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Generalized Compatibility Equations for Tensors of High Ranks in Multidimensional Continuum Mechanics

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Abstract. Compatibility equations are derived for the components of generalized strains of rank m associated with generalized displacements of rank $m-1$ by analogs of Cauchy kinematic relations in n-dimensional space (multi-dimensional continuous medium) ($m \geqslant 1$, $n \geqslant 2$). These relations can be written in the form of equating to zero all components of the incompatibility tensor of rank $m(n-2)$ or its dual generalized Riemann–Christoffel tensor of rank $2m$. The number of independent components of these tensors is found; this number coincides with that of compatibility equations in terms of generalized strains or stresses. The inequivalence of the full system of compatibility equations to any of its weakened subsystems is discussed, together with diverse formulations of boundary value problems in generalized stresses in which the number of equations in a domain can exceed the number of unknowns.

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One cannot definitely claim that the theory of compatibility equations, in conformity with applications to mathematical physics and continuum mechanics, would be nowadays a specific direction of the general theory of PDEs. In every specific problem, the derivation of compatibility equations for the components of a tensor field of rank m in n-dimensional space is quite special (see, e.g., [1, 2]). The knowledge of these equations is necessary first of all to create a mathematically perfect statements of boundary value problems in mechanics and to prove the equivalence of these statements in terms of some quantities. A class of statements of boundary value problems, which is well-known and widely discussed in the last decades, is the formulation of problem is stresses in solid mechanics [3–7].

1. COMPATIBILITY EQUATIONS FOR A VECTOR FIELD

As is well known in vector analysis, if a scalar function, $u(\mathbf{x})$, $\mathbf{x} \in R^n$, $n \geqslant 2$, and a vector function, $\boldsymbol{\varepsilon}^{\{\mathbf{1}\}}(\mathbf{x}) = \varepsilon_i(\mathbf{x})\mathbf{e}_i^{\{\mathbf{1}\}}$, satisfy $a_{1;n} = n$ differential relations

$$\varepsilon_i = u_{,i} \equiv \frac{\partial u}{\partial x_i}, \quad \boldsymbol{\varepsilon}^{\{\mathbf{1}\}} = \operatorname{grad} u, \tag{1.1}$$

then the Cartesian coordinates ε_i are connected by $b_{1;n} = n(n-1)/2$ compatibility equations

$$\varepsilon_{j,i} - \varepsilon_{i,j} = 0, \tag{1.2}$$

which mean that all $b_{1;n}$ components of the rotor (the curl) $(\operatorname{Rot} \boldsymbol{\varepsilon}^{\{\mathbf{1}\}})^{\{\mathbf{n-2}\}}$ are zero. Recall [8] that, in n-dimensional space, the rotor of a vector is the tensor of rank $n-2$, antisymmetric with respect to every pair of indices, with the components

$$(\operatorname{Rot} \boldsymbol{\varepsilon}^{\{\mathbf{1}\}})_{k_1 \ldots k_{n-2}} = \epsilon_{k_1 \ldots k_{n-2} ij} \varepsilon_{j,i}, \tag{1.3}$$

where $\epsilon_{k_1 \ldots k_n}$ is an n-dimensional Levi-Civita symbol[1]. The object dual to $(\operatorname{Rot} \boldsymbol{\varepsilon}^{\{\mathbf{1}\}})^{\{\mathbf{n-2}\}}$, which is obtained from $(\operatorname{Rot} \boldsymbol{\varepsilon}^{\{1\}})^{\{n-2\}}$ by the complete contraction with respect to $n-2$ indices with

[1]The following notation and agreements are used in formulas (1.1)–(1.3) and below: (a) the superscripts in upper curly braces used in an index-free notation for a tensor mean their rank; they are omitted for the components of a tensor, because the rank is always equal to the number of subscripts; (b) the commas in the subscripts mean partial differentiation with respect to coordinates; (c) the summation from 1 to n with respect to repeating twice (in a monomial) Latin subscripts; here the symbol of sum is omitted for brevity; (d) there is no summation with respect to Greek subscripts, independently of the number of their repetition; (e) the Latin subscripts are mutually independent, while different Greek subscripts always take different values; (f) the notation $(\overrightarrow{\alpha_1, \ldots, \alpha_n})$ means that

the Levi-Civita symbol, is an antisymmetric tensor of rank two for every n. Up to factor, the components of this tensor coincide with the left-hand side of (1.2). The number of independent computations of these two tensors is the same (because they are dual to each other) and is equal to $C_n^{n-2} = C_n^2 = b_{1;n} = n(n-1)/2$.

Along with the vector field $\boldsymbol{\varepsilon}^{\{\mathbf{1}\}}(\mathbf{x})$, we consider another field $\boldsymbol{\sigma}^{\{\mathbf{1}\}}(\mathbf{x})$ related to $\boldsymbol{\varepsilon}^{\{\mathbf{1}\}}$ by a nondegenerate algebraic vector-operator $\check{\mathbf{G}}^{\{\mathbf{1}\}}$,

$$\boldsymbol{\varepsilon}^{\{\mathbf{1}\}} = \big(\check{\mathbf{G}}(\boldsymbol{\sigma}^{\{\mathbf{1}\}})\big)^{\{\mathbf{1}\}}, \quad \boldsymbol{\sigma}^{\{\mathbf{1}\}} = \big(\check{\mathbf{G}}^{-1}(\boldsymbol{\varepsilon}^{\{\mathbf{1}\}})\big)^{\{\mathbf{1}\}}. \tag{1.4}$$

Let us single out important special cases of defining an operator $\check{\mathbf{G}}^{\{\mathbf{1}\}}$ which is homogeneous (i.e., without explicit coordinates $\mathbf{x}$), local (i.e., obtaining no derivatives with respect to x_i), and scleronomic (i.e., containing no derivatives with respect to time).

