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Asymptotic Theory of Linear Water Waves in a Domain with Nonuniform Bottom with Rapidly Oscillating Sections

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Abstract. A linear problem for propagation of gravity waves in the basin having the bottom of a form of a smooth background with added rapid oscillations is considered. The formulas derived below are asymptotic ones; they are quite formal, and we do not discuss the problem concerning their uniformness with respect to these parameters.

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1. INTRODUCTION

In this paper, we consider a linear problem for propagation of gravity waves in a basin having the bottom of a form of a smooth background with added rapid oscillations. Namely, we assume that the function D describing the depth in the basin with characteristic size L in the physical variables is

$$D(x) = d\Big(D_0\Big(\frac{x}{L}\Big) + \frac{d_1}{d} D_1\Big(\frac{x}{L}, \frac{\Theta(x/L)}{l_1/L}\Big)\Big).$$

Here x is a 2D vector column with components (x_1, x_2), d is the characteristic basin depth, d_1 is the characteristic height of short oscillations, and l_1 is the characteristic horizontal size of short oscillations. The scalar positive functions $D_0(x)$ and $D_1(x, y)$ and the vector column function Θ with components $\Theta_1(x), \Theta_2(x)$ are assumed to be smooth, and $D_1(x, y)$ is 2π-periodic with zero average with respect to each variable y_1, y_2 of vector y. We assume also that $\nabla\Theta_1$ and $\nabla\Theta_2$ are not parallel at each point x. In the simplest case $\Theta_j(x) = x_j$ (in which $D_1 = D_1(\frac{x}{L}, \frac{x}{l_1})$), a nonlinear dependence $\Theta_j(x)$ of x means that the bottom oscillation could be different at different places of the bottom. Assume that, for $|x|$ large enough, the function D becomes a constant, although our future consideration are more or less formal, and we do not use this assumption.

In terms of physical variables, the linearized system for the gravity waves is written for the potential Φ (see, e.g., [4, 5]) as follows:

$$\Delta\Phi + \Phi_{zz} = 0, \quad -D \leqslant z \leqslant 0, \quad \Phi_z + \langle\nabla D, \nabla\Phi\rangle = 0 \quad \text{for} \quad z = -D, \quad \Phi_{tt} + g\Phi_z = 0 \quad \text{for} \quad z = 0.$$

The function η describing the free surface elevation can be easily found from the formula $\eta = -(1/g)\Phi_t$ for $z = 0$, where g is the gravity acceleration. In the nonstationary case, we consider the Cauchy–Poisson problem at $t = 0$ for this system, $\Phi|_{t=0} = \phi(x/l)$, $\partial\Phi/\partial t|_{t=0} = \phi^1(x/l) \equiv -g\eta^0(x/l)$, where $\Phi^0(y)$ and $\eta(y)$ are smooth functions decaying at infinity more rapidly than $1/|y|^{1+\beta}$; $\beta > 0$ and l characterize the size of the initial perturbation.

In addition to the characteristic values L, d, d_1, l_1, introduce the characteristic time T during which the wave passes the distance L, and let a be the characteristic amplitude of the wave. Now we introduce the parameters

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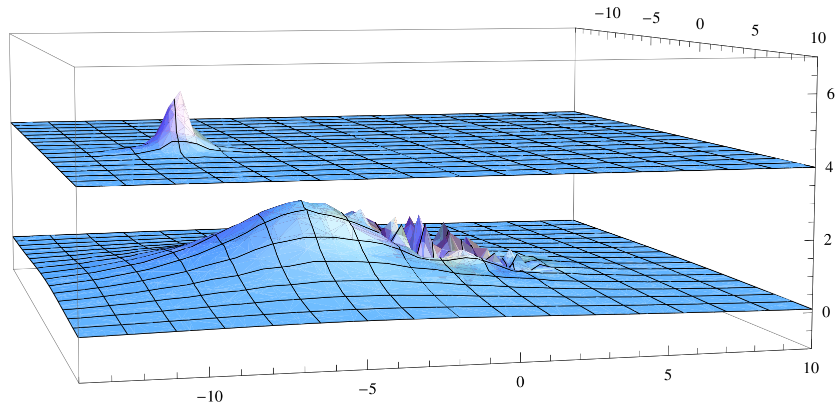

Fig. 1. The initial perturbation (on the left-hand side over the water surface) and the part of the ocean bottom in the form of slow varying underwater bank with a domain of rapid oscillations.

$$h=\frac{d}{L},\quad \varepsilon=\frac{l_1}{L},\quad \delta=\frac{d_1}{d},\quad \lambda=\frac{d}{l_1}=\frac{h}{\varepsilon},\quad \delta\lambda=\frac{\delta h}{\varepsilon}=\frac{d_1}{l_1},\quad \mu=\frac{l}{L},\tag{1.1}$$

and pass to the new dimensionless variables $z=dz'$, $x=Lx'$, $t=\frac{L}{\sqrt{gd}}t'$, $D'=\frac{1}{d}D(x)$, $\Phi'=\frac{\Phi}{a\sqrt{gd}}$, $\eta'=\eta/a$. Rewrite the above system in the new variables, omitting the superscript (the prime) to simplify the notation,

$$h^2\Delta\Phi+\Phi_{zz}=0,\qquad -D\leqslant z\leqslant 0,\tag{1.2}$$

$$\Phi_z+h^2\langle\nabla D,\nabla\Phi\rangle=0,\qquad \text{for}\quad z=-D,\tag{1.3}$$

$$h^2\Phi_{tt}+\Phi_z=0,\qquad \text{for}\quad z=0,\tag{1.4}$$

$$D=D_0+\delta D_1\left(x,\frac{\theta(x)}{\varepsilon}\right),\tag{1.5}$$

$$\Phi\Big|_{z=0,t=0}=\phi\left(\frac{x}{\mu}\right),\quad h\frac{\partial\Phi}{\partial t}\Big|_{z=0,t=0}=-\eta\Big|_{t=0}=-\eta^0\left(\frac{x}{\mu}\right).\tag{1.6}$$

For the free elevation, we have $\eta=-h\Phi_t|_{z=0}$. Let us make some assumptions about the parameters. Assume that h, ε, and δ and, as a rule, μ are small. It does not mean that we study waves on a shallow water only. The smallness of h means that the background D_0 of the depth function D is a quite smooth (slow varying) function. On the contrary, the smallness of ε means that there are fast oscillations on the bottom, and we assume that the space-frequency of these oscillations is not bigger than the wavelength of the wave under consideration, $\lambda\leqslant 1$. The smallness of μ means that we consider waves which are short with respect to the size of the basin. Considering of long waves means that waves are long with respect to the depth, $1\gg\mu\gg h$. The parameter δ characterizes the amplitude of the depth oscillation. The formulas which we derive below are asymptotic ones, they are quite formal and we do not discuss the question about their uniformness with respect to these parameters. Below we shall use the assumption $\mu=O(h^\alpha)$, and we shall analyze it at an appropriate place. We want to say that, in real situation, the parameters listed above are just numbers, and assumptions of our type are more or less artificial.

Our main aim is to study solutions to Eqs. (1.2)–(1.6). It is impossible to obtain exact formulas for these solutions in the general situation suitable for real applications; one can speak only about some asymptotics with respect to (small) parameters h, ε, δ, and μ. Needless to say, these asymptotics could depend on the relationship between these parameters. Our strategy is as follows. Like in [8], we split our consideration into two steps: (1) using the operator version of adiabatic approximation [8, 9, 11], we derive the reduced (homogenized) pseudodifferential equation, eliminating the vertical variable z and fast oscillations in the coefficients; (2) we show that, choosing the small parameters in an appropriate form, we can simplify this equation, presenting it in the form of a generalized Boussinesq equation with variable velocity and dispersion and reduce the original problem to this one, which was studied in [10]. The step (1) is technically complicated, and the most part of this paper is devoted to it. We split it into two steps: (1a) we eliminate the vertical variable z and (1b) we eliminate the fast oscillations. In spite of complicated calculations, the final formulas and physical conclusions are quite simple and understandable. Thus, we present and discuss these final formulas before the long calculations. To make our considerations clear, along with condition (1.6), we consider the initial condition of WKB-type functions with the small parameter μ,

$$\Phi\Big|_{z=0,t=0} = A_1^0(x)e^{\frac{i}{\mu}S^0(x)}, \quad h\frac{\partial\Phi}{\partial t}\Big|_{z=0,t=0} = -\eta\Big|_{t=0} = -A_2^0(x)e^{\frac{i}{\mu}S^0(x)}, \tag{1.7}$$

where S^0, A^1, A^2 are smooth functions, $\nabla S^0 \neq 0$, and A^1, A^2 are compactly supported.

The paper is organized as follows. First, in Section 2, we present the reduced equation in general form, briefly describe its properties, and present the main qualitative result concerning the long wave, the Boussinesq type equation, discuss the influence of dispersion of various type, and estimate its influence in the case of tsunami wave. In Section 3, we describe general ideas of constructing a pseudodifferential equation on the surface. In Section 4, we discuss the expansion of the symbols of operators with respect to the small parameters. We use the theory of functions of noncommuting operators to eliminate the vertical variable in Section 5 and to eliminate the fast phase in Section 6.

2. REDUCED HOMOGENIZED AND LINEARIZED BOUSSINESQ TYPE EQUATIONS AND THEIR SOLUTIONS

As was mentioned above, one of our main objectives is the construction of an asymptotic solution to (1.2)–(1.4) with initial conditions in the form of (1.6) and (1.7). From the point of view of applications, the most interesting object here is the function $\eta(x,t)$ describing the free elevation.

Assume for a moment that the bottom function D is constant. Denote by $\tilde\eta(p,t)$ the Fourier transform of the function η:

$$\tilde\eta = \frac{1}{2\pi\mu}\int_{\mathbb{R}^2} e^{-\frac{i}{\mu}\langle p,y\rangle}\eta(y,t)dy.$$

Here $p=(p_1,p_2)$ are the momenta conjugate to (x_1,x_2). As is well known (and easy to show), the function $\tilde\eta$ satisfies the following equation:

$$\mu^2\tilde\eta_{tt} = \frac{\mu}{h}|p|\tanh\Big(\frac{h}{\mu}|p|D\Big)\tilde\eta, \quad \tilde\eta|_{t=0} = u^0(p), \quad \mu\tilde\eta_t|_{t=0} = u^1(p), \quad \frac{h}{\mu} = \frac{d}{l}, \tag{2.1}$$

where the initial functions $u^0(p), u^1(p)$ are reconstructed from the functions $\eta^0(y)$ and $\phi(y)$. Assume that these functions decay as $|p|\to\infty$. The solution of the ODE (2.1) is

$$\tilde\eta = e^{it\frac{\omega(p)}{\mu}}A_+^0(p) + e^{-it\frac{\omega(p)}{\mu}}A_-^0(p), \quad \omega(p) = \sqrt{\frac{\mu}{h}|p|\tanh\Big(\frac{h}{\mu}|p|D\Big)}, \quad A_\pm^0(p) = \frac{(u^0(p)\pm iu^1(p))}{2}. \tag{2.2}$$

Using the inverse Fourier transform, one can construct exact solutions in an integral form and then study their asymptotics for $\mu \ll 1$, which is actually the asymptotics for large x with respect to the dimensional variables. It is good to take into account the ratio $\frac{h}{\mu} = \frac{d}{l}$ for the parameters h and μ. If $\frac{h}{\mu}$ is small, then we can use the Taylor expansion for ω^2 and write

$$\omega^2 = p^2D - \frac{h^2}{3\mu^2}p^4D^4 + O\Big(\frac{h^4}{\mu^4}\Big), \quad \frac{\omega}{\mu} = \frac{|p|\sqrt{D}}{\mu} - \frac{\gamma}{6}|p|^3D^{3/2} + \frac{1}{\mu}O\Big(\frac{h^4}{\mu^4}\Big). \tag{2.3}$$

where $\gamma = \frac{h^2}{\mu^3}$. If t is bounded and $h^2 \leqslant \mathrm{const}\,\mu^3$, then

$$e^{\pm it\frac{\omega(p)}{\mu}}A^0(p) = e^{\pm it\frac{p^2D}{\mu}}\Big(A_\pm(p,t) + \frac{t}{\mu}O\Big(\frac{h^4}{\mu^4}\Big)\Big), \quad A_\pm(p,t) = A_\pm^0(p)e^{\mp it\frac{\gamma}{6}|p|^3D^{3/2}}. \tag{2.4}$$

Of course, if $h^2 = o(\mu^3)$ (or $\gamma = o(1)$), then one can write $A_\pm(p,t) = A_\pm^0(p) + O(\frac{h^2}{\mu^3})$. Under the assumption that $A_\pm^0(p)$ are rapidly decaying functions as $|p|\to\infty$, one can use (2.2) and this elementary consideration to obtain various asymptotic formulas for the solutions of the original Cauchy–Poisson problem. Needless to say that these formulas are not uniform with respect to parameters μ, h, and are trivial like the expansions (2.3) and (2.4).

