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Mathematical Quantum Yang–Mills Theory Revisited

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Abstract. A mathematically rigorous relativistic quantum Yang–Mills theory with an arbitrary semisimple compact gauge Lie group is set up in the Hamiltonian canonical formalism. The theory is nonperturbative, without cut-offs, and agrees with the causality and stability principles. This paper presents a fully revised, simplified, and corrected version of the corresponding material in the previous papers Dynin ([11] and [12]). The principal result is established anew: due to the quartic self-interaction term in the Yang–Mills Lagrangian along with the semisimplicity of the gauge group, the quantum Yang–Mills energy spectrum has a positive mass gap. Furthermore, the quantum Yang–Mills Hamiltonian has a countable orthogonal eigenbasis in a Fock space, so that the quantum Yang–Mills spectrum is point and countable. In addition, a fine structure of the spectrum is elucidated.

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1. INTRODUCTION

1.1. Context

I address *both items* of the Clay Mathematics Institute "Quantum Yang–Mills theory" problem requiring a mathematical proof that,

for any compact semisimple global gauge group, a nontrivial quantum Yang–Mills theory exists on the four-dimensional Minkowski spacetime and has a positive mass gap. Existence includes establishing axiomatic properties at least as strong as Gårding–Whigtman axioms (see Jaffe–Witten [1]).

As such this is a problem of mathematical existence. It does not require a reconstruction of the conventional quantum Yang–Mills theory. (Notably the famous LHC experimental discovery of a "Higgs scalar field" has not verified the hypothetical Higgs mechanism for the origin of a positive mass of classical Yang–Mills fields.)

The proposed mathematically rigorous quantum Yang–Mills theory is relativistic, nonperturbative, and constructive. It does imply a positive mass gap in the spectrum of quantum Yang–Mills Hamiltonian. Actually, the whole spectrum is described qualitatively.

Mathematical foundations of quantum mechanics are the von Neumann theory of (unbounded) Hermitian operators in a Hilbert space and H. Weyl canonical quantization rule. The latter has led to Wigner's quantum statistical mechanics and then to the calculus of annihilation and creation operators in quantum optics (see Agarval–Wolf [2]) on the physics side, and to the theory of pseudodiffefential operators (see Shubin [32, Chap. 4] and Folland [15, Chap. 2]) on the mathematics side.

The symbolic calculus is a far reaching generalization of the classical Heaviside operational calculus for ordinary differential equations. The *nonlinear* Hamiltonian function on the phase space of a classical mechanical system in an Euclidian space may serve, e.g., as normal, or Weyl, or anti-normal symbol of the *linear* Schrödinger partial differential operator for a quantum analogue of the system. The symbol choice depends on the ordering of annihilation and creation operators or equivalently of operators of multiplication and partial differentiation in the Schrödinger operator.

In quantum field theory, the first infinite-dimensional functional Schrödinger equation was introduced by P. Jordan and W. Pauli (*Zur Quantumelectrodynamik ladungsfreier Felder,* Zeitung für Physik, **47** (1928)). Much later, such operators have been used by Schwinger during the 1950's. Yet

"Mathematically, quantum field theory involves integration, and elliptic operators, on infinite-dimensional spaces. Naive attempts to formulate such notions in infinite dimensions lead to all sorts of trouble. To get somewhere, one needs the very delicate constructions considered in physics, constructions that at first sight look rather specialized to many mathematicians. For this reason, together with inherent analytical difficulties that the subject presents, rigorous understanding has tended to lag behind development of physics" (WITTEN [34, p. 346]).

In 1954, GELFAND–MINLOS [16] proposed to solve the Schwinger infinite-dimensional partial differential equations via approximations by finite-dimensional ones with large number of independent variables (see BEREZIN [5, Preface]). Such approximations drastically differ from the customary lattice approximations. Afterwards, a rigorous mathematics of infinite-dimensional partial differential operators has been developed along Gelfand–Minlos lines, in particular, by KREE–RACZKA [25] in the cylindrical formalism. Simultaneously, P. Kree found an alternative formalism of Gelfand nuclear triples (see KREE [24]). It yields a mathematically rigorous symbolic calculus similar to the heuristic real-analytic Agarwal–Wolf calculus (see AGARWAL–WOLF [2]) in quantum optics. An important difference with the standard symbolic calculus of finite-dimensional pseudo-differential operators is that it is based on convergent series expansions rather than asymptotic ones. In general terms, Martineau analytic functionals replace Schwartz distributions. This is a natural framework for infinite-dimensional generalization of Agarwal–Wolf calculus of creation and annihilation operators.

1.2. Issues

1.2.1. Conventions. We use natural units in quantum field theory: Planck's $\hbar$ (relevant for quantum effects), Einstein's c (relevant for relativistic effects), and the energy unit GeV, or the reciprocal length Fermi unit fm (relevant for nuclear physics).

Dimensional homogeneity is maintained carefully. In particular, Fock spaces are built over Hilbert spaces with dimensionless scalar products.

The scaleless Yang–Mills coupling constant is suppressed.

1.2.2. Classical Yang–Mills fields. In the global Hamiltonian (aka temporal) gauge relativistic Yang–Mills equations are known to form a nonlinear hyperbolic system of the 2nd order partial differential equations on the Minkowski space $\mathbf{R}^{1,3}$ with the finite propagation speed $\leqslant 1$ property of solutions dubbed classical Yang–Mills fields. The relativistic invariance of the Yang–Mills Lagrangian implies the relativistic covariance of Yang–Mills fields.

In the 1st order formalism, the Yang -Mills equations form an infinite-dimensional Hamiltonian system (see FADDEEV–SLAVNOV [14, Sec. III.2]). The Hamiltonian equations have a unique global solution for the Cauchy problem with given initial data constrained by dynamically invariant nonlinear partial differential equations (see GOGANOV–KAPITANSKII [21]). Because of the finite propagation speed, global solutions are generated by the solutions with the constrained initial data restricted to the central balls $\mathbb{B} = \mathbb{B}(r)$ of the radius $0 < r < \infty$ in $\mathbb{R}^3$.

In general, initial Cauchy data may be assigned on any space-like hyperplane in Minkowski space with Lorentz orthogonal time axis presumed to be future oriented. Since the proper Lorentz transformations act transitively on the future oriented time-like causal cone, the assigned Cauchy problems are relativistically equivalent.

1.2.3. Classical Yang–Mills Hamiltonian. The relativistic invariance of the classical Yang–Mills action functional of solutions with compactly supported initial data yields the Noether energy-momentum 4-vector $P = (P^\mu)$ on Minkowski space. The functional time-component P^0 is the classical Yang–Mills Hamiltonian and the Euclidean 3-vector (P^1, P^2, P^3) is the Yang–Mills momentum functional.

Constrained initial data with compact supports generate all solutions of Yang–Mills equations and the classical Yang–Mills Hamiltonian is completely defined by them. Hence it is uniquely defined by classical Yang–Mills fields compactly supported by balls $\mathbb{B}$.

By the coordinate scaling invariance of the Yang–Mills action functional, the energy functional P^0 is inversely proportional to r. The functional $H := rP^0$ is the *scaleless Yang–Mills Hamiltonian.* Since H is invariant under scaling transformation, it is the same on all balls $\mathbb{B}(r)$.

The time-independent gauge invariance of Yang–Mills Hamiltonian H in the temporal gauge allows to reduce the nonlinear phase space of constrained initial data to infinite-dimensional *vector space of transversal initial data* in $\mathbb{B}$. Thus the calculus of operators of creation and annihilation in a Fock space over a Hilbert space of such transversal data is used for a quantum Yang–Mills theory.

1.2.4. Canonical quantization. Classical mechanics has provided such a successful framework in physics that it is natural to rephrase physical systems in terms of fixed time degree of freedom which evolve in time (see Reed–Simon [30, p. 215]).

Hamiltonian formulation of classical Yang–Mills theory (see Faddeev–Slavnov [14, Chap. III, Eq. (2.64)]) is such a paraphrase. Its relativistic covariance is implied by the relativistic covariance of the equivalent Yang–Mills equations.

The Hilbert spaces $\mathcal{L}^2_\perp(\mathbb{B}(r))$ of transversal initial data with the scaleless inner products are scaling invariant, and so are the rigged Fock spaces over them.

The scaleless Yang–Mills Hamiltonian functional is considered as the *anti-normal symbol* of the scaleless quantum Yang–Mills Hamiltonians in a rigged Fock spaces over transversal initial data in $\mathbb{B}(r)$. The operators are densely defined in the Fock space over $\mathcal{L}^2_\perp(\mathbb{B}(r))$ and are unitarily equivalent.

Since scaleless the Yang–Mills Hamiltonian is nonnegative, the proposed quantum Yang–Mills Hamiltonians have unique nonnegative Friedrichs operator extensions[1]. By the unitary equivalence, the operators have the same spectrum, the Yang–Mills spectrum the paper‘s title.

Due to gauge invariance, the Yang–Mills Hamiltonian does not contain a positive quadratic form term. However, the *quartic term* of the Hamiltonian functional along with the semisimplicity of the gauge group entail such a quadratic form in the Weyl symbol of quantum Yang–Mills Hamiltonian[2]. The arising mass quadratic form implies that the Friedrichs Hermitian extension of the quantum Yang–Mills Hamiltonian has a countable orthonormal eigenbasis in a Fock space, so that the quantum Yang–Mills spectrum is point and countable.

It is shown that monomial multiparticle eigenstates form a countable eigenbasis in the Fock Hilbert space, so that the Yang–Mills quantum energy-mass spectrum is a countable set of eigenvalues. Furthermore, there is a positive mass gap at the spectrum‘s bottom.

Since scaleless quantum Yang–Mills Hamiltonian is not relativistically invariant, the Yang–Mills mass depends on Lorentz coordinate frame (as in classical special relativity). Restoration of the physical dimension $[L]^{-1}$ of Yang–Mills Hamiltonian implies that the mass gap is proportional to the classical energy level.