(1) If $\check{\mathbf{G}}^{\{\mathbf{1}\}}$ is isotropic, then the most general form of the constraint (1.4) is as follows:

$$\boldsymbol{\varepsilon}^{\{\mathbf{1}\}} = G\big(|\boldsymbol{\sigma}^{\{\mathbf{1}\}}|\big)\boldsymbol{\sigma}^{\{\mathbf{1}\}}, \quad \boldsymbol{\sigma}^{\{\mathbf{1}\}} = F\big(|\boldsymbol{\varepsilon}^{\{\mathbf{1}\}}|\big)\boldsymbol{\varepsilon}^{\{\mathbf{1}\}}, \tag{1.5}$$

where $G\big(|\boldsymbol{\sigma}^{\{\mathbf{1}\}}|\big)$ and $F\big(|\boldsymbol{\varepsilon}^{\{\mathbf{1}\}}|\big)$ are two arbitrary functions, satisfying the condition $GF = 1$, of independent invariants of the vectors $\boldsymbol{\sigma}^{\{\mathbf{1}\}}$ and $\boldsymbol{\varepsilon}^{\{\mathbf{1}\}}$, namely, their lengths. In this case, the fields $\boldsymbol{\varepsilon}^{\{\mathbf{1}\}}$ and $\boldsymbol{\sigma}^{\{\mathbf{1}\}}$ are collinear.

(2) If the operator $\check{\mathbf{G}}^{\{\mathbf{1}\}}$ is physically linear (i.e., sustains the superposition principle), then

$$\boldsymbol{\varepsilon}^{\{\mathbf{1}\}} = \boldsymbol{\Lambda}^{\{\mathbf{2}\}} \cdot \boldsymbol{\sigma}^{\{\mathbf{1}\}}, \quad \boldsymbol{\sigma}^{\{\mathbf{1}\}} = (\boldsymbol{\Lambda}^{-1})^{\{\mathbf{2}\}} \cdot \boldsymbol{\varepsilon}^{\{\mathbf{1}\}}, \tag{1.6}$$

where $\boldsymbol{\Lambda}^{\{\mathbf{2}\}}$ is a positive-definite tensor independent of $\mathbf{x}$ and defining some type of anisotropy.

(3) The intersection of the above two cases corresponds to the spherical tensor $\boldsymbol{\Lambda}^{\{\mathbf{2}\}} = \boldsymbol{\Lambda}\mathbf{I}^{\{2\}}$, $\Lambda = \text{const}$, $\mathbf{I}^{\{2\}} = \mathbf{e}_i^{\{1\}} \otimes \mathbf{e}_i^{\{1\}}$, and to the constraints

$$\boldsymbol{\varepsilon}^{\{\mathbf{1}\}} = \Lambda\boldsymbol{\sigma}^{\{\mathbf{1}\}}, \quad \boldsymbol{\sigma}^{\{\mathbf{1}\}} = \frac{1}{\Lambda}\,\boldsymbol{\varepsilon}^{\{\mathbf{1}\}}. \tag{1.7}$$

We refer to the tensors $\mathbf{u}^{\{\mathbf{m-1}\}}(\mathbf{x})$, $\boldsymbol{\varepsilon}^{\{\mathbf{m}\}}(\mathbf{x})$ and $\boldsymbol{\sigma}^{\{\mathbf{m}\}}(\mathbf{x})$, $m \geqslant 1$, as the generalized displacements of rank $m-1$, the generalized strains of rank m, and generalized stresses of rank m, respectively, and call the constraints (1.4) *generalized constitutive relations*, which can contain material functions and constants G, $\boldsymbol{\Lambda}^{\{\mathbf{2}\}}$, and Λ.

Let a vector field $\boldsymbol{\sigma}^{\{\mathbf{1}\}}$ in a bounded domain $V \in R^3$ with smooth boundary ∂V each of whose points is equipped with a unit outward pointing normal $\mathbf{n}(\mathbf{x})$ satisfy some physical law or a postulate of continuum mechanics,

$$\operatorname{div}\boldsymbol{\sigma}^{\{\mathbf{1}\}} + X = 0, \quad \mathbf{x} \in V, \tag{1.8}$$

where $X(\mathbf{x})$ is a given scalar function (external volume force). A static boundary condition holds on ∂V,

$$\boldsymbol{\sigma}^{\{\mathbf{1}\}} \cdot \mathbf{n} = P, \quad \mathbf{x} \in \partial V, \tag{1.9}$$

where $P(\mathbf{x})$ is a given external surface force.

Traditionally, in mechanics, two equivalent statements of a boundary value problem are distinguished, namely, in terms of displacements and stresses. It follows from (1.1) and (1.4) that problem (1.8), (1.9) in generalized displacements consists of solving a single equation

$$\operatorname{div}\big(\check{\mathbf{G}}^{-1}(\operatorname{grad} u)\big)^{\{1\}} + X = 0, \quad \mathbf{x} \in V, \tag{1.10}$$

with a single boundary condition

$$\big(\check{\mathbf{G}}^{-1}(\operatorname{grad} u)\big)^{\{1\}} \cdot \mathbf{n} = P, \quad \mathbf{x} \in \partial V. \tag{1.11}$$

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the relation in question is to be amplified to all even permutations of the subscripts $\alpha_1, \ldots, \alpha_n$; (g) in quantities $a_{m;n}$ and $b_{m;n}$, the first subscript coincides with the rank of the object $\boldsymbol{\varepsilon}^{\{\mathbf{m}\}}$ and the other with the dimension n of the space n.