Let us discuss (2.1)–(2.4) from the point of view of differential equations. First, we come back to the variable (x_1,x_2); then equation (2.1) takes a form of the *pseudodifferential equation*

$$\mu^2\eta_{tt} + H'(-i\mu\nabla, \mu, h)\eta = 0, \quad H' = \frac{\mu}{h}|p|\tanh\Big(\frac{h}{\mu}|p|D\Big). \tag{2.5}$$

The function H' is the *symbol* of this μ-pseudodifferential operator. Using the Taylor expansion of H' and replacing it by $Dp^2 - \frac{\gamma\mu}{3}D^4p^4$, $\gamma = \frac{h^2}{\mu^3}$, means the approximation of the pseudodifferential equation by the differential (linearized Boussinesq) equation

$$\eta_{tt} - D^2\eta_{xx} - \frac{D^4h^2}{3}\eta_{xxxx} = 0 \quad (a) \quad \Longleftrightarrow \quad \mu^2\eta_{tt} - \mu^2D^2\eta_{xx} - \gamma\mu^5\frac{D^4}{3}\eta_{xxxx} = 0 \quad (b). \tag{2.6}$$

We want to stress the following important facts. First, there are two parameters in the problem: h and μ, and the asymptotic formulas crucially depend on their relationship. Second, operations with operators could be replaced by the operations with their symbols. The parameter μ comes from the initial data, and the parameter h comes from the equation. Both of them play an important role. Of course, equations (a) and (b) in (2.6) are equivalent; however, from the asymptotic point of view, the form (b) is more informative than (a). For instance, if $\gamma = o(1)$, then we can omit the term with the fourth derivatives and replace the Boussinesq equation by the wave equation $\eta_{tt} - D^2\eta_{xx} = 0$. The term with the fourth derivatives $\gamma\mu^5\frac{D^4}{3}\eta_{xxxx}$ is known as the weak dispersion. We want to note that one may view equations (2.5) and (2.6) as h-pseudodifferential equations

$$h^2\eta_{tt} + H(-ih\nabla)\eta = 0 \tag{2.7}$$

with the symbols $H = |p|\tan(D|p|)$ and $H = Dp^2 - \frac{D^3}{3}p^4$; a passage from the h-pseudodifferential equation with the symbol H to μ-pseudodifferential equations (2.5), (2.6) (b), with the symbols H' is

$$H'(p, h, \mu) = \frac{\mu}{h}H\Big(\frac{h}{\mu}p\Big). \tag{2.8}$$

Note that this "play" with the parameters and the passage from h-pseudodifferential equations to μ-pseudodifferential make sense just in the construction of asymptotic solutions.

Consider now the case in which the bottom is slowly varying and can be described by a smooth function $D = D_0(x)$. Using the technique of pseudodifferential operators, one can derive the pseudodifferential equation for the free elevation similar to (2.7),

$$\mu^2\eta_{tt} = H'(x, -i\mu\nabla, \mu, h)\eta; \tag{2.9}$$

however, now with the symbol $H'(x, p, \mu, h)$ determined by some asymptotic expansion. It is more reasonable to construct at first the symbol of h-pseudodifferential operator, because the h-pseudodifferential equation occurs in the original system (without the initial data and parameter μ), and then pass to the μ-pseudodifferential equation by a formula similar to (2.8). The operators $-i\mu\frac{\partial}{\partial x_j}$ and x_j do not commute, and it is necessary to agree about their ordering. There exist many reasons to suppose that the differential operators $-ih\frac{\partial}{\partial x_j}$ act first, and the operators x_j act after them. Using the Feynman–Maslov notation [16, 18], we can now write $H(\overset{2}{x}, -i\mu\overset{1}{\nabla}, \mu, h)$. We shall use this ordering as a rule below. From the point of view of further asymptotic construction with respect to parameters μ and h, it is sufficient to find an explicit expression for the leading term first, and probably, a part of the second correction. As was shown in [18–20],

$$H(x,\, p,\, h) = H^0(x,\, p) + hH^1(x,\, p) + O(h^2), \quad H^0 = |p|\tanh(D_0(x)|p|), \tag{2.10}$$
$$H^1 = -\frac{i}{2}\mathrm{tr}\frac{\partial^2 H_0}{\partial x\partial p} = i\frac{D_0|p|\tanh(D_0|p|) - 1}{\cosh^2(D_0|p|)}\langle\nabla D_0,\, p\rangle.$$

Replacing $H(p, x, h)$ by $H'(p, x, h, \mu)$ and reexpanding $H'(p, x, h, \mu)$ with respect to the parameter μ, we obtain a linearized Boussinesq-type equation with variable coefficients.

Now let us consider the case in which some parts of the bottom include fast oscillations and the function D is of the form (1.5), and introduce new parameters δ and ε characterizing the size and the height of these oscillations. We want to derive pseudodifferential (or differential) equations describing the leading term of the free elevation with coefficients smoothly depending on x. Generally speaking, the asymptotic solution should be presented like a series of different modes (similar, e.g., to interior modes in a stratified liquid).

Important remark. Moreover, these modes can intersect, and one faces the effect of the so-called mode conversion, or intersection of characteristics (see, e.g., [7]). Fortunately, these effects are not crucial for "long waves" under some physically reasonable assumptions about the relationship between the parameters μ, h, δ, ϵ, etc. (or about the relationship between the dimensional values l, L, l_1, d, d_1).

In this case, one can use the Boussinesq-type equation with variable coefficients and take into account two types of weak dispersion: the standard dispersion coming from the water wave theory and another one implying the rapidly oscillating parts of the bottom.

We want to present below final simple formulas; however, we are to mention some points in advance. The problem with fast oscillations is a problem of homogenization. In the case in question, we cannot use standard methods of homogenization (see, e.g., [1–3]) because the actual wave is small. For this reason, classical methods are not applicable here, and we consider the averaging in the frame of adiabatic approximation combined with operator methods and use the Maslov theory of functions of noncommuting operators [8, 9, 15].

An analogous computation gives us a pseudodifferential equation of the type (2.9); however, the fact that we have a series of parameters in our problem and consider long waves (as compared with the depth) enables us to make an expansion and obtain a linearized Boussinesq-type equation with variable coefficients and fourth derivatives showing the effects of fast oscillations. Note that the influence of fast bottom oscillations is similar to that obtained from changing of the full system by the Boussinesq-type equation $\eta_{tt}(x, t) = (g\langle\nabla, D(x)\nabla\rangle + (g/3)D^3(x)\Delta^2)\eta(x, t)$. From the physical point of view, this means that the propagating long wave induces all kinds of waves, but only the long wave approximation affects the main part of the asymptotic solution.

Now let us introduce the main formulas.

Theorem 1. *Denote by $D_{1k}(x)$ the Fourier coefficients of the function $D_1(x,y)$ on the torus $\mathbb{T}^2 = \{y_1 \in [0, 2\pi], y_2 \in [0, 2\pi]\}$. Thus,*

$$D_1(x,y) = \sum_{k\in\mathbb{Z}^2, k\neq 0} D_{1k}(x)e^{ik\cdot y}. \tag{2.11}$$

Then the symbol of the reduced equation (2.9) *for the free surface elevation has the following form*:

$$\begin{aligned} H'(x, p) \approx{}& D_0|p|^2 - \delta^2 D_0 \sum_{|k|\neq 0} \frac{|D_{1k}|^2}{D_0^2|\Theta_x k|^2}\langle\Theta_x k, p\rangle^2 \\ &- \frac{h^2}{3\mu^2}D_0^3|p|^4 - \frac{\varepsilon^2\delta^2}{\mu^2}\sum_{|k|\neq 0}\frac{|D_{1k}|^2}{D_0^2|\Theta_x k|^2}\Big(p^2 - 2\frac{\langle\Theta_x k, p\rangle^2}{|\Theta_x k|^2}\Big)^2 - i\mu\langle\nabla D_0, p\rangle. \end{aligned} \tag{2.12}$$

In the physical variables, the Cauchy problem for this linearized Boussinesq-type equation is

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\eta(x, t) ={}& \Bigg[-gD_0(x)\Delta + igD_0(x)\sum_{|k|\neq 0}\frac{|D_{1k}|^2}{D_0^2(x)|\Theta_x k|^2}\langle\Theta_x k, \nabla\rangle^2 - \frac{g}{3}D_0^3(x)\Delta^2 \\ &- gdl_1^2L^2\sum_{|k|\neq 0}\frac{|D_{1k}|^2}{D_0^2(x)|\Theta_x k|^2}\left(\Delta - 2\frac{\langle\Theta_x k, \nabla\rangle^2}{|\Theta_x k|^2}\right)^2 - \frac{g}{L}\langle\nabla D_0, \nabla\rangle\Bigg]\eta(x, t), \\ \eta|_{t=0} ={}& \eta_0(x/l), \qquad \eta_t|_{t=0} = 0. \end{aligned} \tag{2.13}$$

Remark. If we eliminate the terms including the forth degree with respect to p_j in (2.12) (and the corresponding derivatives in (2.13)), which is possible for a sufficiently long wave, then we obtain a wave equation [9].

As was shown in [10], the asymptotic solution for problem (2.13) is given by the formula

$$\eta(x, t) = \sqrt{\frac{l}{|X_\psi(\psi, t)|}}\sqrt{\frac{D_0(0)}{D_0(X(\psi, t))}}Re\left[e^{-i\pi m(\psi, t)/2}aF\left(\frac{S(t, x, \psi)}{l}, \psi, q(\psi, t)\right)\right]. \tag{2.14}$$

The function $q(\psi, t)$ describes the dispersion effects and can be presented as a combination of two functions $q(\psi, t) = q_1(\psi, t) + q_2(\psi, t)$. The first function is connected with the standard water dispersion and is defined by the formula

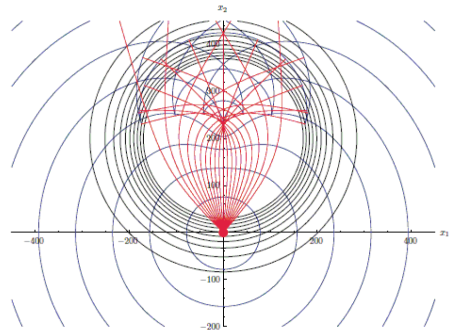

Fig. 2. Trajectories for variable depth.

$$q_1(\psi,\, t) = \frac{D^{3/2}(0)}{6l^3}\sqrt{g}\int_0^t D(X(\psi,\, \tau))d\tau. \tag{2.15}$$

The other function describes the influence of fast oscillations of the bottom on the wave profile. This function is given by the following expression:

$$q_2(\psi,\, t) = \frac{l_1^2 d_1^2}{2l^3}\sqrt{\frac{g}{D_0(0)}}\sum_{|k|\neq 0}\int_0^t \Big(P^2 - 2\frac{\langle P, \Theta_x(X)k\rangle^2}{\langle \Theta_x(X)k, \Theta_x(X)k\rangle}\Big)^2 \frac{|D_{1k}(x)|^2}{D_0^2(X)\langle \Theta_x(X)k, \Theta_x(X)k\rangle}\, d\tau. \tag{2.16}$$

If we choose the initial function $\eta^0(x/l)$ in the special case of "the simple piston model" given by the formula

$$\eta_0(x/l) = \frac{a}{\Big(1 + \big(\frac{x_1}{lb_1}\big)^2 + \big(\frac{x_2}{lb_2}\big)^2\Big)^{3/2}}, \tag{2.17}$$

then the function $F(y,\, \psi,\, q)$ can be represented in the simple form via a combination of Airy function and its derivatives,

$$F(y,\, \psi,\, q) = -\frac{\pi b_1 b_2}{\sqrt{6q}}\frac{d}{dz}\Big(Ai^2(z) + iAi(z)Bi(z)\Big)\Big|_{z=\frac{y+i\beta(\psi)}{(12q)^{1/3}}}, \tag{2.18}$$

where $\beta(\psi) = \sqrt{b_1^2\cos^2\psi + b_2^2\sin^2\psi}$. Otherwise, the form of the function F can be much more complicated. Here $S(t,\, x,\, \psi)$ is the action function which can be evaluated by the formula

$$S(t,\, x,\, \psi) = \left(\frac{D_0(0)}{D(X(\psi,\, t))}\right)^{1/4} y(x),$$

where $y(x)$ is the distance to the wave front. If the point x lies outside the wave front area, then we take positive sign for $y(x)$, and otherwise, the negative sign.