1.3. Acknowledgements

I am thankful to Clifford Taubes for warning that the Yang–Mills Hamiltonian was oversimplified in Dynin [11] and Dynin [12]. The error is corrected in the present paper.

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2. ANALYSIS IN BARGMANN–FOCK SPACE

Basic references are Kree [23], Kree [24], and Kree–Raczka [25].

2.1. Kree Rigging of a Bargmann–Fock Space

The complexification $\mathcal{H}^0 := \mathcal{X} \oplus i\mathcal{X}$ of a separable real Hilbert space $\mathcal{X}$ carries the complex conjugation $z := x + iy \mapsto \overline{z} := x - iy$, an antilinear isometric involution.

The conjugation converts the Hilbert space $\mathcal{H}^0$ into the anti-dual space $\overline{\mathcal{H}^0}$. The Hermitian scalar product of $\overline{z}$ and w in $\mathcal{H}^0$ is denoted by $\overline{z}w$ (as in $\mathbb{C}$), a shorthand for Dirac's $\langle z|w\rangle$.

A *nuclear Gelfand sesquilinear rigging* of the Hilbert space $\mathcal{H}^0$ is a triple of dense continuous embeddings (see Gelfand–Vilenkin [17])

$$\mathcal{H} \subset \mathcal{H}^0 \simeq \overline{\mathcal{H}^0} \subset \mathcal{H}^*, \tag{2.1}$$

[1]By Glimm–Jaffe [19], the normal quantization of a nonnegative functional is not necessary a nonnegative operator.
[2]This argument cannot be applied to the photonic Maxwell–Schrödinger operator!

where

(1) A nuclear countably Hilbert space $\mathcal{H}$ is the intersection of a countable nested family of Hilbert spaces

$$\bigcap \mathcal{H}^n, \quad n \geqslant 0, \quad \mathcal{H}^{n+1} \subset \mathcal{H}^n,$$

where the embedding are nuclear linear operators with dense ranges (see Gelfand–Vilenkin [16]). The topology is defined by the simultaneous convergence in all $\mathcal{H}^n$. In fact, $\mathcal{H}$ is a Frechet nuclear space (see Treves [33]).

(2) The strong anti-dual $\mathcal{H}^*$ of $\mathcal{H}$ of continuous antilinear functionals $z^*w := z^*(w)$ on $\mathcal{H}^n$ of continuous antilinear functionals $z^*w := z^*(w)$ is the union of the anti-dual Hilbert spaces $\mathcal{H}^{n*}$ of $\mathcal{H}^n$ with the topology defined by convergence in a $\mathcal{H}^{n*}$. In fact, $\mathcal{H}^*$ is a nuclear LF-space (see Treves [33]).

(3) The equivalence $\mathcal{H}^0 \simeq \overline{\mathcal{H}^0}$ is defined via the Riesz representation of antilinear functionals (see Treves [33]). In particular, $z^* = \overline{z}$ for $z \in \mathcal{H}^0$.

The Bargmann–Fock space $\mathcal{K}(\mathcal{H}^0)$ is the Hilbert space of entire analytic functionals $\Psi(\zeta^*)$ on $\mathcal{H}^*$ that are square integrable with respect to the Gaussian probability measure $d\gamma$ on $\mathcal{H}^*$. The latter is uniquely defined through its pull-offs via finite rank Hermitian projectors $p : \mathcal{H}^* \to \mathcal{H}$ from the finite-dimensional Gaussian probability measures on finite-dimensional complex subspaces $p(\mathcal{H}^*)$

$$d\gamma(p\overline{\zeta}, p\zeta) \;=\; c(p)^{-1} d(p\overline{\zeta})(dp\zeta) e^{-\overline{p\zeta} p\zeta}, \quad c(p) := \int_{p\mathcal{H}^*} d\gamma(p\overline{\zeta}, p\zeta) e^{-\overline{p\zeta} p\zeta} \tag{2.2}$$

(see Gelfand–Vilenkin [17]).

The complex conjugation in $\mathcal{K}(\mathcal{H}^0)$ is $\overline{\Psi}(\zeta) := \overline{\Psi(\overline{\zeta})}$, and the Hermitian scalar product is

$$\langle\, \overline{\Psi} |\, \Phi \,\rangle \;:=\; \int_{\mathcal{H}^*} d\gamma \quad \overline{\Psi}(\zeta)\Phi(\zeta^*).$$

In view of the Fernique theorem (see Kuo [26, Chap. 3, Th. 2.4], the functionals are integrable with respect to the Gaussian measure on $\mathcal{H}^*$.[3]

The *Kree rigging* of the Fock space $\mathcal{K}(\mathcal{H}^0)$ consists of the dense continuous embeddings of complex topological vector spaces (see Kree [24] and [25, Subsec. (5.9])

$$\mathcal{K}(\mathcal{H}^*) \;\subset\; \mathcal{K}(\mathcal{H}^0) \;\subset\; \mathcal{K}(\mathcal{H}), \tag{2.3}$$

where

(1) The nuclear countably-Hilbert space $\mathcal{K}(\mathcal{H}^*)$ is the space of entire holomorphic functionals $\Phi(z^*)$ on $\mathcal{H}^*$ of the first order exponential growth on every $\mathcal{H}^{n*}$.[4]

By Boland [9] and Colombeau [10, Chap. 8, Abstract], the countably Hilbert space $\mathcal{K}(\mathcal{H})$ is nuclear. In particular, it is reflexive.

(2) The space $\mathcal{K}(\mathcal{H})$ is the space of all continuous Gateaux holomorphic functionals $\Psi(w)$ on $\mathcal{H}$ with the topology of uniform convergence on compact subsets of $\mathcal{H}$.

By Boland [9], the space $\mathcal{K}(\mathcal{H})$ is the strong dual of $\mathcal{K}(\mathcal{H}^*)$. Furthermore, $\mathcal{K}(\mathcal{H})$, is a nuclear space (see Treves [33, Proposition 50.6]).

[3]The Fernique theorem per se involves the Wiener space $\mathcal{H}^{1*}$, a carrier of Gaussian probability measure $d\gamma$ in the bigger space $\mathcal{H}^*$.

[4]A functional on $\mathcal{H}$ is *holomorphic* if it is holomorphic on every finite-dimensional subspace and is continuous on every $\mathcal{H}^{n*}$. Since $\mathcal{H}^*$ is a nuclear Silva space, this property is equivalent to the Silva-analyticity in Colombeau [10, Chap. 2].

2.2. Borel Transform

Coherent (*aka exponential*) *states* $e^z(\zeta^*) := e^{\zeta^* z}$, $z \in \mathcal{H}$, have the following well known basic properties[5]

(1) Any $\Psi \in \mathcal{K}(\mathcal{H}^*)$ where it has a unique coherent states expansion

$$\Phi(\overline{\zeta}) \ = \ \int_{\mathcal{K}(\mathcal{H}^*)} d\gamma(\zeta^*, \zeta)\overline{e^z(\zeta^*)}\Phi(\zeta^*). \tag{2.4}$$

This equation is for $\Phi(z^*)$ on the dense subspace $\mathcal{H} \subset \mathcal{H}^*$ where $z^* = \overline{z}$, which defines $\Phi(z^*)$ uniquely.

(2) *Overlap identity*

$$\langle \ e^z \mid e^w \ \rangle \ = \ e^{\overline{z}w}, \quad z, w \in \mathcal{H}. \tag{2.5}$$

Proof. It follows from FOLLAND [15, Chap. 1, Th. (1.63)] that for nonnegative integers m, n we have $\langle z^n | w^n \rangle = n!(\overline{z}w)^n$ and that $\langle z^m | w^n \rangle = 0$ if $m \neq n$. Then

$$\langle e^z \mid e^w \ \rangle \ = \left\langle \ \sum_{m=0}^{\infty} z^m/m! \mid \sum_{n=0}^{\infty} w^n/n! \right\rangle \ = \ e^{\overline{z}w}.$$

(3) *Borel transform F* (see COLOMBEAU [10, Chap. 7], as well as TREVES [33, Chap. 22]) is

$$\Phi(\overline{\zeta}) \mapsto (F\Phi)(\overline{z}) \ := \ \langle \ e^z \mid \Phi \ \rangle, \qquad \Psi(\zeta) \mapsto (F\Psi)(z) \ := \ \langle \ \Psi \mid e^z \ \rangle. \tag{2.6}$$

By (2.5), the Borel transform $F(e^w(\overline{\zeta^*})) = e^w(\overline{z^*}))$. Thus Borel transform induces a linear topological isomorphism of the corresponding riggings of the Fock space (see KREE [24, Th. (2.15)]).

(4) Borel transform intertwines the directional differentiation $\partial_{\overline{z}}$, $z \in \mathcal{H}$, and the multiplication with $\overline{\zeta}z$ in $\mathcal{K}(\mathcal{H})$.

(5) *Sesquilinear Borel transform* in the sesquilinear Kree triple

$$\mathcal{K}(\overline{\mathcal{H}^*} \times \mathcal{H}^*) \ \subset \ \mathcal{K}(\overline{\mathcal{H}^0} \times \mathcal{H}^0) \ \subset \ \mathcal{K}\overline{\mathcal{H}} \times \mathcal{H}) \tag{2.7}$$

is the tensor product of Borel transforms defined by sesquiholomorphic coherent states

$$e^{\overline{\eta_1^* z_1} + \eta_2^* z_2} \ = \ e^{\overline{z_1}}(\overline{\eta_1^*})e^{z_1}(\eta_1^*).$$

2.3. Calculus of Creation and Annihilation Operators

Operators of creation and annihilation are continuous operators of multiplication and complex directional differentiation in $\mathcal{K}(\mathcal{H}^*)$ and $\mathcal{K}(\mathcal{H})$,

$$\widehat{z} \quad \Phi(\zeta^*) \ := \ (\zeta^* z)\Phi(\zeta^*), \quad \widehat{z^*}\Phi(\zeta^*) \ := \ \partial_{z^*}\Phi(\zeta^*), \tag{2.8}$$

$$\widehat{z} \quad \Psi(\zeta) \ := \ \partial_z \Psi(\eta)), \quad \widehat{z^*}\Psi(\eta) \ := \ (z^*\eta)\Psi(\eta). \tag{2.9}$$

(1) The adjoint of a creation operator $\widehat{z}$ is the annihilation operator $\widehat{\overline{z}}$.

