$. We have $K_0(B) = K_0(A) \otimes K_0(A)$ and $K_1(B) = Tor(K_0(A), K_0(A))$. Since $x \otimes 1 = 1 \otimes x \in K_0(A) \otimes K_0(A)$ by the naturalness of the Künneth sequence, we have $\sigma_* = 1 : K_*(B) \to K_*(B)$.

By the UMCT theorem, we have the exact sequence

$$0 \to Pext(K_*(B), K_*(B)) \to KK(B, B) \to Hom_\Lambda(\underline{K}(B), \underline{K}(B)) \to 0,$$

where $\underline{K}(B) = \oplus_{m=0}^{\infty} K_*(B; \mathbb{Z}_m)$. Assume that $K_*(B; \mathbb{Z}_m) = K_*(B)$ for $m = 0$. Let us denote by $Hom_\Lambda(\underline{K}(B), \underline{K}(B)))$ the set of all sequences of homomorphisms $\gamma_j^m\colon K_j(B; \mathbb{Z}_m) \to K_j(B; \mathbb{Z}_m)$ that commute with the natural Bockstein operations, i.e., $\gamma_j^m \circ \rho_m^j = \rho_m^j \circ \gamma_j^0$, $\gamma_{j+1}^0 \circ \beta_m^j = \beta_m^j \circ \gamma_j^m$ and $\gamma_j^{nm} \circ \kappa_{nm,m}^j = \kappa_{nm,m}^j \circ \gamma_j^m$, where $\rho_m^j\colon K_j(B) \to K_j(B; \mathbb{Z}_m)$, $\beta_m^j\colon K_j(B; \mathbb{Z}_m) \to K_{j+1}(B)$, and $\kappa_{nm,m}^j\colon K_j(B; \mathbb{Z}_m) \to K_j(B; \mathbb{Z}_{nm})$ are given by the Kasparov product with the KK-classes of obvious homomorphisms $I_m \to \mathbb{C}$, $SM_m \hookrightarrow I_m$, and $I_m \hookrightarrow I_{nm}$, respectively, where $I_m = \{f \in C[0,1] \otimes M_m : f(0) = 0, f(1) \in \mathbb{C}\}$ (see [3] for details). Since $K_*(B)$ is finitely generated by Proposition 53.4 of [10], we have $Pext(K_*(B), K_*(B)) = 0$, i.e.,

$$KK(B, B) = Hom_\Lambda(\underline{K}(B), \underline{K}(B)).$$

By definition, $K_*(B; \mathbb{Z}_m) = KK_*(I_m, B)$, where $I_m = \{f \in C[0,1] \otimes M_m : f(0) = 0, f(1) \in \mathbb{C}\}$. By the naturalness of UCT-theorem, for every m, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \overset{\phi}{\longleftarrow} & KK(I_m, B) & \overset{\delta}{\longrightarrow} & L & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \sigma_*} & & \uparrow{\scriptstyle id} & & \\
0 & \longrightarrow & H & \overset{\phi}{\longleftarrow} & KK(I_m, B) & \overset{\delta}{\longrightarrow} & L & \longrightarrow & 0
\end{array}$$