Here $X(\psi,\, t)$ and $P(\psi,\, t)$ are the solutions of the Hamilton system

$$\dot{X} = \mathcal{H}_p(X,\, P), \quad \dot{P} = -\mathcal{H}_x(X,\, P), \quad \mathcal{H}(x,\, p) = |p|\sqrt{D_0(x)}, \quad X|_{t=0} = 0,\, P|_{t=0} = (\cos\psi,\, \sin\psi).$$

The trajectories $X(\psi,\, t)$ and wave fronts for different instants of time are shown on the Fig. 2.

The function $m(\psi,\, t)$ is the Morse index of the trajectory $X(\psi,\, t)$, and it counts the focal points on this trajectory till the time moment t.

We express the coefficients $\frac{h^2}{3\mu^2}$, $\frac{h^2\delta^2}{\mu^2\lambda^2}$ in (2.12) in terms of the original variables of the problem. Using formulas (1.1), we find

$$\frac{h^2}{3\mu^2} = \frac{1}{3}\Big(\frac{d}{l}\Big)^2, \qquad \frac{\varepsilon^2\delta^2}{\mu^2} = \frac{h^2\delta^2}{\mu^2\lambda^2} = \frac{d_1^2}{d^2}\frac{l_1^2}{l^2}. \tag{2.19}$$

We want to discuss the influence of "dispersion" terms (the coefficients of the fourth degrees of p_j). These terms began to play role during a dimensionless time $t \sim 1$ $\Big($the real time is $T = \frac{L}{\sqrt{gd}}\Big)$ if they are proportional to $\mu = \frac{l}{L}$. During the time T, the initial perturbation passes the distance $\sim L$. Thus, consider the equations $\mu \approx \frac{h^2}{3\mu^2} = \frac{1}{3}\big(\frac{d}{l}\big)^2$ and $\mu \approx \frac{2h^2\delta^2}{\mu^2\lambda^2} = 2\frac{d_1^2}{d^2}\frac{l_1^2}{l^2}$. It is easy to see that they are equivalent to

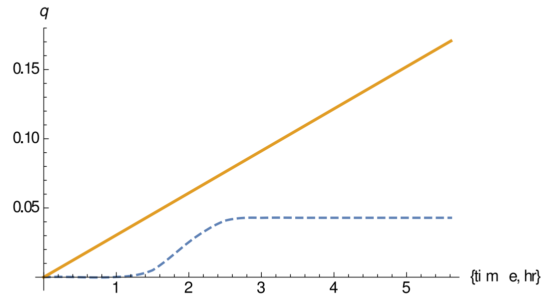

Fig. 3. Graphs of $q_1(\pi/3,\, t)$ (thick line), and the function $q_2(\pi/3,\, t)$ (dashed line).

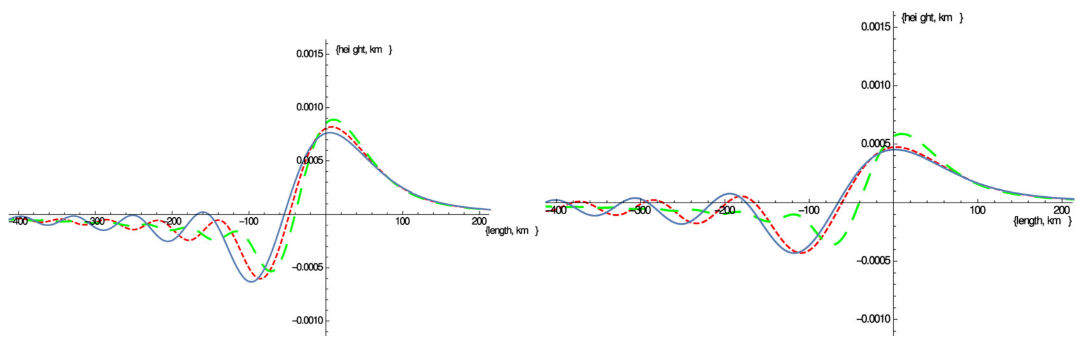

Fig. 4. Graphs of the wave profiles for the angle $\psi = \pi/3$ and the times 2.5 hr (left) and 5.5 hr (right). The short dashed line corresponds to the water oscillation, and the long dashed line to the bottom oscillations. The solid line corresponds to a combination of these dispersion effects.

$$L^{\mathrm{wat}} \approx \frac{3l^3}{d^2}, \quad L^{\mathrm{osc}} \approx \frac{d^2 l^3}{2d_1^2 l_1^2}. \tag{2.20}$$

Let us point out the scales when the standard water dispersion and the dispersion implied by rapid oscillations of the bottom are comparable. Write $L^{\mathrm{wat}} \approx L^{\mathrm{osc}} \iff d \approx (6)^{1/4}\sqrt{l_1 d_1} \approx 1.56\sqrt{l_1 d_1}$. If one takes $d_1 = 0.1\,\mathrm{km}$, $l_1 = 50\,\mathrm{km}$, then $d \approx 1.56\sqrt{5} = 3.5\,\mathrm{km}$.

As an example, we consider a basin with constant depth of $D_0 = 4\,\mathrm{km}$ and the area $L = 4000$. The initial source is symmetric ($b_1 = b_2 = 1$) with $l = 50\,\mathrm{km}$, and is located at the point with coordinates $(-1000\,\mathrm{km},\, -1000\,\mathrm{km})$. The fast oscillations are described by the following formulas:

$$D_1\left(x,\, \frac{\Theta(x)}{\varepsilon}\right) = A(x)\left(\cos\frac{x_1}{\varepsilon} + \cos\frac{x_2}{\varepsilon}\right),\ A(x) = e^{-4|x|^2}. \tag{2.21}$$

The dispersion is described by the formulas

$$q_1(\psi,\, t) = \frac{1}{3}\frac{D_0^{5/2}\sqrt{g}}{l^3}t, \quad q_2(\psi,\, t) = \frac{d_1^2 l_1^2}{l^3}\frac{\sqrt{g}}{\sqrt{D_0}}\int_0^t (P_1^2(\psi,\, \tau) - P_2^2(\psi,\, \tau))^2 d\tau. \tag{2.22}$$

In Fig. 3, we have presented two graphs of dispersion coefficients for given $\psi = \pi/3$ and with dependance of time.

The wave profiles given by (2.14), (2.18) at the time moments of 2.5 and 5.5 hours are shown on Fig. 4

3. THE PSEUDODIFFERENTIAL EQUATION ON THE FREE SURFACE

First, we regularize the problem and remove the dependence on the small parameter ε in the coefficients. Following [12–14, 8], we seek the potential Φ in the form

$$\Phi = \Psi(x, \theta(x)/\varepsilon, z, t), \tag{3.1}$$

where $\Psi(x, y, z, t)$ is a 2π-periodic function, of the variables y_1, y_2, depending on parameters ε, δ, etc., which are omitted to simplify the notation. The substitution of this function into (1.2)–(1.4) gives the equations

$$(h\nabla + \lambda\nabla_y^\theta)^2\Psi + \Psi_{zz} = 0, \qquad -D \leqslant z \leqslant 0, \tag{3.2}$$

$$\Psi_z + \langle h\nabla D + \delta\lambda\nabla_y^\theta D_1, (h\nabla + \lambda\nabla_y^\theta)\rangle\Psi = 0, \qquad \text{for} \quad z = -(D_0 + \delta D_1), \tag{3.3}$$

$$h^2\Psi_{tt} + \Psi_z = 0, \quad \eta = -h\Psi_t \qquad \text{for} \quad z = 0, \tag{3.4}$$

$$D = D_0 + \delta D_1(x, y). \tag{3.5}$$

Here $\nabla_y^\theta = \Theta_x\nabla_y$, Θ_x is the 2×2 matrix consisting of the vector columns $\nabla\Theta_1$, and $\nabla\Theta_2$. Recall that $\lambda = \frac{h}{\varepsilon}$. Write also $\hat{p} = -ih\frac{\partial}{\partial x}$. To begin with, assume that $\lambda \sim 1$.

The main aim of this section is to express the solution $\Psi(x, y, z, t)$ using a certain function $v(x, t)$ which does not depend on the variables z and y and satisfies a simpler equation.

Introduce the function

$$\phi(x, y, t) = \Psi(x, y, z, t)|_{z=0}. \tag{3.6}$$

Assume that one had constructed a solution $\Psi(x, y, z, t)$ to problem (3.2)–(3.4). Then the derivative $\frac{\partial\Psi}{\partial z}(x, y, z, t)|_{z=0}$ defines a linear operator $\hat{\mathbf{L}}$,

$$\frac{\partial\Psi}{\partial z}(x, y, z, t)|_{z=0} = \hat{\mathbf{L}}\phi,$$

known as Dirichlet-to-Neumann mapping. As soon as the operator $\hat{\mathbf{L}}$ is constructed, one can reduce the original system (3.2)–(3.4) to a 2D equation on the plane $z = 0$ for the function $\phi(x, t)$,

$$h^2\phi_{tt} + \hat{\mathbf{L}}\phi = 0. \tag{3.7}$$

Lemma 1. *The operator $\hat{\mathbf{L}}$ acting on the space $L_2[\mathbb{R}_x^2 \times \mathbb{T}_y^2]$, where $\mathbb{T}^2$ is the 2D torus $\{y_1 \in [0, 2\pi], y_2 \in [0, 2\pi]\}$, is at least symmetric.*

Proof. Let us fix two functions $\phi_1(x, y)$ and $\phi_2(x, y)$ and, solving system (3.2), (3.3), (3.6), construct two functions $\Psi_1(x, y, z)$ and $\Psi_2(x, y, z)$. Multiply the Laplace equation (3.3) for $\Psi_1(x, y, z)$ by $\Psi_2(x, y, z)$, the Laplace equation (3.3) for $\Psi_2(x, y, z)$ by $\Psi_1(x, y, z)$, subtract the second product from the first one, and integrate the result over x, y, z in the space $x \in \mathbb{R}_x^2, y \in \mathbb{T}_y^2, z \in [-D(x, y), 0]$. Using the Green formula, we obtain

$$\begin{aligned}\int_{\mathbb{R}_x^2}\int_{\mathbb{T}_y^2}\Big(\frac{\partial\Psi_1}{\partial z}(x, y, 0)\Psi_2(x, y, 0) - \frac{\partial\Psi_2}{\partial z}(x, y, 0)\Psi_1(x, y, 0)\Big)dx\,dy \\ = \int_{\mathbb{R}_x^2}\int_{\mathbb{T}_y^2}(\hat{\mathbf{L}}\phi_1(x, y)\phi_2(x, y) - \hat{\mathbf{L}}\phi_2(x, y)\phi_1(x, y))dx\,dy = 0.\end{aligned}$$

We want to show that $\hat{\mathbf{L}}$ is a pseudodifferential operator, and it could be presented in the form

$$\hat{\mathbf{L}} = L(\overset{2}{x}, \overset{1}{\hat{p}}, \overset{2}{y}, -i\overset{1}{\nabla}_y, \delta, h). \tag{3.8}$$

Here $\hat{p} = -ih\frac{\partial}{\partial x} = -ih\nabla$, and we use the Feynman–Maslov notation (see [15, 9]) for ordering operators. The symbol $L(x, p, y, \xi, \delta, h)$ (as well as other symbols) also depends on ε; however, to simplify the notation, we do not mention this dependence. Following [18, 19, 12], we seek a solution Ψ in a form of a pseudodifferential operator $\hat{\mathbf{R}} = R(\overset{2}{x}, \overset{1}{\hat{p}}, z, \overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}, \delta, h)$, with the symbol $R(x, p, z, y, \xi, \delta, h)$, acting on the function $\phi(x, y, t)$:

$$\Psi(x, z, y, t) = \hat{\mathbf{R}}\phi(x, y, t). \tag{3.9}$$

Then

$$\frac{\partial\Psi}{\partial z}(x, y, t)|_{z=0} = \frac{\partial\hat{\mathbf{R}}}{\partial z}\Big|_{z=0}\phi(x, y, t) \qquad \text{and} \qquad \hat{\mathbf{L}} = \frac{\partial\hat{\mathbf{R}}}{\partial z}\Big|_{z=0} \Leftrightarrow L = \frac{\partial R}{\partial z}\Big|_{z=0}. \tag{3.10}$$

If one finds a solution ϕ to Eq. (3.7), then one can construct a solution Φ to original system (1.2)–(1.5) by formulas (3.9) and (3.1).