In an isotropic physically linear case (1.7), problem (1.10), (1.11) becomes the Neumann problem for the Poisson equation,

$$\Delta u + \Lambda X = 0, \quad \mathbf{x} \in V; \qquad \frac{\partial u}{\partial n} = \Lambda P, \quad \mathbf{x} \in \partial V. \tag{1.12}$$

In terms of generalized stresses, in the domain V, one should solve a single equation (1.8) and $b_{1;n}$ equations (1.2), where one must substitute the constitutive relations (1.4),

$$\big(\check{\mathbf{G}}(\boldsymbol{\sigma}^{\{1\}})\big)_{j,i} - \big(\check{\mathbf{G}}(\boldsymbol{\sigma}^{\{1\}})\big)_{i,j} = 0. \tag{1.13}$$

Moreover, the static boundary condition (1.9) must hold. In the simplest case (1.7), the compatibility equations are represented similarly to (1.2),

$$\sigma_{j,i} - \sigma_{i,j} = 0, \quad \mathbf{x} \in V. \tag{1.14}$$

Thus, the n components of $\boldsymbol{\sigma}^{\{1\}}(\mathbf{x})$ should satisfy $b_{1,n} + 1 = (n^2 - n + 2)/2$ equations (1.8), (1.14) in the domain V and one boundary condition (1.9). The number of equations in V coincides with the number of unknowns only if $n = 2$, and, for $n \geqslant 3$, the number of equations exceeds that of unknowns. Certainly, the discrepancy between these two numbers poses the question: Are there dependent equations among the $b_{1;n}$ compatibility equations? At the "vector level" (Section 1 of the present paper is devoted to the "vector level" to answer this very question), the answer is obviously negative.

Indeed, the general solution of (1.14) admits a representation

$$\sigma_i = \Lambda u_{,i}, \quad \Lambda = \mathrm{const}, \tag{1.15}$$

i.e., for every solution $\boldsymbol{\sigma}^{\{1\}}(\mathbf{x})$ of system (1.14), there is a scalar field $u(\mathbf{x})$ such that relations (1.15) hold. Let us now weaken the system (1.14) by removing from it, for example, the equation $\sigma_{1,2} - \sigma_{2,1} = 0$. Then the weakened system obtains a particular solution

$$\sigma_1 = g_1(x_2), \quad \sigma_2 = g_2(x_1), \quad \sigma_3 \equiv 0, \quad \ldots\ , \ \sigma_n \equiv 0, \tag{1.16}$$

where g_1 and g_2 are arbitrary functions of their arguments. If we choose these functions in such a way that $dg_1/dx_2 \neq dg_2/dx_1$ then, obviously, there is no scalar field $u(\mathbf{x})$ such that the representation (1.15) holds. Hence, system (1.14) is not equivalent to any weakened subsystem of the system, and all $b_{1;n}$ equations (1.14) must enter the statement of the problem in generalized stresses [9].

However, there is a statement equivalent to (1.8), (1.14), (1.9) in which the number of equations on V coincides with the number of unknowns σ_i. This system is to solve n unbound Poisson equations on the domain V,

$$\Delta\sigma_i = -X_{,i}, \quad \mathbf{x} \in V, \tag{1.17}$$

assuming that the condition (1.9) and

$$\operatorname{div}\boldsymbol{\sigma}^{\{1\}} + X = 0, \quad \sigma_{j,i} - \sigma_{i,j} = 0, \quad \mathbf{x} \in \partial V, \tag{1.18}$$

are satisfied on the boundary.

Let us establish the equivalence of the statements (1.8), (1.14), (1.9) and (1.17), (1.18), (1.9). If we adopt the first of them, then, differentiating (1.14) with respect to x_j, summing over j, and taking (1.8) into account, we obtain (1.17). Moreover, the equations (1.8) and (1.14), which are given on V, hold also on the boundary ∂V due to the assumed continuity of the quantities entering these equations, which implies (1.18).

Suppose now that the setting (1.17), (1.18), (1.9) is chosen. Let us differentiate (1.17) with respect to x_i, sum over i, and see that the function $\operatorname{div}\boldsymbol{\sigma}^{\{1\}} + X$ is harmonic on V. Since it vanishes on ∂V (see (1.18)), we arrive at the desired equation (1.8). Further, let us differentiate (1.17) with respect to x_j, make the alternation operation with respect to the indices i and j, and arrive at the fact that all $b_{1;n}$ functions $\sigma_{j,i} - \sigma_{i,j}$ are harmonic on V. Since they vanish on ∂V (see (1.18)), equations (1.14) follow again.

2. COMPATIBILITY EQUATIONS FOR TENSORS OF RANK TWO IN n-DIMENSIONAL CONTINUUM MECHANICS

In this section, we omit the adjective "generalized" for the displacements $\mathbf{u}^{\{1\}}(\mathbf{x})$, strains $\boldsymbol{\varepsilon}^{\{\mathbf{2}\}}(\mathbf{x})$, and stresses $\boldsymbol{\sigma}^{\{\mathbf{2}\}}(\mathbf{x})$, because, for $m=2$, their physical meaning corresponds completely to the key concepts of continuum mechanics, namely, to the vector of displacements, to the symmetric tensor of small strains, and to the symmetric Cauchy stress tensor.

Instead of (1.1), we now have $a_{2;n}=n(n+1)/2$ differential relations

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\quad \boldsymbol{\varepsilon}^{\{\mathbf{2}\}}=\operatorname{Def}\mathbf{u}^{\{1\}}\equiv\operatorname{Sym}(\operatorname{Grad}\mathbf{u}^{\{1\}})^{\{2\}} \tag{2.1}$$

which are called Cauchy relations in the geometrically linear theory.