(2) Operators $\widehat{z}$ and $\widehat{\overline{z}}$ are continuous in $\mathcal{K}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})^*$ where they act.

[5]They are straightforward on cylindrical states $p\zeta^*$ and then, by strong limits, are extended to all states. Note also that the coherent states are cylindrical.

(3) Canonical bosonic commutation relations (CCR) take the form

$$[\widehat{z_1^*}, \widehat{z_2}] \ = \ z_1^* z_2, \quad [\widehat{z_1^*}, \widehat{z_2}] \ = \ 0, \quad [\widehat{z_1}, \widehat{z_2}] \ = \ 0. \tag{2.10}$$

(4) *Coherent states* e^z are the eigenstates of the annihilation operators: $\widehat{z^*} e^w \ = \ (z^* w) e^z$.

(5) Creators and annihilators generate strongly continuous commutative operator groups in $\mathcal{K}(\mathcal{H}^*)$ and $\mathcal{K}(\mathcal{H})$

$$\widehat{e^z} \Phi(\zeta^*) \ = \ e^{w^* z} \ \Phi(\zeta^*), \quad \widehat{e^{z^*}} \Phi(\zeta^*) \ = \ \Phi(\zeta^* + z^*), \tag{2.11}$$

$$\widehat{e^z} \Psi(\zeta) \ = \ \Psi(\zeta + z), \qquad \widehat{e^{z^*}} \Psi(\zeta) \ = \ e^{\zeta^* z} \Psi(\zeta). \tag{2.12}$$

The operator products $e^{\widehat{z}} e^{\widehat{z^*}}$ and $e^{\widehat{z^*}} e^{\widehat{z}}$ are invertible continuous operators in $\mathcal{K}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H}^*)$.

By Baker–Campbell–Hausdorff formula and the canonical commutation relations (2.10), we have

$$e^{\widehat{z}} e^{\widehat{z^*}} \ = \ e^{\widehat{z} + \widehat{z^*}} e^{z^* z/2}, \qquad e^{\widehat{z^*}} e^{\widehat{z}} \ = \ e^{\widehat{z} + \widehat{z^*}} e^{-z^* z/2}. \tag{2.13}$$

Therefore, the operator $e^{\widehat{z} + \widehat{z^*}}$ is also continuous and invertible in $\mathcal{K}(\mathcal{H})$.

The sesquiholomorphic Borel transform of $\tilde{\Theta} \in \mathcal{K}(\overline{\mathcal{H}} \times \mathcal{H})$

$$\Theta(\overline{z_1}, z_2) \ = \ \langle \tilde{\Theta}(\overline{\eta_1}, \eta_2) \mid e^{\eta_1}(\overline{z_1}) e^{\overline{\eta_2}}(z_2) \ \rangle \tag{2.14}$$

is quantized as the continuous *normal, Weyl, and anti-normal operators* from $\mathcal{K}(\mathcal{H}^*)$ to $\mathcal{K}(\mathcal{H})$ defined by the corresponding normal, Weyl, and anti-normal ordering of creators and annihilators

$$\widehat{\Theta}_\nu \ = \ \Theta_\nu(\widehat{\overline{\eta_1}}, \widehat{\eta_2}) \ := \ \Big\langle \ \tilde{\Theta}_\nu(\overline{\eta_1}, \eta_2) \ \Big| e^{\widehat{\eta_1}} e^{\widehat{\overline{\eta_2}}} \Big\rangle, \tag{2.15}$$

$$\widehat{\Theta}_\omega \ = \ \Theta_\omega(\widehat{\overline{\eta_1}}, \widehat{\eta_2}) \ := \ \Big\langle \ \tilde{\Theta}_\omega(\eta_1, \overline{\eta_2}) \Big| \ e^{\widehat{\eta_1} + \widehat{\overline{\eta_2}}} \ \Big\rangle, \tag{2.16}$$

$$\widehat{\Theta}_\alpha \ = \ \Theta_\alpha(\widehat{\overline{\eta_1}}, \widehat{\eta_2}) \ := \ \Big\langle \ \tilde{\Theta}_\alpha(\eta_1, \overline{\eta_2}) \Big| \ e^{\widehat{\overline{\eta_2}}} e^{\widehat{\eta_1}} \ \Big\rangle. \tag{2.17}$$

The coherent matrix elements

$$\begin{aligned}
\Big\langle \ e^{z_1} \ \Big| \ \widehat{\Theta}_\nu \ \Big| e^{z_2} \ \Big\rangle \ &= \ \Big\langle \ \tilde{\Theta}\nu(\eta_1, \overline{\eta_2}) \ \Big| \ \langle \ e^{z_1} \mid e^{\widehat{\eta_1}} e^{\widehat{\overline{\eta_2}}} \ | e^{z_2} \ \rangle \Big\rangle, \\
\Big\langle \ e^{z_1} \ \Big| \ \widehat{\Theta}_\omega \ \Big| e^{z_2} \ \Big\rangle \ &= \ \Big\langle \ \tilde{\Theta}_\omega(\eta_1, \overline{\eta_2}) \ \Big| \ \langle \ e^{z_1} \mid e^{\widehat{\eta_1} + \widehat{\overline{\eta_2}}} \ | e^{z_2} \ \rangle \Big\rangle, \\
\Big\langle \ e^{z_1} \ \Big| \ \widehat{\Theta}_\alpha \ \Big| e^{z_2} \ \Big\rangle \ &= \ \Big\langle \ \tilde{\Theta}_\alpha(\eta_1, \overline{\eta_2}) \ \Big| \ \langle \ e^{z_1} \mid e^{\widehat{\overline{\eta_2}}} e^{\widehat{\eta_1}} \ | e^{z_2} \ \rangle \Big\rangle
\end{aligned}$$

are well defined. For starters

$$\langle \ e^{z_1} \mid e^{\widehat{\eta_1}} e^{\widehat{\overline{\eta_2}}} \ | e^{z_2} \ \rangle \ = \ \langle \ e^{\widehat{\overline{\eta_1}}} e^{z_1} \mid e^{\widehat{\overline{\eta_2}}} e^{z_2} \ \rangle \ = \ \langle \ e^{\overline{\eta_1} z_1} e^{z_1} \mid e^{\overline{\eta_2} z_2} e^{z_2} \ \rangle \ \overset{(2.5)}{=} \ e^{\overline{z_1} \eta_1 + \overline{\eta_2} z_2} e^{\overline{z_1} z_2}$$

imply via (2.14) that

$$\Big\langle \ e^{z_1} \ \Big| \ \widehat{\Theta}_\nu \ \Big| e^{z_2} \ \Big\rangle \ = \ \langle \tilde{\Theta}_\nu(\overline{\eta_1}, \eta_2) \mid e^{\eta_1}(\overline{z_1}) e^{\overline{\eta_2}}(z_2) \ e^{\overline{z_1} z_2} \ \rangle \ = \ \Theta_\nu(\overline{z_1}, z_2) e^{\overline{z_1} z_2}. \tag{2.18}$$

Since $\Theta_\nu(\overline{z_1}, z_2) e^{\overline{z_1} z_2}$ is sesquiholomorphic on $\overline{\mathcal{H}} \times \mathcal{H}$, it follows that the normal coherent state matrix $\langle \ e^{z_1} \mid e^{\widehat{\eta_1}} e^{\widehat{\overline{\eta_2}}} \ | e^{z_2} \rangle$ is the Grothendieck kernel of a continuous linear operator $\widehat{\Theta}_\nu : \ \mathcal{K}(\mathcal{H}^*) \to \mathcal{K}(\mathcal{H})$ (see Treves [33]).

Vice versa, for any continuous linear operator $Q:\ \mathcal{K}(\mathcal{H}^*) \to \mathcal{K}(\mathcal{H})$, the sesquiholomorphic functional

$$\Theta^Q_\nu(\overline{z_1}, z_2) \ := \ \langle \ e^{z_1} \mid Q \mid e^{z_2} \ \rangle e^{-\overline{z_1} z_2} \tag{2.19}$$

belongs to $\mathcal{K}(\overline{\mathcal{H}} \times \mathcal{H})$ and satisfies

$$\langle \ e^{z_1} \mid Q \mid e^{z_2} \ \rangle \ = \ \Big\langle \ \tilde{\Theta}^Q_\nu(\overline{z_1}, \eta_2) \ \Big| \ \langle \ e^{z_1} \mid e^{\widehat{\eta_1}} e^{\widehat{\overline{\eta_2}}} \mid e^{z_2} \ \rangle \Big\rangle. \tag{2.20}$$

As in (2.13),

$$e^{\widehat{\eta_1}} e^{\widehat{\overline{\eta_2}}} \ = \ e^{\widehat{\eta_1} + \widehat{\overline{\eta_2}}} e^{-\overline{\eta_2}\eta_1/2}, \qquad e^{\widehat{\overline{\eta_2}}} e^{\widehat{\eta_1}} \ = \ e^{\widehat{\eta_1} + \widehat{\overline{\eta_2}}} e^{-\overline{\eta_2}\eta_1/2}, \tag{2.21}$$

so that

$$\tilde{\Theta}_\nu(\overline{\eta_1}, \eta_2) \ = \ e^{\overline{\eta_1}\eta_2/2} \tilde{\Theta}_\omega(\overline{\eta_1}, \eta_2), \qquad \tilde{\Theta}_\omega(\overline{\eta_1}, \eta_2) \ = \ e^{\overline{\eta_1}\eta_2/2} \tilde{\Theta}_\alpha(\overline{\eta_1, \eta_2}). \tag{2.22}$$

Applying the intertwining property of sesquiholomorphic Borel transform, we obtain

$$\Theta_\nu(\overline{z_1}, z_2) \ = \ e^{\overline{\partial_{z_1}}\partial_{z_2}/2} \Theta_\omega(\overline{z_1}, z_2), \quad \Theta_\omega(\overline{z_1}, z_2) \ = \ e^{\overline{\partial_{z_1}}\partial_{z_2}/2} \Theta_\alpha(\overline{z_1}, z_2). \tag{2.23}$$

These equations define $\Theta^Q_\omega(\overline{z_1}, z_2)$ and $\Theta^Q_\alpha(\overline{z_1}, z_2)$ via $\Theta_n u^Q(\overline{z_1}, z_2)$ for all continuous linear operators $Q:\ \mathcal{H} \to \mathcal{H}^*$.