Equation (3.7) is still not absolutely standard for the semiclassical analysis, because the operator ∇ has the factor h (e.g., we have $\hat{p} = -ih\nabla$) and the operator ∇_y has not. This type of equations is known as equations with operator-valued symbol, which are considered using the adiabatic approximation, which we use in an operator form again (see [9, 8, 11]). Namely, we seek some asymptotic solutions to (3.7) in the form

$$\phi = \hat{\chi} v(x,t), \quad \hat{\chi} = \chi(\overset{2}{x}, \overset{1}{\hat{p}}, y, \delta, h), \tag{3.11}$$

and assume that the function $v(x,t)$ satisfies the reduced equation (the so-called Peierls substitution)

$$h^2 \frac{\partial^2 v}{\partial t^2} + \hat{H} v(x,t) = 0, \quad \hat{H} = H(\overset{2}{x}, \overset{1}{\hat{p}}, \delta, h). \tag{3.12}$$

Here the symbols $\chi(x,p,y,\delta,h)$ and $H(x,p,\delta,h)$ are smooth functions depending regularly on the parameters h and δ (see below). It is reasonable to think that the operator $\hat{H}$ is at least symmetric.

If one finds a solution v to (3.12), then

$$\Psi = R\left(\overset{2}{x}, \overset{1}{\hat{p}}, z, \overset{2}{y}, -i\overset{1}{\frac{\partial}{\partial y}}, \delta, h\right) \chi(\overset{2}{x}, \overset{1}{\hat{p}}, y, \delta, h)\Big|_{y=\theta/\varepsilon} v \tag{3.13}$$

is a solution to system (1.2)–(1.5). Of course, one can try to construct the operator $\hat{\mathbf{R}}\hat{\chi}$ (or its symbol) directly, but it seems to be more convenient technically and pragmatic to use the above type of factorization. We must also say that, actually, there are infinitely many operators $\hat{\chi}$ and corresponding equations (3.12) (which correspond to the so-called modes or terms). Thus, generally speaking, the solution to the Cauchy–Poisson problem is a sum over all modes; however, a nice fact is that the principal part of the solution with initial data (1.6), (1.7) uses only the single "main" mode. We shall discuss this question below and first focus on providing formulas for symbols R, L, χ, H.

4. PERTURBATION THEORY FOR THE OPERATORS $\hat{H}$ AND $\hat{\mathbf{L}}$

One cannot obtain effective formulas for the symbols of the operators R, L, χ, and H, and it is possible to speak about finding coefficients of the expansion of these operators with respect to small parameters h, ε, and δ. There is a regular procedure (an algorithm) of such calculations using perturbation theory (needless to say, it is a nontrivial technical problem even to find explicit formulas for the first coefficients). Thus, it is better to analyze at first equation (3.12) to understand how many terms of the expansion of R, L, χ, H one should find to obtain a reasonable asymptotic result.

Let us first discuss the question about a reasonable number of terms in expansions $\chi(x,p,y,\delta,h)$ and $H(x,p,\delta,h)$. Assume that $\hat{H}$ is at least a symmetric operator in $L^2(\mathbb{R}^2)$ and that H has the following expansions:

$$H = H^0(x,p,\delta) + hH^1(x,p,\delta) + h^2H^2(x,p,\delta) + \cdots, \tag{4.1}$$

$$H^j(x,p,\delta) = H_0^j(x,p) + \delta H_1^j(x,p) + \delta^2 H_2^j(x,p) + \cdots, \tag{4.2}$$

$$H^0(x,p,\delta) = \tilde{\mathcal{H}}(p,x,\delta) + \mathcal{H}_{\mathrm{corr}}(p,x,\delta) + O(p^6), \tag{4.3}$$

here $\tilde{\mathcal{H}}(p,x,\delta) = \langle p, Q(x)p\rangle$ is a homogeneous polynomial of second degree with smooth coefficients which are entries of 2×2 symmetric real matrix function $\|Q_{kn}(x)\|$, and $\mathcal{H}_{\mathrm{corr}} = O(|p|^4)$.

Here we present some arguments close to [8, 9, 21]. Suppose that we want to construct an asymptotic solution of WKB wave packets type $A(x,t)e^{i\frac{S(x,t)}{\mu}}$. This type of asymptotic solutions occurs if one considers the Cauchy problem with WKB-type initial data

$$v|_{t=0} = A^0(x)e^{i\frac{S^0(x)}{\mu}}, \qquad v_t|_{t=0} = A^1(x)e^{i\frac{S^0(x)}{\mu}}, \tag{4.4}$$

where $S^0(x), A^0, A^1$ are smooth real functions and μ is a *new* small parameter. The family of several (small) parameters in the original problem makes it not "standard" from the mathematical point of view, but these parameters exist in real physical problems and, fortunately, their presence gives opportunity to find some constructive asymptotic formulas; without them, all formulas are just "mathematical hocks," useless for applications. We do not want to discuss the question about uniform expansions of asymptotic solutions of original equation with respect to all parameters h, δ, μ and assume that there are connections among them and that μ and δ are connected with h by equations $\mu = h^\alpha, \delta = h^\beta$, $0 \leqslant \alpha \leqslant 1$, $\beta \geqslant 0$. The main difference between the parameters μ and δ is that μ is singular and δ is regular. Note also that all these parameters h, μ, δ in real applications are just numbers.

Substituting the WKB asymptotic solution

$$v = A(x,t,h)e^{i\frac{S(x,t)}{\mu}} \tag{4.5}$$

into (3.12) and following the WKB-method (or the ray method), one obtains [16]

$$\rho \equiv \left(h^2\frac{\partial^2}{\partial t^2} + \hat{H}\right)(A(x,t)e^{i\frac{S(x,t)}{\mu}}) = e^{i\frac{S(x,t)}{\mu}}\Big[-\left(\frac{h}{\mu}S_t - ih\frac{\partial}{\partial t}\right)^2 A + H\left(\frac{h}{\mu}\nabla S, x, \delta, h\right)A(x,t)$$
$$-ih\Big(\Big(\Big\langle\nabla_p H(\frac{h}{\mu}\nabla S, x, \delta, h\Big), \nabla\Big\rangle A + \frac{h}{2\mu}\mathrm{tr}\Big(\frac{\partial^2 H}{\partial p^2}\Big(\frac{h}{\mu}\nabla S, x, \delta, h\Big)\frac{\partial^2 S}{\partial x^2}\Big)\Big)A + h^2\hat{G}A\Big]. \tag{4.6}$$

Here $\hat{G} = G(\frac{h}{\mu}\overset{2}{\nabla}S, \overset{2}{x}, \overset{2}{t}, \overset{1}{\nabla}, \delta, h)$ is a pseudodifferential operator such that its action on a smooth function A gives a smooth bounded function if $h/\mu \leqslant \mathrm{const}$, and $G(p,x,t,\varsigma,\mu)|_{p=0} = 0$. Using the relation $\mu = O(h^\alpha)$, we separate out the main terms with respect to parameters μ and μ/h on the right-hand side of (4.6). We distinguish three cases.

(1) *"Short waves."* In this case, $\mu \sim h$; to simplify the notation, we put $\mu = h$. To derive the equations for the phase S and amplitude A, according to WKB, method we preserve terms up to $h^0 = 1$ and h and then put them separately to zero. These two terms are

$$-\big(S_t^2 - H^0(\nabla S, x, \delta)\big)A,$$
$$ih\Big(2S_t\frac{\partial A}{\partial t} - \Big\langle\nabla_p H^0\Big(\frac{h}{\mu}\nabla S, x, \delta\Big), \nabla\Big\rangle A - \frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2 H^0}{\partial p^2}(\nabla S, x, \delta)\frac{\partial^2 S}{\partial x^2}\Big)\Big)A - iH^1(\nabla S, x, \delta)\Big)A.$$

The first term gives the Hamilton–Jacobi equation for the phase S (more precisely, two equations, because of the square of S_t), and the other one gives the transport equation for the amplitude A. However, there is a small parameter δ and, assuming a relationship $\delta = O(h^\beta)$, one can split $H^0(\nabla S, x, \delta)$ into two parts: the "main part" $\tilde{H}^0(\nabla S, x, \delta)$ and the correction $H^0_{\mathrm{corr}}(\nabla S, x, \delta)$; suppose that $H^0_{\mathrm{corr}}(\nabla S, x, \delta) = O(h)$. The main part defines the Hamilton–Jacobi equations

$$S_t \pm \sqrt{\tilde{H}^0(\nabla S, x, \delta)} = 0. \tag{4.7}$$

Include the correction $H^0_{\mathrm{corr}}(\nabla S, x, \delta)/h$ into the transport equation, replace $H^0(\nabla S, x, \delta)$ by $\tilde{H}^0(\nabla S, x, \delta)$ in (4.2), and put $\delta = 0$ in H^1. The transport equation becomes

$$\Big(2S_t\frac{\partial}{\partial t} - \Big\langle\nabla_p\tilde{H}^0\Big(\frac{h}{\mu}\nabla S, x, \delta\Big), \nabla\Big\rangle - \frac{1}{2}\mathrm{tr}\left(\frac{\partial^2\tilde{H}^0}{\partial p^2}(\nabla S, x, t, \delta)\frac{\partial^2 S}{\partial x^2}\right)$$
$$-\frac{i}{h}H^0_{\mathrm{corr}}(\nabla S, x, \delta) - iH^1(\nabla S, x, 0)\Big)A = 0. \tag{4.8}$$

If S and A satisfy to (4.7) and (4.8), then $\rho \equiv (h^2\frac{\partial^2}{\partial t^2} + \hat{H})(A(x,t)e^{i\frac{S(x,t)}{h}}) = O(h\min(\delta, h\delta))$. This enables one to prove the asymptotic formula $v = A(x,t)e^{i\frac{S(x,t)}{h}} + O(\min(\delta, h\delta))$ for the solution v to (3.12),(4.4).

The important conclusion for this case is: *to construct the leading term of the WKB-solution, one needs* $H^0(x,p,\delta)$ *and* $H^1(x,p,0)$.

(2) *"Middle waves."* If h/μ is a small parameter, we can take off the factor $(h/\mu)^2$ in (4.6) and, using (4.3), write

$$\Big[-\Big(\frac{h}{\mu}S_t - ih\frac{\partial}{\partial t}\Big)^2 A + H\Big(\frac{h}{\mu}\nabla S, x, \delta, h\Big)A$$
$$-ih\Big(\Big\langle\nabla_p H\Big(\frac{h}{\mu}\nabla S, x, \delta, h\Big), \nabla\Big\rangle A + \frac{h}{2\mu}\frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2 H^0}{\partial p^2}\Big(\frac{h}{\mu}\nabla S, x, \delta, h\Big)\Big)\Big)A + h^2\hat{G}A\Big]$$
$$= \frac{h^2}{\mu^2}\Big[-\Big(S_t - i\mu\frac{\partial}{\partial t}\Big)^2 A + \tilde{\mathcal{H}}(\nabla S, x, \delta)A + \frac{\mu^2}{h^2}\mathcal{H}_{\mathrm{corr}}\Big(\frac{h}{\mu}\nabla S, x, \delta\Big)A + \mu\langle\nabla_p H^1(0, x, \delta), \nabla S\rangle A$$
$$-i\mu\Big(\langle\nabla_p\tilde{\mathcal{H}}(\nabla S, x, \delta), \nabla\rangle A + \frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2\tilde{\mathcal{H}}}{\partial p^2}(\nabla S, x, \delta)\frac{\partial^2 S}{\partial x^2}\Big)\Big)A + O\Big(\mu\max\Big(\mu, \frac{h}{\mu}, \frac{h^3}{\mu^4}\Big)\Big)\Big].$$