In the n-dimensional continuum mechanics, the Kröner incompatibility tensor $\boldsymbol{\eta}^{\{\mathbf{2}(n-\mathbf{2})\}}(\boldsymbol{\varepsilon}^{\{\mathbf{2}\}})\equiv$ $\operatorname{Ink}\boldsymbol{\varepsilon}^{\{\mathbf{2}\}}$ of rank $2(n-2)$ with the Cartesian components

$$\eta_{p_1\ldots p_{n-2}q_1\ldots q_{n-2}}(\boldsymbol{\varepsilon}^{\{\mathbf{2}\}})=\epsilon_{p_1\ldots p_{n-2}li}\,\epsilon_{q_1\ldots q_{n-2}jk}\,\varepsilon_{ij,lk} \tag{2.2}$$

is well known [9–12]. It is clear from the definition in (2.2) that, for $n\geqslant 4$, it is antisymmetric with respect to every pair of both the first $n-2$ indices and the $n-2$ last ones and, for $n\geqslant 3$, it is symmetric with respect to the permutation of families of $n-2$ first indices and $n-2$ last ones. As is known from differential geometry, the number of independent components $\boldsymbol{\eta}^{\{\mathbf{2(n-2)}\}}(\boldsymbol{\varepsilon}^{\{\mathbf{2}\}})$ is equal to $b_{2;n}=n^2(n^2-1)/12$.

For $n=2$, the Kröner tensor is the scalar

$$\boldsymbol{\eta}(\boldsymbol{\varepsilon}^{\{\mathbf{2}\}})=\epsilon_{li}\,\epsilon_{jk}\,\varepsilon_{li,jk}=2\varepsilon_{12,12}-\varepsilon_{11,22}-\varepsilon_{22,11} \tag{2.3}$$

and, for $n=3$, it is a tensor of rank two with six independent components

$$\eta_{pq}(\boldsymbol{\varepsilon}^{\{\mathbf{2}\}})=\epsilon_{pli}\,\epsilon_{qjk}\,\varepsilon_{ij,lk}=(\varepsilon_{ij,ji}-\varepsilon_{ii,jj})\delta_{pq}+\varepsilon_{pq,ii}+\varepsilon_{ii,pq}-\varepsilon_{pi,iq}-\varepsilon_{qi,ip}, \tag{2.4}$$

$$\eta_{\alpha\alpha}(\varepsilon^{\{\mathbf{2}\}})=2\varepsilon_{\beta\gamma,\gamma\beta}-\varepsilon_{\beta\beta,\gamma\gamma}-\varepsilon_{\gamma\gamma,\beta\beta},\quad(\overrightarrow{\alpha,\beta,\gamma}), \tag{2.5}$$

$$\eta_{\alpha\beta}(\varepsilon^{\{\mathbf{2}\}})=\varepsilon_{\alpha\beta,\gamma\gamma}+\varepsilon_{\gamma\gamma,\alpha\beta}-\varepsilon_{\alpha\gamma,\gamma\beta}-\varepsilon_{\beta\gamma,\gamma\alpha},\quad(\overrightarrow{\alpha,\beta,\gamma}). \tag{2.6}$$

If an n-dimensional continuous medium belongs to an Euclidean space, then all components of the Kröner tensor vanish, i.e., the following compatibility equations for deformations hold:

$$\boldsymbol{\eta}^{\{\mathbf{2(n-2)}\}}(\boldsymbol{\varepsilon}^{\{\mathbf{2}\}})=0. \tag{2.7}$$

The fact that the strain tensor satisfies $b_{2;n}$ equations (2.7) is a sufficient condition for the integrability of system (2.1) with respect to n unknown functions u_i on an n-dimensional domain in which every closed contour contained in the domain can be contracted to a point by a continuous deformation.

Mathematically, systems (2.1) and (2.7) are quite related. Along with the above considerations concerning compatibility, expressions (2.1) are a parametrization of a general solution of system (2.7). Here one can regard $u_i(\mathbf{x})$ as arbitrary three times differentiable functions of coordinates, without equipping them with the meaning of displacements in advance.

A wide class of continuous media is described by the constitutive relations

$$\boldsymbol{\varepsilon}^{\{\mathbf{2}\}}=\big(\check{\mathbf{G}}(\boldsymbol{\sigma}^{\{\mathbf{2}\}})\big)^{\{\mathbf{2}\}},\quad \boldsymbol{\sigma}^{\{\mathbf{2}\}}=\big(\check{\mathbf{G}}^{-1}(\boldsymbol{\varepsilon}^{\{\mathbf{2}\}})\big)^{\{\mathbf{2}\}} \tag{2.8}$$

with nondegenerate tensor-operator $\check{\mathbf{G}}^{\{2\}}$. As in Section 1, we concentrate our attention at important special cases of defining a homogeneous local scleronomic operator $\check{\mathbf{G}}^{\{2\}}$.

(1) *Isotropy.* The most general form of isotropic constraint (2.8) is as follows:

$$\begin{aligned}\boldsymbol{\varepsilon}^{\{\mathbf{2}\}}&=G_0\mathbf{I}^{\{\mathbf{2}\}}+G_1\boldsymbol{\sigma}^{\{\mathbf{2}\}}+\cdots+G_{n-1}(\boldsymbol{\sigma}^{\{\mathbf{2}\}})^{n-1},\\ \boldsymbol{\sigma}^{\{\mathbf{2}\}}&=F_0\mathbf{I}^{\{\mathbf{2}\}}+F_1\boldsymbol{\varepsilon}^{\{\mathbf{2}\}}+\cdots+F_{n-1}(\boldsymbol{\varepsilon}^{\{\mathbf{2}\}})^{n-1},\end{aligned} \tag{2.9}$$

where the families of material functions $G_0,\ldots,G_{n-1}$ and $F_0,\ldots,F_{n-1}$ depend on the families of n independent invariants $I_{\sigma k}$ and $I_{\varepsilon k}$, $k=1,\ldots,n$, of the corresponding tensors. For these invariants, one can take, for example, the traces of the first n powers of the tensors $\boldsymbol{\sigma}^{\{\mathbf{2}\}}$ and $\boldsymbol{\varepsilon}^{\{\mathbf{2}\}}$:

$$I_{\sigma k} = \sqrt[k]{\operatorname{tr}\left(\boldsymbol{\sigma}^{\{2\}}\right)^k}, \quad I_{\varepsilon k} = \sqrt[k]{\operatorname{tr}\left(\boldsymbol{\varepsilon}^{\{2\}}\right)^k}. \tag{2.10}$$

(2) *Physical linearity.* We have the Hooke law for an anisotropic n-dimensional elastic solid,

$$\boldsymbol{\varepsilon}^{\{2\}} = \mathbf{J}^{\{4\}} \cdot \boldsymbol{\sigma}^{\{2\}}, \quad \boldsymbol{\sigma}^{\{2\}} = (\mathbf{J}^{-1})^{\{4\}} \cdot \boldsymbol{\varepsilon}^{\{2\}} \tag{2.11}$$

with the material tensor of elastic pliability $\mathbf{J}^{\{4\}}$ defining the type of anisotropy.