A sesquiholomorphic functional $\Theta(\overline{z_1}, z_2)$ is uniquely defined by its restriction $\sigma(\overline{z}, z)$ to the real diagonal of $\overline{\mathcal{H}} \times \mathcal{H}$. The restrictions

$$\sigma^Q_\nu(\overline{z}, z), \quad \sigma^Q_\omega(\overline{z}, z), \quad \sigma^Q_\alpha(\overline{z}, z) \tag{2.24}$$

are the *normal, Weyl, anti-normal symbols* of an operator Q. The symbols are arbitrary real analytic functionals on the real diagonal of $\overline{\mathcal{H}} \times \mathcal{H}$). Furthermore, under the corresponding ordering of creators and annihilators,

$$Q = \sigma^Q_\nu(\widehat{\overline{z}}, \widehat{z}) \ = \ \sigma^Q_\omega(\widehat{\overline{z}}, \widehat{z}) \ = \ \sigma^Q_\alpha(\widehat{\overline{z}}, \widehat{z}). \tag{2.25}$$

Formulas (2.19) and (2.25) reproduce well-known formulas for finite-dimensional pseudo-differential operators (see Folland [15, Chap. 3, Sec. 7]).

In terms of their matrix elements, the operators $Q:\ \mathcal{K}(\mathcal{H}^*) \to \mathcal{K}^(\mathcal{H})$ are strongly convergent series

$$e^{\widehat{z}} e^{\widehat{\overline{z}}} \ = \ \sum_{m,n=1}^\infty \frac{\widehat{z}^m \widehat{\overline{z}}^n}{m!n!}, \quad e^{\widehat{z} + \widehat{\overline{z}}} \ = \ \sum_{n=1}^\infty \frac{(\widehat{z} + \widehat{\overline{z}})^n}{n!}, \quad e^{\widehat{\overline{z}}} e^{\widehat{z}} \ = \ \sum_{m,n=1}^\infty \frac{\widehat{z}^n \widehat{\overline{z}}^m}{m!n!}. \tag{2.26}$$

Thus any operator $Q:\ \mathcal{K}(\mathcal{H}^*) \to \mathcal{K}^(\mathcal{H})$ is a *partial differential operator of infinite order with holomorphic coefficients* (see Berezin [5, Chap. 1]).

If a symbol $\sigma^Q(\overline{z}, z)$ is a sesquiholomorphic polynomial (i.e., $\Theta^Q(\overline{z_1}, z_2)$ is polynomial on finite-dimensional sesquiholomorphic subspaces of $\overline{\mathcal{H}} \times \mathcal{H}$), then Q is a differential operator of finite order and, therefore, a continuous linear operator in $\mathcal{K}(\mathcal{H}^*)$.

2.3.1. Number operator. Let $\{e_j\} \subset \mathcal{H}$ be an orthonormal basis in $\mathcal{H}^0$, so that any $z \in \mathcal{H}$ has a unique orthogonal expansion $z = \sum_j z_j$ in $\mathcal{H}^0$. This defines a continuous linear *number operator* in $\mathcal{H}$

$$\mathbf{N} \ := \ \sum_j \widehat{\overline{z_j}} \widehat{z_j} : \mathcal{H} \to \mathcal{H} \tag{2.27}$$

that does not depend on orthonormal basis choice.

Further, the number operator is nonnegative the domain $\mathcal{H}$. Its Friedrichs Hermitian extension (again denoted $\mathbf{N}$) has an eigenbasis of monomial functionals $\prod_{k=1}^n (z^* z_{j_k})$ with eigenvalues $n = 0, 1, \ldots$. Thus the spectrum of $\mathbf{N}$ is a point spectrum, the fundamental eigenvalue 0 is simple but all others are infinitely degenerate.

It follows that the diagonal matrix elements satisfy the equality $\langle e^z | \mathbf{N} | e^z \rangle = (\overline{z} z) e^{\overline{z} z}$, so that the symbols of $\mathbf{N}$ are

$$\sigma^{\mathbf{N}}_\nu \ \overset{(2.19)}{=} \ \overline{z} z, \quad \sigma^{\mathbf{N}}_\omega \ \overset{(2.16)}{=} \ \overline{z} z \ - \ 1/2, \quad \sigma^{\mathbf{N}}_\alpha \ \overset{(2.17)}{=} \ \overline{z} z \ - \ 1. \tag{2.28}$$

2.4. Fock Representations of Canonical Commutation Relations

A *Fock representation of canonical commutation relations* over a pre-Hilbert space $\mathcal{H}$ is a set of linear annihilation and creation operators $\widehat{\overline{z}},\ \widehat{z},\ , z \in \mathcal{H}$, in a complex Hilbert space $\mathcal{K}(\mathcal{H}^0)$ defined on a dense subspace $\mathcal{K}(\mathcal{H})$ such that

(1) $\widehat{\overline{z}},\ \widehat{z} : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$.
(2) The nonzero canonical commutators relations

$$[\widehat{\overline{z}}, \widehat{w}] \ = \ \overline{z}w \tag{2.29}$$

are satisfied.
(3) There is a unit fiducial $\Omega \in \mathcal{K}(\mathcal{H})$ such that $\mathcal{K}(\mathcal{H})$ is the linear span of the monomial vectors $\widehat{z}^k\Omega,\ k = 0, 1, \ldots$
(4) $\widehat{\overline{z}}\Omega = 0$.

As is well known (see Glimm–Jaffe [20, Th. 6.3.4.]) for a given $\mathcal{H}$, the Fock representations are irreducible and unitarily equivalent. Furthermore, unitary operators defining the equivalence are completely defined by the correspondence between the Ω's.

By Subsec. 2.3, any Fock representation of canonical commutation relations sets up the corresponding nuclear Gelfand sesquilinear rigging

$$\mathcal{K}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H}^0) \subset \mathcal{K}(\mathcal{H}^*)$$

over the nuclear Gelfand sesquilinear rigging $\mathcal{H} \subset \mathcal{H}^0 \subset \mathcal{H}^*$. Thus, the theory of partial differential operators is transferred to all Fock representations. By the unitary equivalence, the operator calculus formulas are the same (though the unitarily equivalent realizations of the operators depend on the representation).

For a Gelfand sesquilinear rigging $\mathcal{H} \subset \mathcal{H}^0 \subset \mathcal{H}^*$, the real parts $x := (1/\sqrt{2}(z^* + z))$ of $z \in \mathcal{H}^0$ define the real Hilbert spaces $X^0 := \Re\,\mathcal{H}^0$ and then the Gaussian real nuclear Gelfand triple

$$X \ \subset \ X^0 \ \subset \ X'. \tag{2.30}$$

By Glimm–Jaffe [20, Th. 6.3.3]), the creation annihilation operators

$$\widehat{z} \quad \Psi(w') := \big(x \cdot w' - \partial_x\big)\Psi(w'), \qquad \widehat{z^*}\Psi(w') = \partial_x\Psi(w') \tag{2.31}$$

together with $\Omega(x) = 1$ define a irreducible Fock representation of the commutation relations in the complex Hilbert space $\mathcal{K}(\mathcal{H}^0) \ := \ \mathcal{L}^2(\mathcal{H}^*,\ d\gamma$. Then the unitary equivalence with the representation of the commutation relations in $\mathcal{K}(\mathcal{H}^0) \ := \ \mathcal{L}^2(\mathcal{H}^*,\ d\gamma)$ produces the equivalent corresponding Gelfand nuclear triple $\mathcal{K} \subset \ \mathcal{K}(\mathcal{H}^0) \ \subset \ \mathcal{K}(\mathcal{H}^*)$.

By the fundamental von Neumann-Stone theorem, all irreducible representations of canonical commutation relations on a finite-dimensional space $\mathcal{H} = \mathbb{C}^n$ are unitary equivalent, in particular to Fock representations.

The Schrödinger irreducible representation is defined by the Hermitian unbounded operators $\hat{x},\ x \in \mathbb{R}^n = \Re\mathbb{C}^n$ of the directional multiplications and the operators $\hat{y},\ y \in \mathbb{R}^n = \Im\mathbb{C}^n$, of the directional derivatives in the complex Hilbert space $\mathcal{L}^2(\mathbb{R}^n, dw)$:

$$\hat{x}f(w) := (x \cdot w)f(w), \qquad \hat{y}f(w) := -i(\partial_e \cdot w)f(w). \tag{2.32}$$

The nonzero commutation relation

$$[\hat{y}, \hat{x}] = -i(y \cdot x)1 \tag{2.33}$$

leads to the creation and annihilation operators $\widehat{z}$ and $\widehat{z^*}$ for $z := (1/\sqrt{2})(x + iy),\ z^* := (1/\sqrt{2})(x - iy)$ in $\mathcal{L}^2(\mathbb{R}^n, dx)$ are

$$\widehat{z} \ : = (1/\sqrt{2})(\hat{x} + i\hat{y}), \quad \widehat{z^*} \ : = (1/\sqrt{2})(\hat{x} - i\hat{y}). \tag{2.34}$$

One may choose a Schrödinger unit fiducial state as $\Omega(w) := (2\pi)^{-n/2}e^{-w \cdot w/2}$.

The fiducial transformation $(2\pi)^{-n/2}e^{-w \cdot w/2} \mapsto 1$ extends to the unitary isomorphism between $\mathcal{L}^2(\mathbf{R}^n,\ d\gamma)$ and $\mathcal{L}^2(\mathbf{R}^n,\ d\gamma)$.