Now we recall the smallness of the parameter δ; then we can continue the equation

$$\frac{h^2}{\mu^2}\Big[-\big(S_t^2 - \tilde{\mathcal{H}}^2(\nabla S, x, \delta)\big)A + i\mu\Big(2S_t\frac{\partial}{\partial t} - \langle\nabla_p\tilde{\mathcal{H}}(\nabla S, x, \delta), \nabla\rangle - \frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2\tilde{\mathcal{H}}}{\partial p^2}(\nabla S, x, \delta)\frac{\partial^2 S}{\partial x^2}\Big)$$
$$-\frac{i\mu}{h^2}\mathcal{H}_{\mathrm{corr}}\Big(\frac{h}{\mu}\nabla S, x, t, \delta\Big) - i\langle\nabla_p H^1(0, x, t, 0), \nabla S\rangle\Big)A + O\Big(\mu\max\Big(\mu, \frac{h}{\mu}, \delta, \frac{h^3}{\mu^4}\Big)\Big)\Big].$$

Note that at least $\frac{\mu}{h^2}\mathcal{H}_{\mathrm{corr}}(\frac{h}{\mu}\nabla S, x, \delta) = O(\frac{h^2}{\mu^3})$. Thus, if we assume that this term is $O(1)$, then we can write the Hamilton–Jacobi and the transport equations in the form

$$S_t \pm \sqrt{\tilde{\mathcal{H}}(\nabla S, x, \delta)} = 0, \tag{4.9}$$

$$\Big(2S_t\frac{\partial}{\partial t} - \Big\langle\nabla_p\tilde{\mathcal{H}}\Big(\frac{h}{\mu}\nabla S, x, \delta\Big), \nabla\Big\rangle - \frac{1}{2}\mathrm{tr}\Big(\frac{\partial^2\tilde{\mathcal{H}}}{\partial p^2}(\nabla S, x, \delta)\frac{\partial^2 S}{\partial x^2}\Big)$$
$$-\frac{i\mu}{h^2}\mathcal{H}_{\mathrm{corr}}\Big(\frac{h}{\mu}\nabla S, x, \delta\Big) - i\langle\nabla_p H^1(0, x, 0), \nabla S\rangle\Big)A = 0. \tag{4.10}$$

If the phase S and the amplitude A are the solutions to these equations, then it is possible to prove that $v = A(x,t)e^{i\frac{S(x,t)}{\mu}} + O(\max(\mu, \frac{h}{\mu}, \delta, \frac{h^3}{\mu^4}))$ for the solution to (3.12), (4.4). *We see that, as in the short-wave case, to construct the leading term of an asymptotic solution, we need H^0 and H_0^1* again; moreover, in this case, we can expand these terms with respect to momentum p and replace H^0 and H_0^1 by their Taylor polynomials. We shall return to this problem below, when we shall present explicit formulas for $H^0(x, p, \delta)$ and $H_0^1(x, p)$.

(3) *"Long waves."* In this case, $\mu = 1$. We have the standard homogenization theory, and there are no fast oscillations in the solution v. Thus, since $H|_{p=0} = 0$, we can represent (3.12) in the form

$$h^2\Big(v_{tt} + \tilde{\mathcal{H}}\Big(\overset{2}{x}, -i\overset{1}{\frac{\partial}{\partial x}}, \delta\Big)v - i\langle\nabla_p H^1(x, 0, \delta), \nabla\rangle v + O(h)v\Big) = 0. \tag{4.11}$$

Now we can divide this equation by h^2, and assuming that the limit $v_0 = \lim_{h\to 0} v$ exists, obtain the equation for v_0,

$$v_{0tt} + \tilde{\mathcal{H}}(\overset{2}{x}, -i\overset{1}{\nabla}, \delta)v_0 - i\langle\nabla_p H^1(x, 0, \delta), \nabla\rangle v_0 = 0, \tag{4.12}$$

or

$$v_{0tt} - \sum_{j,k=1}^{2} Q_{j,k}(x, \delta)\frac{\partial^2}{\partial x_j\partial x_k}v_0 - \sum_{j=1}^{2} b_j(x)\frac{\partial}{\partial x_j}v_0 = 0, \quad b = i\nabla_p H^1(x, 0, \delta). \tag{4.13}$$

We see that, in this case, one needs to found $\nabla_p H^1(x, 0, \delta)$ with the same accuracy as $\tilde{\mathcal{H}}(x, p, \delta)$. It is nontrivial to find this term by a direct calculation, but this can be avoided using the symmetricity (or even self-adjointness) in $L^2(\mathbb{R}^2)$ of the operator in (4.13). Obviously, there exists only one possibility with the prescribed term with second derivatives, namely, this equation is

$$v_{0tt} - \sum_{j,k=1}^{2}\frac{\partial}{\partial x_j}\Big(Q_{j,k}(x, \delta)\frac{\partial v_0}{\partial x_k}\Big) = 0 \quad\Longleftrightarrow\quad v_{0tt} - \mathrm{div}\big(Q(x, \delta)\mathrm{grad} v_0\big) = 0. \tag{4.14}$$

The conclusion of this section is as follows: to construct the leading term of the asymptotic solution of reduced equation, one should found $H^0(x, p, \delta)$ and $H^1(x, p, 0)$. Taking into account this fact and analyzing the construction of the symbols L, R, χ, one can easy formulate the same conclusion for the symbols L, R, χ.

5. ELIMINATING THE VERTICAL VARIABLES AND A PSEUDODIFFERENTIAL EQUATION FOR THE SURFACE WAVES

Substituting the function (3.9) into (3.2)–(3.4) and passing from the operators to their symbols, we obtain the following system for the *function* $R(x,p,z,y,\xi,\delta,h)$:

$$-(P-ih\nabla-i\lambda\nabla_y^{\theta})^2R+\frac{\partial^2R}{\partial z^2}=0,\qquad -D\leqslant z\leqslant 0,\tag{5.1}$$

$$\frac{\partial R}{\partial z}+i\langle q,P-ih\nabla-i\lambda\nabla_y^{\theta}\rangle R=0\quad\text{for}\quad z=-(D_0+\delta D_1),\tag{5.2}$$

$$R=1\quad\text{for}\quad z=0.\tag{5.3}$$

Here
$$P=P(p,x,\xi)\equiv p+\lambda\theta_x(x)\xi,\quad q=h\nabla D+\lambda\nabla_y^{\Theta}D\equiv h\nabla D+\delta\lambda\nabla_y^{\Theta}D_1$$
are just vector *functions*. Write $\mathbf{A}^2=\mathbf{A}_0^2+h\mathbf{a}-h^2\Delta=(P-ih\nabla-i\lambda\nabla_y^{\theta})^2$, where

$$\mathbf{A}_0^2=(P-i\lambda\nabla_y^{\theta})^2,\tag{5.4}$$

$$\mathbf{a}=-2i\langle P-i\lambda\Theta_x\nabla_y,\,\nabla\rangle-\lambda\langle\Delta\Theta,\nabla_y\rangle-i\lambda\langle\Delta\Theta,\xi\rangle,\qquad\Delta\Theta=\begin{pmatrix}\Delta\Theta_1\\ \Delta\Theta_2\end{pmatrix}.\tag{5.5}$$

We need also the commutator $\mathbf{C}=[\mathbf{a},\mathbf{A}_0^2]$. We have
$$\mathbf{C}=\mathbf{a}\mathbf{A}_0^2-\mathbf{A}_0^2\mathbf{a}=-2i\langle P,\,\nabla\rangle(P-i\lambda\nabla_y^{\theta})^2=-4i\langle(P-i\lambda\nabla_y^{\theta}),\,\langle P,\nabla\rangle(P-i\lambda\nabla_y^{\theta})\rangle.\tag{5.6}$$
One can also give another formula for $\mathbf{C}$ introducing the function $g=\langle P,\,x\rangle+\lambda\langle\Theta,\,\xi-i\nabla_y\rangle$. Then
$$\mathbf{C}=-2i\langle\nabla g,\,\nabla|\nabla g|^2\rangle.\tag{5.7}$$
Now we can formally present the solution of (5.1), (5.3) in the form
$$R=\cosh(z\mathbf{A})1+\frac{\sinh(z\mathbf{A})}{\mathbf{A}}L,\tag{5.8}$$
which contains a still unknown symbol $L(x,p,y,\xi)$ of the operator $\hat{\mathbf{L}}$ in the form
$$L=\partial R/\partial z\Big|_{z=0},\tag{5.9}$$
where $\cosh(z\mathbf{A})1$ and $\frac{\sinh(z\mathbf{A})}{\mathbf{A}}L$ mean that the operator $\cosh(z\mathbf{A})$ acts on the function $f=1$ and the operator $\frac{\sinh(z\mathbf{A})}{\mathbf{A}}$ acts on the function $L(x,p,y,\xi)$. Recall that, if one finds the function (symbol) L, then it is possible to recover the operator $\hat{\mathbf{R}}$ and, in turn, the function $\Psi(x,y,z,t)$ using the function $\phi(x,y,t)=\Psi(x,y,z,t)|_{z=0}$, which should satisfy (3.7) on the free surface $z=0$.

As was said above, it is impossible to find effective formulas for L, and one can speak only on an asymptotic expansion L with respect to small parameters h and δ. The aim of this section is to find the first coefficients of this expansion.

Substituting the anzatz (5.8) into the boundary condition (5.2) gives

$$(\mathbf{A}\sin(z\mathbf{A})+i\langle q,\,P-ih\nabla-i\lambda\nabla_y^{\Theta}\rangle\cosh(z\mathbf{A}))|_{z=-D}1$$
$$+\left(\cosh(z\mathbf{A})+i\langle q,\,P-ih\nabla-i\lambda\nabla_y^{\Theta}\rangle\frac{\sinh(z\mathbf{A})}{\mathbf{A}}\right)\Bigg|_{z=-D}L=0.$$

We replace here z by D after acting by the operator $\mathbf{A}$. Write

$$U=(-\overset{1}{\mathbf{A}}\sinh(\overset{2}{D}\overset{1}{\mathbf{A}})+\overbrace{i\langle q,\,P-ih\nabla-i\lambda\nabla_y^{\Theta}\rangle}^{2}\cosh(\overset{3}{D}\overset{1}{\mathbf{A}}))1,\tag{5.10}$$

$$\hat{V}=\left(\cosh(\overset{2}{D}\overset{1}{\mathbf{A}})-\overbrace{i\langle q,\,P-ih\nabla-i\lambda\nabla_y^{\Theta}\rangle}^{2}\frac{\sinh(\overset{3}{D}\overset{1}{\mathbf{A}})}{\overset{1}{\mathbf{A}}}\right)\tag{5.11}$$

Using this notation, we can present the equation for the symbol L in a more compact form $U+\hat{V}L=0$ and write
$$L=-(\hat{V})^{-1}U.\tag{5.12}$$
This is a formal formula, and it is not useful in specific problems; thus, we use the perturbation theory taking into account the presence of parameters h and δ.

Lemma 2. *Assume that the operator $\hat{V}$ and the function U have the following expansions*:

$$\begin{aligned}\hat{V} &= \hat{V}_0 + \delta\hat{V}_1 + \delta^2\hat{V}_2 + h\hat{V}_0^1 + O(\delta^3) + O(\delta h) + O(h^2),\\ U &= U_0 + \delta U_1 + \delta^2 U_2 + hU_0^1 + O(\delta^3) + O(\delta h) + O(h^2);\end{aligned} \tag{5.13}$$

then the following expansion holds:

$$L(x,\, p,\, y,\, \xi) = L_0 + \delta L_1 + \delta^2 L_2 + hL_0^1 + \cdots, \tag{5.14}$$

where

$$L_0 = -\hat{V}_0^{-1}U_0, \quad L_1 = -\hat{V}_0^{-1}(U_1 + \hat{V}_1 L_0), \quad L_2 = -\hat{V}_0^{-1}(U_2 + \hat{V}_1 L_1 + \hat{V}_2 L_0), \tag{5.15}$$

$$L_0^1 = -\hat{V}_0^{-1}(U_0^1 + \hat{V}_0^1 L_0). \tag{5.16}$$

Proof. This is standard. Substituting the expansions for U, $\hat{V}$, and L into the equation for L gives $U_0+\delta U_1+\delta^2U_2+hU_0^1+(\hat{V}_0+\delta\hat{V}_1+\delta^2\hat{V}_2+h\hat{V}_0^1)(L_0+\delta L_1+\delta^2 L_2+hL_0^1)+O(\delta^3+h^2+h\delta)=0$. Equating the coefficients at $1, \delta, \delta^2$, and h to 0, we obtain (5.15) and (5.16).

Let us find the coefficients $U_0, \hat{V}_0$, etc.