(3) The intersection of the previous two cases leads to the Hooke law for an isotropic n-dimensional elastic solid ($G_0 = (-\nu/E)I_{\sigma 1}$, $G_1 \equiv (1+\nu)/E$, $G_2 = \cdots = G_{n-1} \equiv 0$):

$$\begin{aligned} \boldsymbol{\varepsilon}^{\{2\}} &= -\frac{\nu}{E} I_{\sigma 1}\mathbf{I}^{\{2\}} + \frac{1+\nu}{E}\boldsymbol{\sigma}^{\{2\}}, \\ \boldsymbol{\sigma}^{\{2\}} &= \frac{E\nu}{(1-(n-1)\nu)(1+\nu)} I_{\varepsilon 1}\mathbf{I}^{\{2\}} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}^{\{2\}}, \end{aligned} \tag{2.12}$$

where E is the Young modulus and ν is the Poisson coefficient, which have, for every n, the same mechanical meaning as in the three-dimensional space.

Let, on a bounded domain $V \in R^n$ satisfying the same assumptions as in Section 1, there be n equations of equilibrium which are differential consequences of the integral postulate on the conservation of momentum,

$$\operatorname{Div}\boldsymbol{\sigma}^{\{2\}} + \mathbf{X}^{\{1\}} = 0, \quad \mathbf{x} \in V, \tag{2.13}$$

where $\mathbf{X}^{\{1\}}(\mathbf{x})$ are known volume forces. On the boundary ∂V, n static boundary conditions hold:

$$\boldsymbol{\sigma}^{\{2\}} \cdot \mathbf{n} = \mathbf{P}^{\{1\}}, \quad \mathbf{x} \in \partial V, \tag{2.14}$$

with known distribution of surface loadings $\mathbf{P}^{\{1\}}(\mathbf{x})$.

Without dwelling, in this section, on the statement of a boundary value problem in terms of displacements, we pass immediately to a more nontrivial statement in stresses. On a domain V, there are n equations (2.13) and $b_{2;n} = n^2(n^2-1)/12$ equations (2.7) into which one should substitute the constitutive relations (2.8):

$$\boldsymbol{\eta}^{\{\mathbf{2(n-2)}\}}\left(\left(\check{\mathbf{G}}(\boldsymbol{\sigma}^{\{2\}})\right)^{\{2\}}\right) = 0. \tag{2.15}$$

Moreover, n static boundary conditions (2.14) must also hold.

For the case in which tensors $\boldsymbol{\varepsilon}^{\{2\}}$ and $\boldsymbol{\sigma}^{\{2\}}$ are connected by the linear isotropic law (2.12), the compatibility equations in stresses are represented as follows [13]:

$$\sigma_{rs,mt} + \sigma_{mt,rs} - \sigma_{ms,rt} - \sigma_{rt,ms} + \frac{\nu}{1+\nu}(I_{\sigma 1,rt}\delta_{ms} + I_{\sigma 1,ms}\delta_{rt} - I_{\sigma 1,mt}\delta_{rs} - I_{\sigma 1,rs}\delta_{mt}) = 0. \tag{2.16}$$

Thus, $a_{2;n} = n(n+1)/2$ components of the symmetric tensor $\boldsymbol{\sigma}^{\{2\}}$ on the domain V must satisfy $b_{2;n} + n$ equations (2.7) and (2.13) and, on the domain ∂V, only n boundary conditions (2.14). As above in Section 1, the numbers $a_{2;n}$ and $b_{2;n} + n$ coincide only for $n = 2$ and, for $n \geqslant 3$, the inequality $a_{2;n} < b_{2;n} + n$ holds, and the discrepancy increases as n increases significantly more rapidly than in the “vector case” $m = 1$.

3. POBEDRYA SETTING OF THE PROBLEM IN STRESSES

The independence problem for some equations in the statement of the problem in stresses for a three-dimensional solid ($a_{2;3} = 6$, $b_{2;3} + 3 = 9$) has a long history (see a survey of this problem, for example, in [14–16]). The seeming discrepancy among the number of unknowns, the equations on a domain, and the boundary conditions is eliminated by the setting, which is equivalent to the classical one, of the problem in stresses in mechanics of deformable solid which was suggested by Pobedrya [5, 14] at the end of the 1970s. It is in solving, on the domain of the solid, six generalized compatibility equations assuming the validity of six equations on the boundary, namely, three equilibrium equations (2.13) and three boundary conditions (2.14). In the isotropic elasticity theory (2.12), these compatibility equations are given by the Beltrami–Michell equations

$$\Delta\sigma_{ms} + \frac{1}{1+\nu} I_{\sigma 1,ms} + \frac{\nu \operatorname{div}\mathbf{X}}{1-\nu}\delta_{ms} + X_{m,s} + X_{s,m} = 0. \tag{3.1}$$

Pobedrya's statement is better suited for the application of numerical methods. It is difficult to use the classical Castigliano variational principle to construct difference schemes of some level, because, in this principle, one speaks of a conditional extremum of the Castiglianian. Therefore, a new variational principle was formulated. During four decades, this statement obtained a wide international fame [17–19]. Using this statement, two- and three-dimensional quasi-static boundary value problems of elasticity theory, plasticity theory, viscoelasticity theory, contact problems, heat problems, and problems of computer mechanics of composites were studied numerically-analytically.