Lemma 2.1. *If the anti-normal symbol σ^Q_α is nonnegative, then Q is a nonnegative operator.*

Proof. It suffices to prove the lemma in the cylindrical formalism. There it is equivalent to Shubin [32, Proposition 24.1] for the Schrödinger representation of operators on $\mathbf{R}^n$.

Lemma 2.2. *If the Weyl symbol $\sigma^Q_\omega = f(\Re z)$ does not depend on $\overline{z}$, then Q is the operator of multiplication with $f \in \mathcal{K}(\mathcal{H})$.*

Proof. It suffices to prove the lemma in the cylindrical formalism. There it is equivalent to Folland [15, Proposition (2.8)] for Schrödinger representation of operators on $\mathbf{R}^n$.

2.5. Cylindrical formalism

A Hermitian finite-dimensional projector $p:\ \mathcal{H}^* \to \mathcal{H}$ of rank n induces the *cylindrical projector*

$$P\Phi(z^*) \ := \ \Phi(pz^*), \quad P\Psi(w) \ := \ \Psi(pw).$$

in $\mathcal{KH}^*$ and $\mathcal{KH}$.

The range of a rank n projector p is naturally isomorphic to $\mathbf{C}^n$. Therefore, the cylindrical nuclear Gelfand triples $P(\mathcal{K}(\mathcal{H}^*) \subset P\mathcal{K}(\mathcal{H}^0) \subset P\mathcal{K}(\mathcal{H})$ are equivalent to the Kree triples over the finite-dimensional triples $\overline{\mathbf{C}}^n \subset \mathbf{C}^n \simeq \overline{\mathbf{C}}^n \subset \mathbf{C}^n$ (see Kree–Raczka [25]).

The compressions of a continuous linear operator $Q : \mathcal{K}(\mathcal{H}^*) \to \mathcal{K}(\mathcal{H})$ are *cylindrical* operators $PQP : P\mathcal{K}(\mathcal{H}^*) \to P\mathcal{K}(\mathcal{H})$. Their coherent matrix elements,

$$\langle e^z \mid PQP \mid e^z \rangle \ = \ \langle \ e^{pz} \mid Q \mid e^{pz} \ \rangle \tag{2.35}$$

define continuous linear operators from $\mathcal{K}(\mathbf{C}^n)$ to $\mathcal{K}(\overline{\mathbf{C}}^n)$, i.e., partial differential operators of infinite order on $\mathbf{C}^n$.

Theorem 2.1. *Operator Q is the strong limit of the cylindrical differential operators PQP as the Hermitian projectors p converge strongly to the unit operator in $\mathcal{K}(\mathcal{H}^*)$.*

Proof. The coherent matrix elements $\langle \Psi^* | Q | \Phi \ \rangle$ are separately continuous sesquilinear forms on the Frechet space $\mathcal{K}(\mathcal{H}^*)$. By a Banach theorem (see [29, Th. V.7]), the sesquilinear form is actually continuous on $\mathcal{K}(\mathcal{H})$. In particular, the operator Q is the weak limit of PQP in $\mathcal{K}(\mathcal{H})$. The nuclear space $\mathcal{K}(\mathcal{H}^*)$ is a Montel space (see Treves [33, Proposition 50.2]). Hence the weak convergence implies the strong one in the topology of $\mathcal{K}(\mathcal{H}^*)$.

As $n \to \infty$, the matrix elements satisfy the equalities

$$\langle e^{\overline{z}} \mid PQP \mid e^w \rangle \ = \ \langle \ e^{p\overline{z}} \mid Q \mid e^{pw} \ \rangle \ \to \ \langle e^{z^*} \mid Q \mid e^w \rangle, \tag{2.36}$$

so that coherent matrix elements of the cylindrical PQP converge to the coherent matrix elements of Q.

Agarwal–Wolf [2] have developed a comprehensive calculus of operators of creation and annihilation in finite dimension. It is quite straightforward to make it mathematically rigorous and then to extend to quantum field theory via cylindrical approximations. This would be another way to deduce the results of this section[6]

3. CLASSICAL YANG–MILLS THEORY

3.1. Yang–Mills Fields

The *global gauge group* $\mathbb{G}$ of a Yang–Mills theory is a connected semisimple compact Lie group with the Lie algebra $\mathfrak{g}$ of skew-symmetric matrices $X = -X'$.

[6] Their formulas are somewhat, different, because they use the symplectic Fourier transform on the phase space. However, the translation of formulas to the language of sesquiholomorphic Borel transform is straightforward (see Folland [15, p. 7]).

The Lie algebra carries the *adjoint representation* $\mathrm{Ad}\,(g)X = gXg^{-1}, g \in \mathbb{G}, X \in \mathfrak{g}$, of the group $\mathbb{G}$ and the corresponding selfrepresentation $\mathrm{ad}(X)Y = [X, Y],\ X, Y \in \mathfrak{g}$. The adjoint representation is orthogonal with respect to the *positive definite* Ad-invariant scalar product

$$X \cdot Y \ := \ \mathrm{trace}(\mathrm{ad}X'\mathrm{ad}Y) \ = \ -\mathrm{trace}(\mathrm{ad}X\mathrm{ad}Y), \tag{3.1}$$

the negative Killing form on $\mathfrak{g}$.

There exists an orthonormal basis $\{T_k\}$ in $\mathfrak{g}$ such that

$$[T_i, T_j] \ = \ c_{ij}^k T_k, \tag{3.2}$$

with the structure constants c_{ij}^k skew-symmetric with respect to the interchanges of all three indices i, j, k. (Summation over repeated indices is assumed throughout.)

Let the Minkowski space $\mathbb{R}^{1,3}$ be oriented and time oriented with the Minkowski metric signature $(1, -1, -1, -1)$. In a Minkowski coordinate system $x^\mu, \mu = 0, 1, 2, 3$, the metric tensor is diagonal. In the natural unit system, the time coordinate x^0 is t. Thus $(x^\mu) = (t, x^i),\ i = 1, 2, 3$.

The *local gauge group* $\widetilde{\mathbb{G}}$ is the group of infinitely differentiable $\mathbb{G}$-valued functions $g(x)$ on $\mathbb{R}^{1,3}$ with the pointwise group multiplication. The *local gauge Lie algebra* $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$-valued functions on $\mathbb{R}^{1,3}$ with the pointwise Lie bracket consists of infinitely differentiable $\mathfrak{g}$-valued functions on $\mathbb{R}^{1,3}$ with the pointwise Lie bracket.

The $\widetilde{\mathbb{G}}$ acts via the pointwise adjoint action on $\widetilde{H}$ and correspondingly on $\mathcal{A}$, the real vector space of *gauge fields* $A = A_\mu(x) \in \tilde{\mathfrak{g}}$.

The gauge fields A define the *covariant partial derivatives*

$$\partial_{A\mu} X \ := \ \partial_\mu X - \mathrm{ad}(A_\mu)X, \quad X \in \widetilde{H}. \tag{3.3}$$

This definition shows that in, the natural units, *gauge connections have the mass dimension* $1/[L]$.

Any $\tilde{g} \in \widetilde{\mathbb{G}}$ defines the affine *gauge transformation*

$$A_\mu \mapsto A_\mu^{\tilde{g}} : = \ -\mathrm{Ad}\,(\tilde{g})A_\mu - (\partial_\mu \tilde{g})\tilde{g}^{-1}, \quad A \in \mathcal{A}, \tag{3.4}$$

so that $A^{\tilde{g}_1} A^{\tilde{g}_2} = A^{\tilde{g}_1 \tilde{g}_2}$.

The relativistic Yang–Mills *curvature* $F(A)$ is the antisymmetric tensor[7]

$$F(A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]. \tag{3.5}$$

The curvature is gauge invariant:

$$\mathrm{Ad}\,(g)F(A) \ = \ F(A^g), \tag{3.6}$$

as well as the *Yang–Mills Lagrangian*

$$(1/4)F(A)^{\mu\nu} \cdot F(A)_{\mu\nu}. \tag{3.7}$$

The corresponding gauge invariant Euler–Lagrange equation is the 2nd order nonlinear partial differential equation $\partial_{A\mu} F(A)^{\mu\nu} = 0$, called the *Yang–Mills equation*

$$\partial_\mu F^{\mu\nu} \ - \ [A_\mu, F^{\mu\nu}] \ = \ 0. \tag{3.8}$$

Yang–Mills fields are solutions of the Yang–Mills equation.

[7]The scaleless Yang–Mills coupling constant is supressed.

3.2. Yang–Mills Phase Space

In the temporal gauge $A_0(t,x^k)=0$, the 2nd order Yang–Mills equation (3.8) is equivalent to the 1st order hyperbolic system for the time-dependent $A_j(t,x^k)$, $E_j(t,x^k) := F_{0,x^k}$ on $\mathbb{B}$ (see Goganov–Kapitanskii [21, Eq. (1.3)])

$$\partial_t A_k \ = \ E_k, \quad \partial_t E_k \ = \ \partial_j F_k^j - [A_j, F_k^j], \qquad F_k^j \ = \ \partial^j A_k - \partial_k A^j - [A^j, A_k]. \tag{3.9}$$

and the *constraint equations*

$$[A^k, E_k] \ = \ \partial^k E_k, \quad \text{i.e.,} \quad \partial_{k,A} E_k \ = \ 0. \tag{3.10}$$

By Goganov–Kapitanskii [21], the evolution system is a semilinear first order partial differential system with *finite propagation speed* of the initial data, and the initial problem for it with constrained initial data at $t=0$

$$a_k(x) \ := \ A(0,x_k), \ e_k(x) \ := \ E(0,x_k), \qquad \partial^k e_k = [a_k, e_{,k}\,] \tag{3.11}$$

is *globally and uniquely solvable* in local Sobolev spaces on the whole Minkowski space $\mathbb{R}^{1,3}$ (with no restrictions at the space infinity).