Lemma 3. *The following equations hold*:

$$\hat{V}_0 = \cosh(D_0\mathbf{A}_0), \tag{5.17}$$

$$\hat{V}_1 = D_1\mathbf{A}_0\sinh(D_0\mathbf{A}_0) - i\langle\lambda\nabla_y^\Theta D_1,\, P - i\lambda\nabla_y^\Theta\rangle\frac{\sinh(D_0\mathbf{A}_0)}{\mathbf{A}_0}, \tag{5.18}$$

$$\hat{V}_2 = \frac{D_1^2\mathbf{A}_0^2}{2}\cosh(D_0\mathbf{A}_0) - i\lambda D_1\langle\nabla_y^\Theta D_1,\, P - i\lambda\nabla_y^\Theta\rangle\cosh(D_0\mathbf{A}_0), \tag{5.19}$$

$$\begin{aligned}\hat{V}_0^1 =& \frac{D_0\sinh(D_0\mathbf{A}_0)}{2\mathbf{A}_0}\mathbf{a} + \frac{D_0^2\mathbf{A}_0\cosh(D_0\mathbf{A}_0) - D_0\sinh(D_0\mathbf{A}_0)}{8\mathbf{A}_0^3}\mathbf{C}\\ &- i\langle\nabla D_0,\, P - i\lambda\nabla_y^\Theta\rangle\frac{\sinh(D_0\mathbf{A}_0)}{\mathbf{A}_0},\end{aligned} \tag{5.20}$$

$$U_0 = -\,|P|\sinh(D_0|P|), \tag{5.21}$$

$$U_1 = \Big(i\langle\lambda\nabla_y^\Theta D_1,\, P\rangle - D_1|P|^2\Big)\cosh(D_0|P|), \tag{5.22}$$

$$U_2 = i\langle\lambda\nabla_y^\Theta D_1,\, P\rangle D_1|P|\sinh(D_0|P|) - \frac{D_1^2|P|^3}{2}\sinh(D_0|P|), \tag{5.23}$$

$$\begin{aligned}U_0^1 =& i\langle\nabla D_0,\, P\rangle\cosh(D_0|P|) + i\lambda\langle\Delta\Theta,\, \xi\rangle\frac{\sinh(D_0|P|) + D_0|P|\cosh(D_0|P|)}{2|P|}\\ &+ i\frac{D_0|P|\cosh(D_0|P|) - \lambda^2\sinh(D_0|P|) + D_0^2|P|^2\sinh(D_0|P|)}{4|P|^3}(\langle P,\, \nabla\rangle(\theta_x\xi)^2);\end{aligned} \tag{5.24}$$

here the operator $\mathbf{C}$ is defined in (5.6).

Proof. (1) Consider the family of operators

$$\hat{\mathbf{V}}(z) = \Big(\frac{\partial}{\partial z} + i\langle h\nabla D + \delta\lambda\nabla_y^\Theta D_1,\, P - ih\nabla - i\lambda\nabla_y^\Theta\rangle\Big)\frac{\sinh(z\mathbf{A})}{\mathbf{A}}. \tag{5.25}$$

Here and below, the operators act from the right to left in the last expression. It is easy to see that $\hat{V} = \hat{\mathbf{V}}(z)\Big|_{z=-D}$ and $U = (\frac{\partial\hat{\mathbf{V}}}{\partial z}(z)1)\Big|_{z=-D}$. It is sufficient to find an expansion of $\hat{\mathbf{V}}(z)$ and then use the last formulas.

We have

$$\begin{aligned}\hat{\mathbf{V}}(z) =& \Big(\frac{\partial}{\partial z} + i\delta\lambda\langle\nabla_y^\Theta D_1,\, P - i\lambda\nabla_y^\Theta\rangle\\ &+ ih\Big[\langle\nabla D,\, P - i\lambda\nabla_y^\Theta\rangle - \delta\lambda\langle\nabla_y^\Theta D_1,\, \nabla\rangle\Big] + h^2\langle\nabla D,\, \nabla\rangle\Big)\frac{\sinh(z\mathbf{A})}{\mathbf{A}}.\end{aligned} \tag{5.26}$$

Recall that $\mathbf{A}^2 = \mathbf{A}_0^2 + h\mathbf{a} - h^2\Delta$ and $\hat{\mathbf{V}}(z)$ depend only on $\mathbf{A}^2$. Here the operators $\mathbf{A}_0^2$ and $h\mathbf{a} - h^2\Delta$ act simultaneously, and we are to reorder them, namely, $\mathbf{A}_0^2$ acts first, and then $h\mathbf{a} - h^2\Delta$ acts. This can be done with the following general formula (see [15, 17]) for noncommuting operators A and B and arbitrary smooth function $f(z)$:

$$f(A \overset{1}{+} B) = f(\overset{1}{A} + \overset{2}{B}) + \frac{1}{2}[A, \overset{2}{B}]f''(\overset{1}{A} + \overset{3}{B}) + R(A, B). \tag{5.27}$$

Here $[A, B]$ is the commutator of the operators A, B, and $R(A, B)$ is the correction which can be evaluated via the commutators $[A, [A, B]]$ of second order. In our problem, we choose $A = h\mathbf{a} - h^2\Delta$ and $B = \mathbf{A}_0^2$. The operator A is proportional to h, and we use the standard Taylor expansion for the smooth functions $f(\overset{1}{A} + \overset{2}{B})$ at the point $\overset{2}{B}$. Thus, the last formula gives

$$f(A \overset{1}{+} B) = f(\mathbf{A}_0^2) + h\Big(f'(\mathbf{A}_0^2)\mathbf{a} + \frac{1}{2}f''(\mathbf{A}_0^2)[\mathbf{a}, \mathbf{A}_0^2]\Big) + O(h^2). \tag{5.28}$$

Using this formula, we obtain

$$\begin{aligned}\frac{\sinh(z\mathbf{A})}{\mathbf{A}} =& \frac{\sinh(z\mathbf{A}_0)}{\mathbf{A}_0} + h\Bigg(\frac{z\mathbf{A}_0\cosh(z\mathbf{A}_0) - \sinh(z\mathbf{A}_0)}{2\mathbf{A}_0^3}\mathbf{a} \\ &+ \frac{(3 + z^2\mathbf{A}_0^2)\sinh(z\mathbf{A}_0) - 3z\mathbf{A}_0\cosh(z\mathbf{A}_0)}{8\mathbf{A}_0^5}\mathbf{C}\Bigg) + O(h^2).\end{aligned} \tag{5.29}$$

We want to construct the coefficients at h^0 and h of the expansion of the last operator with respect to h; thus, in (5.26), we can omit the last term and replace the operator $\mathbf{A}$ by $\mathbf{A}_0$, according to (5.28). Therefore, we can write

$$\begin{aligned}\hat{\mathbf{V}}(z) =& \Big(\frac{\partial}{\partial z} + i\delta\lambda\langle\nabla_y^\Theta D_1,\, P - i\lambda\nabla_y^\Theta\rangle + ih\Big[\langle\nabla D,\, P + i\lambda\nabla_y^\Theta\rangle + \delta\lambda\langle\nabla_y^\Theta D_1,\, \nabla\rangle\Big]\Big) \\ &\times\Bigg(\frac{\sinh(z\mathbf{A}_0)}{\mathbf{A}_0} + h\Big(\frac{z\mathbf{A}_0\cosh(z\mathbf{A}_0) - \sinh(z\mathbf{A}_0)}{2\mathbf{A}_0^3}\mathbf{a} \\ &+ \frac{(3 + z^2\mathbf{A}_0^2)\sinh(z\mathbf{A}_0) - 3z\mathbf{A}_0\cosh(z\mathbf{A}_0)}{8\mathbf{A}_0^5}\mathbf{C}\Big)\Bigg) + O(h^2),\end{aligned} \tag{5.30}$$

and, using this formula, we find the coefficients $\hat{V}_0, \hat{V}_1, U_0, U_1$, etc.

(2) To find the coefficients $U_0, U_1, U_2, \hat{V}_0, \hat{V}_1, \hat{V}_2$, we study $\hat{V}|_{h=0}$ and $U|_{h=0}$. We have

$$\begin{aligned}\hat{V}|_{h=0} &= \cosh(\overset{2}{D}\overset{1}{\mathbf{A}}_0) - i\delta\lambda\langle\nabla_y^\Theta D_1,\, P - i\lambda\overset{1}{\nabla_y^\Theta}\rangle\frac{\sinh(\overset{2}{D}\overset{1}{\mathbf{A}}_0)}{\overset{1}{\mathbf{A}}_0}, \\ U|_{h=0} &= \Big(-\overset{1}{\mathbf{A}}_0\sinh(\overset{2}{D}\overset{1}{\mathbf{A}}_0) + i\delta\lambda\langle\nabla_y^\Theta D_1,\, P - i\lambda\overset{1}{\nabla_y^\Theta}\rangle\cosh(\overset{2}{D}\overset{1}{\mathbf{A}}_0)\Big)1 \\ &= -|P|\sinh(D|P|) + i\delta\lambda\langle\nabla_y^\Theta D_1,\, P\rangle\cosh(D|P|).\end{aligned}$$

The operators $\mathbf{A}_0$ and ∇_y^Θ act before D, and thus, we can put $D = D_0 + \delta D_1$ and use the Taylor expansion with respect to δ. This gives (5.17)–(5.19) and (5.21)–(5.23). In the last formula, we also use the equations $\nabla_y^\Theta 1 = 0$ and $\mathbf{A}_0 1 = |P|$.

(3) To find the coefficients $\hat{V}_0^1$ and U_0^1, we study $\hat{V}|_{\delta=0}$ and $U|_{\delta=0}$. By (5.30), we have

$$\begin{aligned}\hat{V}|_{\delta=0} =& \cosh(\overset{2}{D}_0\overset{1}{\mathbf{A}}_0) - ih\overbrace{\langle\nabla D_0,\, P - i\lambda\nabla_y^\Theta\rangle}^{2}\frac{\sinh(\overset{3}{D}_0\overset{1}{\mathbf{A}}_0)}{\overset{1}{\mathbf{A}}_0} \\ &+ h\Big(\frac{\overset{3}{D}_0\sinh(\overset{3}{D}_0\overset{2}{\mathbf{A}}_0)}{2\overset{2}{\mathbf{A}}_0}\overset{1}{\mathbf{a}} + \frac{(\overset{3}{D}_0)^2\overset{2}{\mathbf{A}}_0\cosh(\overset{3}{D}_0\overset{2}{\mathbf{A}}_0) - \overset{3}{D}_0\sinh(\overset{3}{D}_0\overset{2}{\mathbf{A}}_0)}{8(\overset{2}{\mathbf{A}}_0)^3}\overset{1}{\mathbf{C}}\Big) + O(h^2),\end{aligned} \tag{5.31}$$

$$U|_{\delta=0} = \left(-\overset{1}{\mathbf{A}}_0 \sinh(\overset{2}{D}_0\overset{1}{\mathbf{A}}_0) + ih\overbrace{\langle \nabla D_0,\, P - i\lambda\nabla_y^{\Theta}\rangle}^{2}\cosh(\overset{3}{D}_0\overset{1}{\mathbf{A}}_0)\right)1$$
$$- h\Big(\frac{\overset{3}{D}_0\cosh(\overset{3}{D}_0\overset{2}{\mathbf{A}}_0) + \sinh(\overset{3}{D}_0\overset{2}{\mathbf{A}}_0)}{2\overset{2}{\mathbf{A}}_0}\overset{1}{\mathbf{a}} + \frac{\big((\overset{3}{D}_0\overset{2}{\mathbf{A}}_0)^2 - 1\big)\sinh(\overset{3}{D}_0\overset{2}{\mathbf{A}}_0) + \overset{3}{D}_0\overset{2}{\mathbf{A}}_0\cosh(\overset{3}{D}_0\overset{2}{\mathbf{A}}_0)}{8(\overset{2}{\mathbf{A}}_0)^3}\overset{1}{\mathbf{C}}\Big)1$$
$$+ O(h^2). \tag{5.32}$$

The operators $\mathbf{A}_0$ and D_0 (a function) commute. Thus, we can omit the indices over operators in the last formulas and get (5.20). Taking into account the equations $\nabla_y 1 = 0$ and $\mathbf{A_0}1 = |P|$, we obtain (5.20) and (5.24).

Using formulas (5.17)–(5.24), we can find the coefficients L_0, L_1, L_2, L_0^1, which are given in the following theorem.