Is it possible to present, using Beltrami–Michell equations generalized for an n-dimensional isotropic elastic solid, an analog of the statement in R^3 in which the number of equations on $V \in R^n$, $n \geqslant 4$, would coincide with the number of unknowns? In [13], these generalized equations were obtained using diverse convolutions of the original $b_{2;n}$ equations (2.16):

$$\Delta\sigma_{ms} + \frac{1+(3-n)\nu}{1+\nu}\, I_{\sigma 1,ms} + \frac{\nu \operatorname{div}\mathbf{X}}{1+(2-n)\nu}\,\delta_{ms} + X_{m,s} + X_{s,m} = 0. \tag{3.2}$$

There are $a_{2;n}$ equations of this kind, which coincides with the number of components of the symmetric tensor $\boldsymbol{\sigma}^{\{\mathbf{2}\}}$; for $n = 3$, the left-hand sides of (3.2) and (3.1) coincide.

The equation $b_{2;3} = a_{2;3}$ enables us to replace six equations (2.16) by an arbitrary system of their six independent linear combinations; in particular, by the system of six equations (3.2). Beginning with $n = 4$, the inequality $b_{2;n} > a_{2;n}$ holds, and therefore, no system of $a_{2;n}$ linear combinations of $b_{2;n}$ equations, including the system of generalized Beltrami–Michell equations (3.2), can be equivalent to (2.16). A counterexample of their inequivalence, for $n = 4$, is presented in [13].

4. GENERALIZATION OF CAUCHY RELATIONS TO TENSORS OF HIGH RANKS

The following relations for $\mathbf{u}^{\{m-1\}}(\mathbf{x})$ and $\boldsymbol{\varepsilon}^{\{\mathbf{m}\}}(\mathbf{x})$ are a natural generalization for $m \geqslant 3$ of relations (1.1) and (2.1) written out for $m = 1$ and $m = 2$, respectively:

$$u_{(i_1\ldots i_{m-1},i_m)} \equiv u_{i_1\ldots i_{m-1},i_m} + u_{i_2\ldots i_m,i_1} + \cdots + u_{i_m i_1\ldots i_{m-2},i_{m-1}} = m\varepsilon_{i_1\ldots i_m}; \tag{4.1}$$

they were called in [12] *generalized Cauchy kinematic relations*. Let us consider extensives $u_{i_1\ldots i_{m-1}}$ symmetric with respect to the transposition of every two indices. Taking into account this symmetry condition, among the total number of n^{m-1} components $\mathbf{u}^{\{\mathbf{m-1}\}}$, there are $C^{m-1}_{m+n-2} = (m+n-2)!/[(m-1)!(n-1)!]$ independent ones.

By the constraints (4.1), including the operation of multidimensional cyclic symmetrization, one can readily conclude that the generalized deformations $\boldsymbol{\varepsilon}^{\{\mathbf{m}\}}$ of rank m, as well as $\mathbf{u}^{\{\mathbf{m-1}\}}$, form an absolutely symmetric tensor which is the symmetric part of the gradient $\mathbf{u}^{\{\mathbf{m-1}\}}$. Let us pose the problem concerning the integrability conditions for the system C^m_{m+n-1} of equations (4.1), or the compatibility equations imposed on C^m_{m+n-1} independent components $\varepsilon_{i_1\ldots i_m}$ for unique determining the C^{m-1}_{m+n-2} independent components $u_{i_1\ldots i_{m-1}}$.

Introduce the generalized tensor of rotations $\boldsymbol{\omega}^{\{\mathbf{m}\}}$ with the components $\omega_{i_1\ldots i_m}$,

$$u_{i_1\ldots i_{m-1},i_m} = \varepsilon_{i_1\ldots i_m} + \omega_{i_1\ldots i_m}. \tag{4.2}$$

Naturally, for $m = 1$, one should set $\omega_i = 0$. It follows from (4.1) that $\boldsymbol{\omega}^{\{\mathbf{m}\}}$ is antisymmetric with respect to the cyclic permutations of the indices,

$$\omega_{(i_1\ldots i_m)} = 0. \tag{4.3}$$

It follows immediately from (4.1) and (4.2) that $\omega_{i_1\ldots i_m}$ is connected with the components of the gradient of the tensor $\mathbf{u}^{\{\mathbf{m-1}\}}$,

$$\omega_{i_1\ldots i_m} = u_{i_1\ldots i_{m-1},i_m} - \frac{1}{m}\,u_{(i_1\ldots i_{m-1},i_m)}. \tag{4.4}$$

To derive differential relations between the components of the tensors $\boldsymbol{\omega}^{\{\mathbf{m}\}}$ and $\boldsymbol{\varepsilon}^{\{\mathbf{m}\}}$, we take partial derivatives of order $m-1$ of the equation (4.4):

$$\omega_{i_1\ldots i_m,j_1\ldots j_{m-1}} = u_{i_1\ldots i_{m-1},i_m j_1\ldots j_{m-1}} - \frac{1}{m}\,u_{(i_1\ldots i_{m-1},i_m),j_1\ldots j_{m-1}}. \tag{4.5}$$

Let us represent the right-hand side of (4.5) as a linear combination of C_{2m-1}^{m-1} summands of the form $A_I\varepsilon_{k_1\dots k_m,l_1\dots l_{m-1}}$, $I=1,\dots,C_{2m-1}^{m-1}$. The summation is ranges over all C_{2m-1}^{m-1} permutations of the $2m-1$ subscripts of $\boldsymbol{\varepsilon}^{\{\mathbf{m}\}}$ (taking into account the absolute symmetry with respect to the first m and the last $m-1$ indices). After this, we substitute the expressions for $\boldsymbol{\varepsilon}^{\{\mathbf{m}\}}$ (using $\mathbf{u}^{\{\mathbf{m-1}\}}$) given by (4.1) and equate the sum to the right-hand side of (4.5). The solution of the system of linear algebraic equations with respect to C_{2m-1}^{m-1} coefficients A_I gives the desired differential relations between the components of $\omega_{i_1\dots i_m}$ and $\varepsilon_{i_1\dots i_m}$.