This fundamental theorem has been derived via Ladyzhenskaya method (see [21]) by a reduction to initial data with compact supports. If the constraint equations are satisfied at $t=0$, then, by (3.9), they are satisfied for all t automatically. Thus the *1st order evolution system along with the constraint equations for initial data is equivalent to the 2nd order Yang–Mills system*. Moreover, the constraint equations are invariant under *time independent* gauge transformations.

Sobolev-Hilbert spaces $\mathcal{A}^s, -\infty < s < \infty$, of (generalized) connections $a(x)$ are the completions of the spaces of smooth connections with compact supports in open balls $\mathbb{B}$ of radius r with respect to the norms

$$|a|^2 := \int_{\mathbb{B}} dx \,\big(a\cdot(1-\triangle)^s a\big) < \infty. \tag{3.12}$$

They define the real Gelfand nuclear triple (see [16])

$$\mathcal{A} : \mathcal{A} \ := \bigcap \mathcal{A}^s \ \subset \mathcal{A}^0 \ \subset \ \mathcal{A}^* := \bigcup \mathcal{A}^s, \tag{3.13}$$

where $\mathcal{A}$ is a *real* nuclear Frechet space with the dual A^*.

Similarly, we define the chain of Sobolev-Hilbert spaces $\mathcal{S}^s, -\infty < s < \infty$, of (generalized) scalar fields $u(x)$ on $\mathbb{B}$ with values in Ad $\mathbb{G}$ and the Hilbert norms $|u|$. Let

$$\mathcal{S} : \mathcal{S} \ := \bigcap \mathcal{S}^s \ \subset \mathcal{S}^0 \ \subset \ \mathcal{S}^* := \bigcup \mathcal{S}^s \tag{3.14}$$

be the corresponding Gelfand rigging.

Let $a \in \mathcal{A}^{s+3}$, $s \geqslant 0$. Then, by Sobolev embedding theorem, a is continuously $s+2$-differentiable on $\mathbb{B}$ and, therefore, the following gauged vector calculus operators are continuous:

(1) *Gauged gradient* $\mathrm{grad}^{[a]} \ : \ \mathcal{S}^{s+1} \to \mathcal{A}^s$,

$$\mathrm{grad}_k^{[a]} u \ := \ \partial_k u - [a_k, u]. \tag{3.15}$$

(2) *Gauged divergence* $\mathrm{div}^{[a]} \ : \ \mathcal{A}^{s+1} \to \mathcal{S}^s$,

$$\mathrm{div}^{[a]}\ b \ := \ \mathrm{div}\ b - [a\ ;\ b], \quad [a\ ;\ b] := [a_k, b_k]. \tag{3.16}$$

(3) *Gauged curl* $\mathrm{curl}^{[a]} \ : \ \mathcal{A}^{s+1} \to \mathcal{A}^s$,

$$\mathrm{curl}^{[a]} b \ := \ \mathrm{curl}\ b - [a \overset{\times}{,} b], \quad [a \overset{\times}{,} b]_i \ := \ \varepsilon_{ij}^k\ [a_j, b_k]. \tag{3.17}$$

(4) *Gauged Laplacian* $\triangle^{[a]} : \ \mathcal{S}^{s+2} \to \mathcal{S}^s$,

$$\triangle^{[a]} u \ := \ \mathrm{div}^{[a]}(\mathrm{grad}^{[a]} u). \tag{3.18}$$

The adjoints of the gauged operators are

$$\mathrm{grad}^{a*} = \ -\mathrm{div}^{[a]}. \tag{3.19}$$

Lemma 3.1. *If* $a \in \mathcal{A}^{s+3}, s \geqslant 0,$ *then the operator* $\operatorname{div}^{[a]} : \mathcal{A}^{s+1} \to \mathcal{A}^s$ *is surjective.*

Proof. Let $\overset{\circ}{\mathcal{S}}{}^{s+2}$, $s \geqslant 0$, denote the closure in $\mathcal{S}^{s+2}$ of the space of u's with compact support in the interior of $\mathbb{B}$. The conventional Laplacian $\triangle^0 : \overset{\circ}{\mathcal{S}}{}^{s+2} \to \mathcal{S}^s$ is an isomorphism (see Agranovich [3]).

The gauged Laplacian $\triangle^{[a]}$ differs from the usual Laplacian $\triangle^0$ by first order differential operators, and, therefore, is a Fredholm operator of zero index from $\overset{\circ}{\mathcal{S}}{}^{s+2}$ to $\mathcal{S}^s$, $s \geqslant 0$.

If $\triangle^{[a]}u = 0$, then the $*$-product $(\triangle^{[a]}u)^*u$ is equal to $(\operatorname{grad}^{[a]} u)^*(\operatorname{grad}^{[a]} u)$ so that $\operatorname{grad} u = [a, u]$. The computation

$$(1/2)\partial_k(u \cdot u) = (\partial_k u \cdot u) = [a_k, u] \cdot u = -\operatorname{trace}(a_k uu - ua_k u) = 0 \tag{3.20}$$

shows that the solutions $u \in \overset{\circ}{\mathcal{S}}{}^{s+2}$ are constant. Because they vanish on the ball boundary, they vanish on the whole ball. Since the index of the Fredholm operator $\triangle^{[a]}$ is zero, its range is a closed subspace of codimension equal to the dimension of its null space. Thus the operator $\operatorname{div}^{[a]}\operatorname{grad}^{[a]}$ is surjective, and so is $\operatorname{div}^{[a]}$.

Consider the bundles $\mathcal{H}^s, s \geqslant 0$, of constraint initial data with the base $\mathcal{A}$ and the null vector spaces $\mathcal{E}_a^{s+1}$ of the operators $\operatorname{div}^{[a]} : \mathcal{E}^{s+1} \to \mathcal{E}^s$ as fibers over $a \in \mathcal{A}$.

Their intersection $\mathcal{H}$ is a vector bundle of nuclear countably Hilbert spaces over the nuclear countably Hilbert base $\mathcal{A}$. Together with the unions of the dual spaces $\mathcal{H}^{-s}$, they form a bundle of nuclear Gelfand triples $\mathcal{H}$ over the same base.

Proposition 3.1. *The bundle* $\mathcal{H}$ *is smoothly*[8] *trivial, so that the total space of* $\mathcal{H}$ *is smoothly isomorphic to the Hilbert direct product of its base* $\mathcal{A}$ *and transversal fiber* $\mathcal{E}_\perp := \operatorname{div}(\mathrm{e}) = 0$ *over* $a = 0$.

Proof. The equation for $(a, e) \in \mathcal{H} \times \mathcal{H}$

$$\operatorname{div}^{[a]}e - \operatorname{div}^{[0]}e \;=\; 0, \quad \operatorname{div}^{[0]}e = \operatorname{div} e, \tag{3.21}$$

is satisfied on the vector space $(0, e) :=\; \operatorname{div} e = 0$.

Furthermore, the mapping $\phi(a, e) : \operatorname{div}^{[a]}e - \operatorname{div}^0 e$ uniformly satisfies the conditions of the Nash–Moser implicit function theorem in the form of Raymond [28] in a neighborhood of $\mathcal{E}_\perp$ (the only nonroutine condition of surjectivity of the partial Frechet derivative $\partial_e\phi(a, e)$ is provided by Lemma 3.1). Hence there exists a smooth explicit mapping $e = e(h)$ on that neighborhood such that $(a, e(a))$ solves equation (3.21).

The mappings $e = e(h)$ are smooth local trivializations of the vector bundle $\mathcal{H}$. It is associated with the locally trivial bundle of smooth $*$-orthonormal frames in the fibers (with respect to the scalar product in $\mathcal{E}_\perp^0$).

Since the Frechet space $\mathcal{A}$ is paracompact, its smooth homothety retraction to the origin $a = 0$ has a homotopy lifting to the space of frames (see Nash–Sen [27, Sec. 7.6]). Thus the bundle $\mathcal{H}$ is trivial, so that the total set of constraint initial data is converted to the Hilbert space $\mathcal{A} \times \; \mathcal{H}_{a=0}$ with the flat parallel transport preserving $e \cdot e$.

3.3. Classical Yang–Mills Hamiltonian

The *scaleless Yang–Mills Hamiltonian* with the concealed coupling constant is (see Glassey–Strauss [18, Eq. (10)])

$$H(a, e) := \; (r/2)\int_{\mathbb{B}(r)} d^3x \quad \big((\operatorname{curl} a - [a \overset{\times}{,} a]) \cdot (\operatorname{curl} a - [a \overset{\times}{,} a]) + e \cdot e\big), \quad (a, e) \in \mathcal{A} \times \mathcal{E}, \tag{3.22}$$

[8]In this paper, smooth = infinitely differentiable.

where $[a\overset{\times}{,}a]$ is the vector field with i-th components $\varepsilon_{ij}^{k}[a_j, a_k]$, is invariant with respect to the scaling $x \to sx,\ x \in \mathbf{R}^3, 0 < s < \infty$. (The factor r makes the functional scaleless.)

The integrand of

$$H_1(a) := \ (r/2) \int_{\mathbb{B}(r)} d^3x \quad \big((\operatorname{curl} a - [a\overset{\times}{,}a]) \cdot (\operatorname{curl} a - [a\overset{\times}{,}a])\big), \tag{3.23}$$

is the curvature of the time-independent gauge fields $a(x)$. Thus $H(a)$ is invariant under time-independent gauge transformations and, therefore, is constant on each orbit of the smooth local time-independent gauge group.

By Dell'antoniio–Zwanziger [13, Proposition 1], the closure of the local gauge Lie group $\widetilde{\mathbb{G}}^1$ in the Sobolev space $\mathcal{A}^1$ is an infinite-dimenssional compact group with a continuous action in the Hilbert space $\mathcal{A}^0$. The action orbits are compact, so that the squared continuous Hilbert norm $\|a\|^2$ has an absolute minimal value on every orbit. The minimal values are attained at transversal $a:$ div a $= 0$ (in the distributional sense).

By the Sobolev embedding theorem, $\mathcal{A}^1 \subset \mathcal{L}^6(\mathbb{B})$. Therefore, the functional $H_1(a)$ has a unique continuation to $\widetilde{\mathbb{G}}^1$-orbits in $\mathcal{A}^0$ and is constant on each of them.