Theorem 2. *The following formulas hold for the coefficients L_0, L_1, L_2, L_0^1 of* (5.14):

$$L_0 = |P|\tanh(D_0|P|), \tag{5.33}$$

$$L_1 = \frac{1}{\cosh(D_0\mathbf{A}_0)}(D_1|P|^2 - i\langle\lambda\nabla_y^{\Theta}D_1,\, P\rangle)\frac{1}{\cosh(D_0|P|)}, \tag{5.34}$$

$$L_2 = -\frac{1}{\cosh(D_0\mathbf{A}_0)}(D_1\mathbf{A}_0^2 - i\langle\lambda\nabla_y^{\Theta}D_1,\, P - i\lambda\nabla_y^{\Theta}\rangle)\frac{\tanh(D_0\mathbf{A}_0)}{\mathbf{A}_0}$$
$$\times (D_1|P|^2 - i\langle\lambda\nabla_y^{\Theta}D_1,\, P\rangle)\frac{1}{\cosh(D_0|P|)}, \tag{5.35}$$

$$L_0^1 = i\frac{D_0|P|\tanh(D_0|P|) - 1}{\cosh^2(D_0|P|)}\langle\nabla D_0,\, P\rangle - i\lambda\langle\Delta\Theta,\, \xi\rangle\frac{\sinh(D_0|P|)\cosh(D_0|P|) + D_0|P|}{2|P|\cosh^2(D_0|P|)}$$
$$- i\langle P,\, \nabla P^2\rangle\frac{D_0|P| - \sinh(D_0|P|)\cosh(D_0|P|) - 2D_0^2|P|^2\tanh(D_0|P|)}{2|P|^3\cosh^2(D_0|P|)}. \tag{5.36}$$

Here, recall that $\mathbf{A}_0^2 = (p + \lambda\theta_x(\xi - i\nabla_y))^2$ and $P^2 = (p + \lambda\theta_x\xi)^2$.

Proof. Formulas (5.33), (5.34) easily follow from (5.17), (5.18), (5.21), and (5.22). To get formula (5.35), it is useful to note that $U_2 + \hat{V}_2 L_0 = 0$ in our case, and hence, $L_2 = -\hat{V}_0^{-1}\hat{V}_1 L_1$. Substituting $\hat{V}_0, \hat{V}_1$, and L_1 into this equality gives (5.35). To prove (5.36), we note that U_0^1 does not depend on D_1, and hence, does not depend on y. This means that one should replace the operator $\mathbf{A}_0$ in the definitions of $\hat{V}_0$ and $\hat{V}_0^1$ by the function $|P| = |p + \theta_x\xi|$. Hence, here we have just a product of ordinary functions, and manipulations give (5.36).

Let us note finally in this section that the symbol R of the operator $\hat{R}$ is equal to 1 for $z = 0$, and this means that $\hat{R}|_{z=0} = \hat{1}$ is the identical operator, and

$$\Psi|_{z=0} = \chi(\overset{2}{x}, \overset{1}{\hat{p}}, y, \delta, h)\Big|_{y=\theta/\varepsilon} v. \tag{5.37}$$

6. ELIMINATING FAST VARIABLES Y AND THE CALCULATION OF EFFECTIVE HAMILTONIANS H. PROOF OF THEOREM 1

The above calculations (approximately) reduce our problem given in a 3D strip to problem (3.7) given on a 2D plane (the unperturbed 2D free surface) with

$$\hat{\mathbf{L}} = \hat{\mathbf{L}}(\overset{2}{x}, \overset{1}{\hat{p}}, \overset{2}{y}, -i\overset{1}{\nabla}_y, \delta, h) = L^0(\overset{2}{x}, \overset{1}{\hat{p}}, \overset{2}{y}, -i\overset{1}{\nabla}_y, \delta) + hL^1(\overset{2}{x}, \overset{1}{\hat{p}}, \overset{2}{y}, -i\overset{1}{\nabla}_y, \delta) + O(h^2). \tag{6.1}$$

As was mentioned above, the fast variables y (corresponding to fast oscillating coefficients in the original problem) still present in this equation, and the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are included into

(3.7) in different way, namely, $\frac{\partial}{\partial x}$ enters with a small parameter h, and $\frac{\partial}{\partial y}$ without any small parameter. So, to find some solution, one can make the "second" reduction, using a variant of the adiabatic approximation in operator form [8, 9, 12]. We follow the scheme presented in these papers. We seek a solution of this equation in the form (3.11) with a new unknown function v satisfying the "second" reduced equation (3.12). Assume that $\hat{\chi}$ and $\hat{H}$ are pseudodifferential operators with symbols $\chi(x, p, y, h, \delta)$ and $H(x, p, h, \delta)$ having asymptotic expansions

$$\chi(x, p, y, h, \delta) = \chi^0(x, p, y, \delta) + h\chi^1(x, p, y, \delta) + \cdots , \tag{6.2}$$

$$H(x, p, h, \delta) = H^0(x, p, \delta) + hH^1(x, p, \delta) + \cdots \tag{6.3}$$

Our aim now is to find some coefficients of expansions of $\chi(x, p, y, h, \delta)$ and $H(x, p, h, \delta)$ which give a proof of Theorem 1. In some sense, $\hat{\chi}$ and $\hat{H}$ are similar to an eigenfunction and eigenvalue, and thus, there exist infinitely many functions χ and H suitable for the "second" reduction; so we fix one of them by the conditions

$$\chi|_{p\to 0} = 1 + O(|h| + |\delta|), \qquad H|_{p\to 0} = O(|h| + |\delta|). \tag{6.4}$$

According to [9, 8], we are to find first the eigenvalues H and eigenfunctions χ^0 of the (family) of self-adjoint operators $L^0(\overset{2}{x}, \overset{1}{\hat{p}}, \overset{2}{y}, -i\overset{1}{\nabla}_y, \delta)$ defined on the $L^2(\mathbb{T})$-space on the 2D torus $\mathbb{T}^2 = [0, 2\pi]^2$ with the standard inner product

$$(f(y), g(y))_{\mathbb{T}^{\not\vDash}} = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} \overline{f(y)}g(y)dy.$$

Of course, we can speak only on (asymptotic) expansions of H and χ^0 with respect to the parameter δ. To find the first terms of this expansions, we use the following lemma from a more general assertion in [11] (see Lemma 1 in this paper).

Lemma 4. *Let a self-adjoint operator B have an expansion $B = A + \delta B_1 + \delta^2 B_2 + \delta^3 \tilde{B}_3(\delta)$, where δ is a small parameter, and let A, B_1, B_2, $\tilde{B}_3(\delta)$ be self-adjoint operators. Consider the spectral problem*

$$(A + \delta B_1 + \delta^2 B_2 + \delta^3 \tilde{B}_3(\delta))\varphi = \lambda\varphi, \qquad \|\varphi\| = 1.$$

Assume that λ_0 is a simple eigenvalue of the operator A. Then $\lambda = \lambda_0 + \delta\lambda_1 + \delta^2\lambda_2 + \cdots$ and $\varphi = \varphi_0 + \delta\varphi_1 + \delta^2\varphi_2 + \cdots$, where $A\varphi_0 = \lambda_0\varphi_0$, $\|\varphi_0\| = 1$, $\lambda_1 = (\varphi_0, B_1\varphi_0)$, $\varphi_1 = -(A-\lambda_0)^{-1}(B_1-\lambda_1)\varphi_0$, $\lambda_2 = (\varphi_0, B_2\varphi_0) - \big((A-\lambda_0)^{-1}(B_1-\lambda_1)\varphi_0, (B_1-\lambda_1)\varphi_0\big)$, and $(\varphi_0, \varphi_1) + (\varphi_1, \varphi_0) = 0$. In particular, if $\lambda_1 = 0$, then

$$\varphi_1 = -(A - \lambda_0)^{-1}B_1\varphi_0, \quad \lambda_2 = (\varphi_0, B_2\varphi_0) - \big((A - \lambda_0)^{-1}B_1\varphi_0, B_1\varphi_0\big). \tag{6.5}$$

Equations (6.5) hold for every isolated eigenvalue. However, if we assume that functions φ_0^k form a normalized base for the operator A, $(\varphi_0^k, \varphi_0^m) = \delta_{k,m}$, $\delta_{k,m}$ is the Kronecker symbol, then the last formulas could be expressed using φ_0^k. Namely, introduce the coefficients $g_j^{k,m} = (\varphi_0^m, B_j\varphi_0^k)$. Then

$$\lambda_2^k = g_2^{k,k} - \sum_{m\neq k} \frac{|g_1^{k,m}|^2}{\lambda^m - \lambda^k}. \tag{6.6}$$

In our problem,

$$A = L_0(x, p, -i\overset{1}{\nabla}_y, \overset{2}{y}) = |\hat{P}|\tanh(D_0\hat{P}|)\big|_{\hat{P}=p-i\lambda\Theta_x\frac{\partial}{\partial y}}.$$

Lemma 5. *The eigenfunctions $\chi^0_{(m)}(x,p,y)$ and eigenvalues $H^0_{(m)}(x,p)$ of this operator are*

$$H^0_{(m)} = |p + \lambda\Theta_x m|\tanh(D_0(x)|p + \lambda\Theta_x k|), \chi^0_{(m)}(x,p,y) = e^{im\cdot y}, m = (m_1, m_2) \in \mathbb{Z}^2. \tag{6.7}$$

Obviously, the functions $e^{im\cdot y}$ form a base in $L^2(\mathbb{T}^2)$. Choose a compact set K in $\mathbb{R}^2$ and assume that $|p| < (\lambda/2)\min_{x\in K}||\Theta_x^{-1}||^{-1}$. Then the eigenvalue $H^0_{(0)} = |p|\tanh(D_0(x)|p|)$ is simple for $x \in K$, and

$$H^0_{(m)} - H^0_{(0)} = |p + \lambda\Theta_x m|\tanh(D_0(x)|p + \lambda\Theta_x k|) - |p|\tanh(D_0(x)|p|) > C > 0. \tag{6.8}$$

Proof. We are going to prove (6.8). For any chosen x, (6.8) equals to zero on the straight line $|p+\lambda\Theta_x m| = |p|$ in the plane (p_1, p_2) which is orthogonal to the interval $(0, -\lambda\Theta_x m)$ and intersects it at the middle of this interval. All these straight lines for $m \neq 0$ divide the plane into separate domains having the form of convex polygons. The domain B_x, which includes the origin, is known as the Brillouin zone, and we use formula (6.8) only inside the domain B_x. If one needs to use (6.8) for each x from some compact set K, then one has to take the intersection of all B_x for x in K, which we denote by B. Moreover, to have a uniform expansion of (6.8), one needs to take p in some closed set $\tilde{B}$ in the interior of B. One can always take $\tilde{B}$ as a disk whose center coincides with the origin. Obviously, the strip between the straight lines $|p+\lambda\Theta_x m| = |p|$ and $|p-\lambda\Theta_x m| = |p|$ includes the disk of radius $\lambda|\Theta_x m|/2$ centered at the origin ($\lambda > 0$), the Brillouin zone B_x is the intersection of all such strips, and B_x includes the disk of radius $r_x = \frac{\lambda}{2}\min_{m\neq 0}|\Theta_x m|$. Since $1 \leqslant |m| = |\Theta_x^{-1}\theta_x m| \leqslant \|\Theta_x^{-1}\||\Theta_x m|$, we have $\min_{m\neq 0}|\Theta_x m| \geqslant \|\Theta_x^{-1}\|^{-1}$, and hence, $r_x \geqslant \frac{\lambda}{2}\|\Theta_x^{-1}\|^{-1}$. Taking the minimum over all x in K, we see that one can choose the set $\tilde{B}$, where one can use the expansion (6.8), in the form of any disk centered at the origin and of radius less than $\frac{\lambda}{2}\min_{x\in K}\|\Theta_x^{-1}\|^{-1}$.