is commutative. Here $H = Ext(K_{*+1}(I_m), K_*(B))$ and $L = Hom(K_*(I_m), K_*(B))$. Note that ϕ is defined only on $ker\delta$ and is bijective on it. The right and the left vertical arrows are equal to id, because $\sigma_* = id : K_*(B) \to K_*(B)$. Assume that there is an $x \in KK(I_m, B)$ such that $\sigma_*(x) \neq x$. Since the diagram is commutative, we have $\delta(x) = \delta(\sigma_*(x))$, i.e., $x - \sigma_*(x) \in ker\delta$. Write $y = x - \sigma_*(x)$. Then $\sigma_*(y) = \sigma_*(x - \sigma_*(x)) = \sigma_*(x) - x = -y$, because $\sigma^2 = id$. By the commutativity, $\phi(y) = \phi(\sigma_*(y)) = -\phi(y)$, i.e., $\phi(2y) = 0$. Since ϕ is an isomorphism, $2y = 0$. As is known, $Ext(K_{*+1}(I_m), K_*(B)) = (Ext(\mathbb{Z}_m, \mathbb{Z}_n), 0) = \mathbb{Z}_{\langle m,n \rangle}$, where $\langle m, n \rangle$ stands for the greatest common divisor of m and n. Since n is odd, $\langle m, n \rangle$ is also odd. Since $2y = 0$ in an odd-order cyclic group, we see that $y = 0$, i.e., $\sigma_* = id\colon K_*(B; \mathbb{Z}_m) \to K_*(B; \mathbb{Z}_m)$. By the UMCT-theorem, $[\sigma] = 1 \in KK(B, B)$. The Bott periodicity readily implies that $[S\sigma] = [\sigma] = 1 \in KK(SB, SB) = KK(B, B)$.

Finally, we have $\omega = \omega[S\sigma] = (S\sigma)_*(\langle C\beta, C\alpha \rangle) = \langle C(\sigma \circ \beta), C(\sigma \circ \alpha) \rangle = \langle C\alpha, C\beta \rangle = -\omega$, where $\omega[S\sigma]$ is the Kasparov product of ω and $[S\sigma]$. Thus, $2\omega = 0$. However, since $KK(CA, SB) = \mathbb{Z}_n$ is of odd order, we have $\omega = 0$. This means that α is homotopic to β, and $TC(\mathcal{O}_{n+1}) = 1$.

It is proved in [11] that $TC(C(S^1) \otimes \mathcal{O}_2) = 1$. Similarly, using Lemma 2 and an isomorphism $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2$, we can show that $TC(C(T^n) \otimes \mathcal{O}_2) = 1$. In the same way, we can prove the following proposition, where $C^*(F_2)$ is the full group C^*-algebra of the free group F_2 on two generators.

Proposition 19. $TC(C^*(F_2) \otimes \mathcal{O}_2) = 1$.

Proof. Since $TC(\mathcal{O}_2) = 1$ by Lemma 2, it is sufficient to construct a homotopy between $\alpha_0, \alpha_1\colon C^*(F_2) \to C^*(F_2) \otimes C^*(F_2) \otimes \mathcal{O}_2$, where $\alpha_0(x) = x \otimes 1 \otimes 1$ and $\alpha_1(x) = 1 \otimes x \otimes 1$. Let u, v be generators of F_2 (they are also generators of $C^*(F_2)$). By the Künneth theorem (see Theorem 4.1 of [15]), $K_1(C^*(F_2) \otimes C^*(F_2) \otimes \mathcal{O}_2) = 0$, and hence there is a path u_t (v_t) connecting

$\alpha_0(u)$ with $\alpha_1(u)$ ($\alpha_0(v)$ with $\alpha_1(v)$, respectively). Then the desired homotopy can be defined on the generators by $\alpha_t(u) = u_t$, $\alpha_t(v) = v_t$. Because of the universal property of $C^*(F_2)$, we can extend the mapping α_t to a well-defined *-homomorphism of $C^*(F_2)$. Moreover, α_t is continuous, because, for every $x \in C^*(F_2)$, we can find an approximation in the group algebra, $y \in \mathbb{C}[F_2]$, and $\|\alpha_t(x)-\alpha_s(x)\| \leqslant \|\alpha_t(x)-\alpha_t(y)\|+\|\alpha_t(y)-\alpha_s(y)\|+\|\alpha_s(y)-\alpha_s(x)\| \leqslant 2\|x-y\|+\|\alpha_t(y)-\alpha_s(y)\|$. The function $\alpha_t(y)$ is continuous for every y, since u_t and v_t are continuous paths.

In view of the previous statements, we would like to conclude with the following conjecture:

Conjecture 20. Let X be a finite CW complex. Then $TC(C(X) \otimes \mathcal{O}_2) = 1$.

This conjecture is not clear even in the case $X = S^2$.

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