All in all, the following proposition holds.

Proposition 3.2. *The constrained Yang–Mills Hamiltonian $H(a, e)$ is completely determined by its restriction to the countably Hilbert vector space $\mathcal{H}:\ \mathcal{A}_\perp \times \mathcal{E}_\perp$ of transversal constrained vector fields (a, e).*

The complex Yang–Mills fields

$$z \ := \ (r/\sqrt{2})(a + ire), \quad \overline{z} \ := \ (r/\sqrt{2})(a - ire) \tag{3.24}$$

are scaleless, as well as their Hermitian product

$$\overline{z}z \ := \ (1/r^3) \int_{\mathbb{B}(r)} d^3x \overline{z}(x) \cdot z(x). \tag{3.25}$$

Let

$$\mathcal{Z} \ := \ \mathcal{A}_\perp + i\mathcal{E}_\perp \ \subset \ \mathcal{Z}^0 \ := \ \mathcal{A}_\perp^0 + i\mathcal{E}_\perp^0 \ \subset \ \mathcal{Z}^* \ := \ \mathcal{A}_\perp^* - i\mathcal{E}_\perp^* \tag{3.26}$$

be the corresponding scaleless Gelfand triple.

4. MATHEMATICAL QUANTUM YANG–MILLS THEORY

4.1. Quantization of Classical Yang–Mills Fields

The Yang–Mills equation (3.9) is equivalent to an infinite-dimensional Hamiltonian system for $a(t) := A(t, x)$ and $e(t)) = \partial_t A(t, x)$ on the reduced phase space (see Faddeev–Slavnov [14, Chap. III, Eqs. (2.64)])

$$\partial_t a(t) \ = \ -\partial_{e(t)} H(a(t), e(t)), \quad \partial_t e(t) \ = \ \partial_{a(t)} H(a(t), e(t)). \tag{4.1}$$

By the equivalence, solutions exist for all t and are uniquely defined by the initial data $a(0), e(0)$.

Let

$$z(t) \ = \big(a(t) + ie(t)\big)/\sqrt{2}, \quad \overline{z}(t) \ = \big(a(t) - ie(t)\big)/\sqrt{2}. \tag{4.2}$$

By the Segal quantization (see Reed–Simon [30, Sec. X.7]), the compactly supported classical fields $a(t)$ and $e(t)$ in $\mathbb{B}$ are quantized as the symmetric operators in $\mathcal{K}(\mathcal{H}^*)$

$$\widehat{a}(t) \ := \ (\widehat{z}(t) + \widehat{\overline{z}}(t))/\sqrt{2}, \quad \widehat{e}(t) \ := \ -i(\widehat{z}(t) - \widehat{\overline{z}}(t))/\sqrt{2} \tag{4.3}$$

understood as local quantum fields on Minkowski space.

Adapted Gårding–Wightman axioms.

(1) For a proof of essential self-adjointness of $\widehat{a}(t), -i\widehat{e}(t)$, irreducibility of the set of all local quantum Yang–Mills fields, uniqueness (up to phase transformations) of the vacuum Fock state, and bosonic canonical commutations relations, see Reed–Simon [30, Th. X.41]. The corresponding Hermitian extensions in $\mathcal{K}(\mathcal{H}^0)$ are still denoted $\widehat{a}(t), -i\widehat{e}(t)$.

(2) Each of normal, Weyl, and anti-normal symbols of the operator $\widehat{a}(t)$ is $\overline{z}(t) + z(t)$ satisfies Yang–Mills equations on Minkowski space. Therefore, the local Yang–Mills quantum fields do the same.

(3) Poincaré covariance of local Yang–Mills quantum fields follows from the covariance of their symbol.

(4) Similarly, the evolution of the symbol has the finite propagation speed property, and so does the evolution of quantum Yang–Mills fields, as required by Einstein's causality principle.

Furthermore, if the compact supports of $z(t_1)$ and $z(t_2)$ on $\mathbb{R}^3$ are disjoint, then the commutators

$$[\widehat{a}(t_1), \widehat{a}(t_2)] \;=\; \left[\widehat{z}(t_1) + \widehat{\overline{z}}(t_1)/\sqrt{2},\; \widehat{z}(t_2) + \widehat{\overline{z}}(t_2)/\sqrt{2}\right] \;=\; i\,\Im(\overline{z}(t_1)z(t_2)) \;=\; 0. \tag{4.4}$$

(5) The spectral positivity of quantum Yang–Mills Hamiltonian follows from the nonnegativity of its anti-normal symbol.

4.2. Symbols of the Quantum Yang–Mills Hamiltonian

Since the Yang–Mills Hamiltonian $H = rP^0$ is scaling invariant, it is sufficient to take $r = 1$.

The quantum Yang–Mills Hamiltonian is the infinite-dimensional partial differential operator $\widehat{H} : \mathcal{K}(\mathcal{H}^*) \to \mathcal{K}(\mathcal{H})$ with the classical Yang–Mills Hamiltonian ((3.22), $r = 1$) as the anti-normal symbol

$$\sigma_\alpha^{\widehat{H}} := H(\overline{z}, z) = H(a, e) =: H_1(a) + H_2(e). \tag{4.5}$$

The following Proposition is crucial (note that $a = (1/\sqrt{2})(\overline{z} + z),\; e = (i/\sqrt{2})(\overline{z} - z)$).

Proposition 4.1. *The Weyl and normal symbols of the anti-normal quantum Yang–Mills Hamiltonian $\widehat{H}$ are equal to*

$$\sigma_\omega^{\widehat{H}} \;=\; H_1(a) \;+\; (1/2)\|a\|^2 \;+\; \overline{z}z \;+\; 9/16, \tag{4.6}$$

$$\sigma_\nu^{\widehat{H}} \;=\; H_(a, e) \;+\; \|a\|^2 \;+\; 24/16. \tag{4.7}$$

Proof. In view of cylindrical approximations, it suffices to verify Eq. (4.6) in the Schrödinger representation of the canonical commutation relations over finite-dimensional Euclidean spaces.

The operator $\partial_{\overline{z}*}\partial_z$ has the form $(1/2)(\Delta_a + \Delta_e)$, where the Laplacians are defined with respect to the Killing form on the gauge Lie algebra $\mathfrak{g}$.

By (2.23),

$$\sigma_\omega^{\widehat{H}}(a, e) \;=\; \Big(1 \;+\; \frac{\Delta_a/2}{2} \;+\; \frac{(\Delta_a/2)^2}{2}\Big)H_1(a) \;+\; \Big(1 + \frac{\Delta_e/2}{2}\Big)H_2(e). \tag{4.8}$$

Differentiation with respect to x is a continuous linear operator in $\mathcal{H}^*$. It acts naturally in $\mathcal{K}(\mathcal{H}^*)$, commuting with Δ_a and Δ_e. Hence, by the Leibniz rule for differentiation with respect to a, we have

$$\begin{aligned} \Delta_a(\operatorname{curl} a \cdot \operatorname{curl} a) &= \Delta_a(\operatorname{curl} a) \cdot \operatorname{curl} a + 2\nabla_a \operatorname{curl} a \cdot \nabla_a \operatorname{curl} a + \operatorname{curl} a \cdot \Delta_a \operatorname{curl} a \\ &= (\operatorname{curl} \Delta_a a) \cdot \operatorname{curl} a + 2\operatorname{curl} \nabla_a a \cdot \nabla_a \operatorname{curl} a + \operatorname{curl} a \cdot \operatorname{curl} \Delta_a a \;=\; 0, \end{aligned} \tag{4.9}$$

where in the third line the partial differentiations with respect to the variables x and a are interchanged and $\Delta_a a = 0$ and $\nabla_a a$ is a constant matrix field.

Similarly,

$$\Delta_a\big(\operatorname{curl} a \cdot [a \overset{\times}{,} a]\big) = \operatorname{curl} \Delta_a a \cdot [a \overset{\times}{,} a] + 2(\operatorname{curl} \nabla_a a) \cdot \nabla_a [a \overset{\times}{,} a] + a \cdot \operatorname{curl} \Delta_a [a \overset{\times}{,} a] \ = \ 0, \tag{4.10}$$

since $\Delta_a a = 0$, curl is a symmetric differential operator, and $\nabla_a [a \overset{\times}{,} a]$ is a constant matrix field.

Next, in the orthonormal basis T_k (3.2) of $\mathfrak{g}$, the structure constants c_{ij}^k of the semisimple Lie algebra are totally anti-symmetric. Thus

$$[a_i, a_j] = \sum_k c_{ijk} a_i^k a_j^k, \quad a_i = a_i^k T_k. \tag{4.11}$$

Then, by SIMON ([31, p. 217]),

$$[a_i, a_j] \cdot [a_i, a_j] \ = \ \sum_k (a_i^k a_j^k c_{ij}^k)^2. \tag{4.12}$$

In the Cartesian coordinates $\alpha_i^k(x)$, the differential operator Δ_a is the standard Laplacian with respect to a independently of x so that, by SIMON ([31, page 217]),

$$\Delta_a H_1(a) \ = \ \int_{\mathbb{B}} dx \quad (1/2)\partial^2/\partial(\alpha_i^k(x))^2 2 \sum_k (\alpha_i^k \alpha_j^k c_{ijk})^2(x). \tag{4.13}$$

The skew-symmetry of c_{ij}^k implies that $\sum_k \alpha_i^k \alpha_j^k c_{ij}^k$ does not contain $(\alpha_i^k)^2$. Then, by the Leibniz rule,

$$\begin{aligned}
&\partial^2/\partial(\alpha_i^k)^2 \sum_k \left(\alpha_i^k \alpha_j^k c_{ijk}\right)^2 \\
&= \ 2\partial^2/\partial(\alpha_i^k)^2 \sum_k \left(\alpha_i^k \alpha_j^k c_{ijk}\right) \sum_k \left(\alpha_i^k \alpha_j^k c_{ijk}\right) + \ 2 \left(\partial/\partial\alpha_i^k \sum_k \alpha_i^k \alpha_j^k c_{ij}^k\right) \left(\partial/\partial\alpha_i^k \sum_k \alpha_i^k \alpha_j^k c_{ij}^k\right) \\
&= \ 2 \sum_{ijkl} \alpha_i^k c_{ij}^k \alpha_j^l c_{ljk}(x) \ = \ 2a(x) \cdot a(x) \quad \text{(see SIMON [31, p. 217])}.
\end{aligned}$$

Thus

$$\Delta_a H_1(a) \ = \ \int_{\mathbb{B}} dx\, a \cdot a \ = \|a\|^2, \quad \Delta_a^2 H_1(a) = \Delta_a \|a\|^2 = 2. \tag{4.14}$$

Furthermore,

$$\Delta_e H_2(e) \ = \ \Delta_e \|e\|^2 \ = \ 2. \tag{4.15}$$

Equations (4.8), (4.9), (4.10), (4.13), (4.14), (4.15) entail equation (4.6) of the Proposition for the Weyl symbol $\sigma_\omega^{\widehat{H}}$.