Due to combinatorial complications, it is rather difficult to present the algorithm described above formally in a closed form for every m. For an illustration, we dwell on the special cases $m=2$ and $m=3$ in more detail.

(1) $m=2$, $n\geqslant 1$; $i_1=p$, $i_2=q$, $j_1=s$. For the Cauchy classical relations (2.1) $u_{(p,q)}\equiv u_{p,q}+u_{q,p}=2\varepsilon_{pq}$, we have $\omega_{pq}=(u_{p,q}-u_{q,p})/2$ from (4.4). By the algorithm described above, write

$$\omega_{pq,s}=\frac{1}{2}(u_{p,qs}-u_{q,ps})=A_1\varepsilon_{pq,s}+A_2\varepsilon_{qs,p}+A_3\varepsilon_{sp,q}. \tag{4.6}$$

Substituting the expressions ε_{pq} using the gradients of displacements into (4.6), we arrive at the system of three ($C_{2m-1}^{m-1}=C_3^1=3$) equations with respect to A_1, A_2, and A_3:

$$A_1+A_3=1,\quad A_1+A_2=-1,\quad A_2+A_3=0.$$

The solution $A_1=0$, $A_2=-1$, $A_3=1$ of this system leads to differential constraints, known in kinematics of continuous medium, for components of the tensor of revolutions and the tensor of small deformations,

$$\omega_{pq,s}=\varepsilon_{sp,q}-\varepsilon_{qs,p}. \tag{4.7}$$

(2) $m=2$, $n\geqslant 1$. Relations (4.1), (4.3), and (4.4) are represented in the following form:

$$u_{(pq,s)}\equiv u_{pq,s}+u_{qs,p}+u_{sp,q}=3\varepsilon_{pqs}, \tag{4.8}$$
$$\omega_{pqs}+\omega_{qsp}+\omega_{spq}=0,\quad \omega_{pqs}=\frac{2}{3}\,u_{pq,s}-\frac{1}{3}\big(u_{qs,p}-u_{sp,q}\big).$$

On the other hand, the second derivatives $\omega_{pqs,rt}$ are representable by a sum of ten ($C_{2m-1}^{m-1}=C_5^2=10$) summands,

$$\begin{aligned}\omega_{pqs,rt}=&A_1\varepsilon_{pqs,rt}+A_2\varepsilon_{pqr,st}+A_3\varepsilon_{pqt,rs}+A_4\varepsilon_{srt,pq}+A_5\varepsilon_{prs,qt}\\&+A_6\varepsilon_{rqs,pt}+A_7\varepsilon_{pst,qr}+A_8\varepsilon_{qst,pr}+A_9\varepsilon_{prt,qs}+A_{10}\varepsilon_{qrt,ps}.\end{aligned}$$

Substituting expressions (4.8) into the right-hand side of the last equation and solving the system of ten equations thus obtained for $A_1,\dots,A_{10}$, we find

$$A_1=0,\quad A_2=A_3=A_4=1,\quad A_5=\dots=A_{10}=-\frac{1}{2}.$$

Note that, in both the above equations,

$$\sum_{I=1}^{C_{2m-1}^{m-1}}A_I=0.$$

Thus, in the case of every m, the differential relations between the symmetric and antisymmetric parts of the gradient $\mathbf{u}^{\{\mathbf{m-1}\}}$ include partial derivatives with respect to coordinates of order $m-1$. They can be represented as follows (using a conditional character to some extent):

$$\omega_{i_1\dots i_m,j_1\dots j_{m-1}}=\sum_{I=1}^{C_{2m-1}^{m-1}}A_I\varepsilon_{k_1\dots k_m,l_1\dots l_{m-1}}, \tag{4.9}$$

where I is the index of the combination $\{l_1\dots l_{m-1}\}$.

The compatibility equations in terms of $\varepsilon_{i_1\dots i_m}$ follow immediately from (4.9) and from the requirements that the mixed derivatives of order m are equal,

$$\omega_{i_1\dots i_m,j_1\dots j_{m-2}j_{m-1}j_m}\big(\boldsymbol{\varepsilon}^{\{\mathbf{m}\}}\big) = \omega_{i_1\dots i_m,j_1\dots j_{m-2}j_m j_{m-1}}\big(\boldsymbol{\varepsilon}^{\{\mathbf{m}\}}\big). \tag{4.10}$$

For example, for $m = 2$ and $m = 3$, using (4.7) and (4.9), we have

$$\omega_{pq,st} = \omega_{pq,ts} \iff \varepsilon_{ps,qt} - \varepsilon_{qs,pt} = \varepsilon_{pt,qs} - \varepsilon_{qt,ps}, \tag{4.11}$$

$$\begin{aligned}\omega_{pqs,rtu} = \omega_{pqs,rut} \iff{}& \varepsilon_{pqt,sru} + \varepsilon_{srt,pqu}\\ &- \frac{1}{2}(\varepsilon_{pst,qru} + \varepsilon_{qst,pru} + \varepsilon_{prt,qsu} + \varepsilon_{qrt,psu})\\ &= \varepsilon_{pqu,srt} + \varepsilon_{sru,pqt} - \frac{1}{2}(\varepsilon_{psu,qrt} + \varepsilon_{qsu,prt} + \varepsilon_{pru,qst} + \varepsilon_{qru,pst}).\end{aligned} \tag{4.12}$$