Equation (4.7) for the normal symbol $\sigma_\omega^{\widehat{H}}$ is derived in the same way but, in view of (2.23), with $(1/2)\Delta_a$, $(1/2)\Delta_e$ in (4.8) replaced by Δ_a, Δ_e.

4.3. Spectrum of the Quantum Yang–Mills Hamiltonian

The anti-normal symbol of the quantum Yang–Mills Hamiltonian $\widehat{H}$ is nonnegative so that, by Lemma 2.1, $\widehat{H}$ is nonnegative and, in particular, symmetric on the dense domain $\mathcal{H}$ in the Hilbert space $\mathcal{H}^0$. As such, it has the Friedrichs Hermitian extension $\widehat{H}$ (by abuse of notation, it is transferred to the Hermitian operator). Now the spectrum of $\widehat{H}$ is uniquely defined.

By definition, a *positive mass gap* means that the lowest eigenvalue is simple and is an isolated spectral value corresponding to the physical vacuum. Then the lowest boundary value of the rest of the spectrum represents the *physical mass*.

Theorem 4.1. *The Yang–Mills Schrödinger spectrum has a positive mass gap $\geqslant 9/16$.*

Proof. The lowest spectral value of $\widehat{H}$

$$\lambda_1(\widehat{H}) \ \leqslant\ \inf \sigma_\nu^{\widehat{H}}(\overline{z}, z) \ \overset{(4.5)}{=}\ 24/16, \tag{4.16}$$

because, by (2.19), the normal symbol $\sigma_\nu^{\widehat{H}}(\overline{z}, z)$ is the expectation value of the operator $\widehat{H}$ on the coherent states.

On the other hand, by Proposition 2.2, $H_1(a) + \|a\|^2/2$ is the Weyl symbol of the multiplication operator with $H_1(a) + \|a\|^2/2 \geqslant 0$.

Therefore,

$$\sigma_\omega^{\widehat{H}} \ \overset{(2.24)}{=}\ (\overline{z}z - 1/2) + 1/2 + \ H_1(a) \ + \ \|a\|^{\ 2}/2 \ + \ 9/16 \ \geqslant\ \sigma_\omega^{\mathbf{N}} + 17/16, \tag{4.17}$$

where $\mathbf{N}$ is the number operator.

The shifted number operator $\mathbf{N} + 17/16$ has the simple and isolated fundamental eigenvalue 17/16 and no other spectral value in the interval $17/16 < \lambda < 17/16 + 1 - 0$.

By the inequality (4.16) and nonnegativity of $\mathbf{N}$, the operator $\widehat{H}$ has a spectral value in the same interval.

Finally, the minimax variational principle (see Berezin–Shubin [8, Supplement, Sec. 3, Corollary 1]) implies, with the help of (4.17) that, in the interval $17/16 < \lambda < 17/16 + 1 - 0$, all spectral values of $\widehat{H} + 17/16$ are its eigenvalues and the sum of their multiplicities is not greater than 1, the sum of multiplicities of the eigenvalues of $\mathbf{N} + 17/16$.

Thus the quantum Yang–Mills Hamiltonian has a positive mass gap greater or equal to $(17/16 + 1) - 24/16 = 9/16$.

Theorem 4.2. *The Yang–Mills energy operator has a countable eigenbasis for $\mathcal{F}(\mathcal{H}^0)$ so that its spectrum is a countable set of eigenvalues.*

Proof. As a complex Hilbert space, the space $\mathcal{H}^0$ is isomorphic to $\mathcal{L}^2_\perp(\mathbb{B}, \mathbf{C}\mathfrak{g}^3)$ of transverse square-integrable vector-valued functions $z(x)$ on $\mathbb{B}$ with values in $\mathbf{C}\mathfrak{g}^3$.

The Fourier series expansions of $z(x) = \big(z_1(x), z_2(x), z_3(x)\big),\ x \in \mathbb{B}$,

$$z_k(x) = \sum_{j_k \in \mathbf{Z}^3} \check{z}(j_k) \exp(2\pi \quad ix \cdot j_k), \quad \check{z}(j_k) \in \mathbf{C}\mathfrak{g}^3, \tag{4.18}$$

define the isomorphism between $\mathcal{L}^2_\perp(\mathbb{B}, \mathbf{C}\mathfrak{g}^3)$ and the Hilbert tensor product $l^2(\mathbf{Z}^3) \otimes \mathbf{C}\mathfrak{g}^3$ of square summable $\mathbf{C}\mathfrak{g}$-valued sequences (with the natural conjugation), subject to the transversality condition

$$j_k \cdot \check{z}(j_k) \ = \ 0, \quad j_k \in \mathbf{Z}^3,\ k = 1, 2, 3. \tag{4.19}$$

The isomorphism converts the Gelfand nuclear triple $\mathcal{H} \subset \mathcal{H}^0 \subset \mathcal{H}^*$ into a Gelfand nuclear triple of $\mathbf{C}\mathfrak{g}$-valued sequences presented as elements of completed topological tensor products

$$\ell(\mathbf{Z}^3) \otimes \mathbf{C}\mathfrak{g} \ \subset\ l^2(\mathbf{Z}^3) \otimes \mathbf{C}\mathfrak{g} \ \subset\ \ell^*(\mathbf{Z}^3) \otimes \overline{\mathbf{C}}\mathfrak{g}^*, \tag{4.20}$$

where $\ell(\mathbf{Z}^3)$ is a nuclear Frechet subspace of $l^2(\mathbf{Z}^3)$.

By the infinite-dimensional version of the Hartogs theorem (see Colombeau [10, Sec. 3.3]), the corresponding Fock space $\mathcal{K}(\mathcal{H}^*)$ is isomorphic to the space of all continuous functionals on $\ell^*(\mathbf{Z}^3) \otimes \overline{\mathbf{C}}\mathfrak{g}^*$ that are exponential entire functions on $\overline{\mathbf{C}}\mathfrak{g}$, and the corresponding Kree space $\mathcal{K}(\mathcal{H}^*)$ is isomorphic to the space of all functionals on $\ell(\mathbf{Z}^3) \otimes \mathbf{C}\mathfrak{g}^*$ that are holomorphic functions on $\mathbf{C}\mathfrak{g}$.

By the unitarity of the Fourier series expansions, the Weyl symbol $\sigma_\omega^{\widehat{H}}$ (4.6) is unitarily equivalent to

$$\sigma_\omega^{\widehat{H}_j}(\check{a},\check{e}) \;=\; \frac{1}{2}\bigoplus_{j\in\mathbf{Z}^3}\Big\|\big(j\times\check{a}(j)-[\check{a}(j)\overset{\times}{,}\check{a}(j)]\big)\Big\|^2 \;+9/16+\frac{1}{2}\bigoplus_{j\in\mathbf{Z}^3}(\check{a}(j)\cdot\check{a}(j)\;+\;\check{e}(j)\cdot\check{e}(j)).$$

At the same time, the transversality equation (4.19) is the direct sum of the transversality equations over $\mathbf{C}\mathfrak{g}$.

The operators $\widehat{H}_j\,:\;\;\mathcal{K}(\mathbf{C}\mathfrak{g}^*)\to\mathcal{K}(\mathbf{C}\mathfrak{g})$ over the finite-dimensional space $\mathbf{C}\mathfrak{g}$ with the Weyl symbols satisfy

$$\sigma_\omega^{\widehat{H}_j}(a,e) \;=\; \frac{1}{2}\Big\|j\times\check{a}(j)-[\check{a}(j)\overset{\times}{,}\check{a}(j)]\Big\|^2 \;+\; \frac{1}{2}\big(\check{a}(j)\cdot\check{a}(j)\big)\;+\;\check{e}(j)\cdot\check{e}(j)+\;9/16. \tag{4.21}$$

By (4.21), the operators $\widehat{H}_j$ and $\mathbf{N}$ are hypo-elliptic (see Shubin [32, Definition 25.1]). Hence, by Shubin [32, Th. 26.3], they have countable orthonormal eigenbases in $\mathcal{S}(\mathbf{C}\mathfrak{g})$ with positive eigenvalues $\lambda_n(\widehat{H}_j)$ and $\lambda_n(\mathbf{N})$ converging to $+\infty$ as $n\to+\infty$.

Let $\mathcal{B}_j\subset\mathcal{K}(\mathcal{H}^*)$ be the eigenbases of $\widehat{H}_j$ of the eigenvectors $v_{k,j}\in\mathcal{B}_j$. Then the finite monomial products of $v_{j,k}$ are eigenvectors of the operator $\widehat{H}$ with the eigenvalue $\sum_{j,k}\lambda_{j,k}(\widehat{H}_j)$ (see Reed–Simon [29, Th. VIII.33]).

Finally, the finite monomial products together with a constant state form a countable basis in $\mathcal{K}(\mathcal{H}^0)$.

As the proof shows, the spectrum of $\widehat{H}$ is the set of finite sums of $\widehat{H}_j$ eigenvalues (see Reed–Simon [29, Chap. VIII]).

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