5. THE RANK AND THE NUMBER OF INDEPENDENT COMPONENTS OF THE INCOMPATIBILITY TENSOR

The tensor $\boldsymbol{\eta}^{\{\mathbf{m(n-2)}\}}(\boldsymbol{\varepsilon}^{\{\mathbf{m}\}})$ of rank $m(n-2)$ with the components

$$\eta_{i_{11}\dots i_{1;n-2}\,\dots\, i_{m1}\dots i_{m;n-2}} = \epsilon_{i_{11}\dots i_{1n}}\dots\epsilon_{i_{m1}\dots i_{mn}}\varepsilon_{i_{1n}i_{2;n-1}i_{3n}\dots i_{ml},\, i_{1;n-1}i_{2n}i_{3;n-1}\dots i_{mk}}, \tag{5.1}$$

$$l = \begin{cases} n & \text{if } m \text{ odd}\\ n-1 & \text{if } m \text{ even}\end{cases}, \quad k = \begin{cases} n & \text{if } m \text{ even}\\ n-1 & \text{if } m \text{ odd}\end{cases},$$

is a generalization of the Kröner incompatibility tensor $\boldsymbol{\eta}^{\{\mathbf{2(n-2)}\}}(\boldsymbol{\varepsilon}^{\{\mathbf{2}\}})$ of rank $2(n-2)$ with Cartesian components (2.2) to every value of m. Thus, the array of indices of this tensor is two-dimensional.

Comparing the definition (5.1) with (1.3), we see that, in the case of $m = 1$, i.e., in vector analysis, we have $\boldsymbol{\eta}^{\{\mathbf{n-2}\}}(\boldsymbol{\varepsilon}^{\{\mathbf{1}\}}) = (\operatorname{Rot}\boldsymbol{\varepsilon}^{\{\mathbf{1}\}})^{\{\mathbf{n-2}\}}$. The number of independent components of $\boldsymbol{\eta}^{\{\mathbf{m(n-2)}\}}$ coincides with the number of independent components of the tensor $\mathbf{R}^{\{\mathbf{2m}\}}$ dual to $\boldsymbol{\eta}^{\{\mathbf{m(n-2)}\}}$, i.e., constructed from $\boldsymbol{\eta}^{\{\mathbf{m(n-2)}\}}$ by contractions with m Levi-Civita symbols,

$$R_{p_1q_1\dots p_mq_m} = \epsilon_{p_1q_1i_{11}\dots i_{1;n-2}}\dots\epsilon_{p_mq_mi_{m1}\dots i_{m;n-2}}\eta_{i_{11}\dots i_{1;n-2}\,\dots\, i_{m1}\dots i_{m;n-2}}. \tag{5.2}$$

Let us use the property of summation with respect to $n-2$ indices,

$$\epsilon_{p_1q_1i_{11}\dots i_{1;n-2}}\epsilon_{i_{11}\dots i_{1n}} = (n-2)!(\delta_{p_1i_{1;n-1}}\delta_{q_1i_{1n}} - \delta_{p_1i_{1n}}\delta_{q_1i_{1;n-1}}) \tag{5.3}$$

and, after the substitution of (5.1) into (5.2), obtain

$$\begin{aligned}R_{p_1q_1\dots p_mq_m} &= [(n-2)!]^m(\delta_{p_1i_{1;n-1}}\delta_{q_1i_{1n}} - \delta_{p_1i_{1n}}\delta_{q_1i_{1;n-1}})\cdot\ldots\cdot\\ &\cdot(\delta_{p_mi_{m;n-1}}\delta_{q_mi_{mn}} - \delta_{p_mi_{mn}}\delta_{q_mi_{m;n-1}})\varepsilon_{i_{1n}i_{2;n-1}i_{3n}\dots i_{ml},\, i_{1;n-1}i_{2n}i_{3;n-1}\dots i_{mk}}\end{aligned} \tag{5.4}$$

In the case of $m = 2$ the tensor $\mathbf{R}^{\{\mathbf{2m}\}}$, up to constant factor, coincides with the Riemann–Christoffel tensor $\mathbf{R}^{\{\mathbf{4}\}}$ whose rank is equal to four in the space of any dimension. The assumption that all components of this tensor vanish, which means that the space R^n (in which the deformed medium is contained) is Euclidean, leads to the $b_{2;n} = n^2(n^2-1)/12$ independent compatibility equations (2.7) mentioned above.

It is natural to interpret the object with the components (5.4) of rank $2m$ as a generalization of the Riemann–Christoffel tensor to an arbitrary m. By definition, this tensor is antisymmetric with respect to the transpositions of indices inside every pair $\{p_i, q_i\}$, $i = 1,\dots,m$, and, moreover, is symmetric with respect to the transpositions of the pairs themselves, $\{p_i, q_i\}$ and $\{p_j, q_j\}$. Since the number of different nonzero contributions in each of these pairs is equal to $b_{1;n} = n(n-1)/2$, it follows that the symmetry indicated above keeps $C^m_{m-1+b_{1;n}}$ components of the tensor $\mathbf{R}^{\{\mathbf{2m}\}}$, and hence, also of the tensor $\boldsymbol{\eta}^{\{\mathbf{m(n-2)}\}}$. However, the family of these components cannot be independent due to the Ricci identities, which, for example, for $m = 2$, are represented in the way well known in differential geometry,

$$R_{p_1q_1p_2q_2} + R_{p_1p_2q_2q_1} + R_{p_1q_2q_1p_2} = 0. \tag{5.5}$$

The analogs of the Ricci identities (5.5) for the generalized Riemann–Christoffel tensor $\mathbf{R}^{\{2\mathbf{m}\}}$ reduce the number if the independent components of the tensor by C_n^{2m}. Certainly, this reduction is nontrivial only if the dimension n of the space is not less than $2m$.

Taking into account what was said above, we finally write

$$b_{m;n} = C^m_{m-1+b_{1;n}} - C_n^{2m}, \quad m \geqslant 2,\ n \geqslant 2. \tag{5.6}$$

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