Taking into account condition (6.4), we restrict ourselves to the case of $m = 0$ and, to simplify the notation, omit the subscript (m) in the symbols $H^0_{(m)}(x,p)$ and $\chi^0_{(m)}(x,p,y)$. Thus, we have the following correspondence with the symbols λ_j and the operators A, B_1, B_2, etc., of the lemma: $A = L_0(x,\, p,\, -i\overset{1}{\nabla}_y,\, \overset{2}{y})$, $B_1 = L_1(x,\, p,\, -i\overset{1}{\nabla}_y,\, \overset{2}{y})$, $B_2 = L_2(x,\, p,\, -i\overset{1}{\nabla}_y,\, \overset{2}{y})$, $\varphi_0 = \chi^0(x,p,y) = 1$, $\lambda_0 = H^0 = |p|\tanh(D_0(x)|p|)$. It is easy to see that

$$\begin{aligned} B_1\varphi_0 = L_1(x,\, p,\, -i\overset{1}{\nabla}_y,\, \overset{2}{y})1 &= \frac{1}{\cosh(D_0\mathbf{A}_0)}(D_1|p|^2 - i\langle\lambda\nabla_y^\Theta D_1,\, p\rangle)\frac{1}{\cosh(D_0|p|)} \\ &= \frac{1}{\cosh(D_0|p|)}\frac{1}{\cosh(D_0\mathbf{A}_0)}(|p|^2 - i\lambda\langle p,\, \nabla_y^\Theta\rangle)D_1. \end{aligned}$$

Here $\mathbf{A}_0^2 = (p - i\lambda\theta_x\nabla_y))^2$, and it acts on the function D_1. Recall the Fourier series (2.11) for D_1; we have

$$\begin{aligned} (\varphi_k, B_1\varphi_0) &= \frac{1}{\cosh(D_0|p|)}\Big(e^{ik\cdot y}, \frac{1}{\cosh(D_0\mathbf{A}_0)}(|p|^2 - i\lambda\langle p,\, \nabla_y^\Theta\rangle)D_1\Big) \\ &= \frac{1}{\cosh(D_0|p|)}\Big((|p|^2 - i\lambda\langle p,\, \nabla_y^\Theta\rangle)\frac{1}{\cosh(D_0\mathbf{A}_0)}e^{ik\cdot y}, D_1\Big) \\ &= \frac{(|p|^2 + \lambda\langle\Theta_x k,\, p\rangle)D_{1k}}{\cosh(D_0|p|)\cosh(D_0|p+\lambda\Theta_x k|)}D_{1k}. \end{aligned} \tag{6.9}$$

We have also

$$B_2\varphi_0 = L_2(x,\, p,\, -i\overset{1}{\nabla}_y,\, \overset{2}{y})1 = L_2(x,\, p,\, 0,\, y)$$

$$= -\frac{1}{\cosh(D_0\mathbf{A}_0)}(D_1\mathbf{A}_0^2 - i\langle\lambda\nabla_y^\Theta D_1,\, p - i\lambda\nabla_y^\Theta\rangle)\frac{\tanh(D_0\mathbf{A}_0)}{\mathbf{A}_0}(D_1|p|^2 - i\langle\lambda\nabla_y^\Theta D_1,\, p\rangle)\frac{1}{\cosh(D_0|p|)}.$$

Here the operator $\mathbf{A}_0 = |p - i\lambda\nabla_y^\theta|$ and the operator $i\lambda\nabla_y^\theta$ act only on the function D_1 again. Note also that the operator $(D_1\mathbf{A}_0^2 - i\langle\lambda\nabla_y^\Theta D_1,\, p - i\lambda\nabla_y^\Theta\rangle)$ is symmetric on $L^2(\mathbb{T})$. Using these properties, we get

$$(\varphi_0, B_2\varphi_0)$$

$$\begin{aligned} &= -\frac{1}{\cosh(D_0|p|)}\Big((D_1\mathbf{A}_0^2 - i\langle\lambda\nabla_y^\Theta D_1,\, p - i\lambda\nabla_y^\Theta\rangle)\frac{1}{\cosh(D_0\mathbf{A}_0)}1, \frac{\tanh(D_0\mathbf{A}_0)}{\mathbf{A}_0}(D_1|p|^2 - i\langle\lambda\nabla_y^\Theta D_1,\, p\rangle)\Big) \\ &= -\frac{1}{\cosh^2(D_0|p|)}\Big((D_1|p|^2 - i\langle\lambda\nabla_y^\Theta D_1,\, p\rangle), \frac{\tanh(D_0\mathbf{A}_0)}{\mathbf{A}_0}(D_1|p|^2 - i\langle\lambda\nabla_y^\Theta D_1,\, p\rangle)\Big). \end{aligned}$$

Using the equation

$$\frac{\tanh(D_0\mathbf{A}_0)}{\mathbf{A}_0}e^{ik\cdot y} = e^{ik\cdot y}\frac{\tanh(D_0|p+\lambda\Theta_x k|)}{|p+\lambda\Theta_x k|},$$

after the integration with respect to the variables y, we obtain

$$(\varphi_0, B_2\varphi_0) = -\sum_{k\neq 0} |D_{1k}|^2 \frac{(|p|^2+\lambda\langle\Theta_x k,\, p\rangle)^2}{\cosh^2(D_0|p|)} \frac{\tanh(D_0|p+\lambda\Theta_x k|)}{|p+\lambda\Theta_x k|}. \tag{6.10}$$

Proof of formula (2.12) **for the symbol** H. For the symbol $H(p,\, x,\, \delta,\, h)$ of the operator, we have the following expansion:

$$H(x,\, p,\, \delta,\, h) = H_0^0(x,\, p) + \delta^2 H_2^0(x,\, p) + hH_0^1(x,\, p) + O(\delta^3) + O(h^2) + O(h\delta).$$

The formulas for H^0 and H^1 were presented above in (2.10).

To find H^1, we use the general formula given in [9] (see (3.16)),

$$H^1 = \big(\chi^0, L_0^1\chi^0\big) - i\Big(\chi^0, \frac{d\chi_0}{dt}\Big) - i\Big(\chi^0, \sum_{j=1}^{2}\Big[\frac{\partial L_0}{\partial p_j} - \frac{\partial H}{\partial p_j}\Big]\frac{\partial \chi^0}{\partial x_j}\Big),$$

$$\frac{d}{dt} = -\sum_{s=1}^{2}\frac{\partial H}{\partial x^s}\frac{\partial}{\partial p_s} + \sum_{s=1}^{2}\frac{\partial H}{\partial p_s}\frac{\partial}{\partial x_s}.$$

Due to the equation $\chi^0 = 1 + O(\delta)$, the two last terms in this formula are equal to $O(\delta)$, and

$$H^1 = \Big(1, \hat{L}_0^1 1\Big) + O(\delta). \tag{6.11}$$

Now we replace ξ in formula (5.36) by $-i\frac{\partial}{\partial y}$ and act by the operator thus obtained on 1. A simple analysis of (5.36) shows that the second and third terms in (5.36) include the factor ξ; thus, after replacing ξ by $-i\frac{\partial}{\partial y}$ and acting on 1, these terms disappear; we also have $P = p$, and finally obtain H^1.

To prove that $H|_{p=0} = 0$, we return to (5.10)–(5.12). Consider the operator $L(x, p, \overset{2}{y}, -i\overset{1}{\nabla}_y)|_{p=0}$ and show that

$$L(x, p, \overset{2}{y}, -i\overset{1}{\nabla}_y)1|_{p=0} = 0.$$

Indeed, as soon as ∇_y acts before any other operators, we can put $\xi = 0$ as well as $p = 0$, in formulas (5.10)–(5.12). Thus, we can put $P = 0$ and $\mathbf{A}^2 = (ih\nabla - i\lambda\nabla_y^{\theta})^2$ in (5.10), and so

$$U|_{p=0,\xi=0} = (-\overset{1}{\mathbf{A}}\sinh(\overset{2}{D}\overset{1}{\mathbf{A}}) + i\overbrace{\langle q,\, -ih\nabla - i\lambda\nabla_y^{\Theta}\rangle}^{2}\cosh(\overset{3}{D}\overset{1}{\mathbf{A}}))1 = 0.$$

Hence,

$$(V^{-1}U)(x, p, \overset{2}{y}, -i\overset{1}{\nabla}_y)|_{p=0}1 = 0.$$

This means that $\chi = 1$ is an eigenfunction of the operator $L(x, p, \overset{2}{y}, -i\overset{1}{\nabla}_y)|_{p=0}$ with the eigenvalue $H|_{p=0} = 0$. As was shown above, the eigenvalue and eigenfunction satisfying condition (6.4) are unique. Thus, $H|_{p=0} = 0$ for any sufficiently small h and δ.

Now we need an appropriate representation for H_2^0. It is given by λ_2 in (6.5). Consider the Taylor expansion near the point $p = 0$,

$$\frac{1}{D_0|p+\lambda\Theta_x k|\tanh(D_0|p+\lambda\Theta_x k|)} = g(\sigma_k) + \lambda\langle\Theta_x k,\, p\rangle D_0^2\frac{g'(\sigma_k)}{\sigma_k} + \frac{D_0^4}{2}\lambda^2\langle\Theta_x k,\, p\rangle^2\Big(\frac{g''(\sigma_k)}{\sigma_k^2} - \frac{g'(\sigma_k)}{\sigma_k^3}\Big) + p^2\frac{D_0^2}{2}\frac{g'(\sigma_k)}{\sigma_k} + O(p^3),$$

where

$$g(\sigma_k) = \frac{1}{\sigma_k\tanh(\sigma_k)}, \quad \sigma_k = \lambda D_0|\Theta_x k|. \tag{6.12}$$

Write $\beta = D_0^2\langle p, \Theta_x k\rangle$, and $\alpha^2 = D_0^2 p^2$; then

$$D_0|p+\lambda\Theta_x k| = \sqrt{D_0^2p^2+2D_0^2\langle p,\Theta_x k\rangle+(D_0\lambda\Theta_x k)^2} = \sqrt{\alpha^2+2\beta+\sigma_k^2}$$

and

$$\begin{aligned} g(D_0|p+\lambda\Theta_x k|) &= g\Big(\sqrt{\alpha^2+2\beta+\sigma_k^2}\Big) \\ &= g(|\sigma_k|)+\frac{g'(\sigma_k)}{\sigma_k}\beta+\Big(\frac{g''(\sigma_k)}{2\sigma_k^2}-\frac{g'(\sigma_k)}{2\sigma_k^3}\Big)\beta^2+\frac{g'(\sigma_k)}{\sigma_k}\alpha^2+O(|\alpha|+|\beta|^3). \end{aligned}$$

These equations, together with formulas (6.9) and (6.10), after equating the terms at p^2, $|p|^4$, $p^2\langle\Theta_x k,\,p\rangle^2$, and $\langle\Theta_x k,\,p\rangle^4$, give the following representation for H_2:

$$\begin{aligned} H_2^0 = \sum_{|k|\neq 0} G_k = -\sum_{|k|\neq 0} |D_{1k}|^2 D_0\Big(\lambda^2\langle\Theta_x k,\,p\rangle^2 g(\sigma)+\frac{(D_0\lambda)^4}{2}\langle\Theta_x k,\,p\rangle^4\Big(\frac{g''(\sigma)}{\sigma^2}-\frac{g'(\sigma)}{\sigma^3}\Big) \\ +p^4 g(\sigma)+p^2\langle\Theta_x k,\,p\rangle^2(D_0\lambda)^2\Big(\frac{5}{2}\frac{g'(\sigma)}{\sigma}+\frac{1}{\sigma^2\sinh^2\sigma}-g(\sigma)\Big)\Big)\Big|_{\sigma=\sigma_k}+O(p^6). \end{aligned} \tag{6.13}$$

Remark. (1) Both the functions H_0 and H_2 are even with respect to p. To prove this, one should make the change $p\to -p$, $k\to -k$ and take into account the equations $D_{1,k}=\overline{D}_{1,-k}$, because $D_1(x,y)$ is a real valued function.

(2) The zero denominators disappear in this expansion due to (weak) resonance effects connected with the so-called intersection of terms or changing of multiplicity of characteristics (as was mentioned above) which take place for big momenta p.

Let us study the behavior of G_k for small and big λ (and proceed similarly for σ). For $\sigma\geqslant 2$, one can put $g(\sigma)\approx 1/\sigma$ and omit the terms $1-\tanh\sigma<0.0359724$ and $\frac{1}{\sigma^2\sinh^2\sigma}<0.0190055$ if $\sigma>2$. Then

$$G_k=-|D_{1k}|^2\Big(\frac{D_0}{\sigma}\lambda^2\langle\Theta_x k,p\rangle^2-\frac{D_0^5}{2}\lambda^4\langle\Theta_x k,p\rangle^4\frac{1}{\sigma^5}+\frac{D_0}{\sigma}p^4-p^2\lambda^2\langle\Theta_x k,p\rangle^2D_0^3\Big(-\frac{5}{2\sigma^3}-\frac{1}{\sigma}\Big)\Big)+O(p^6).$$

For small λ (this means that σ_k is small), with the equation

$$\sigma\tanh\sigma=\sigma^2+O(\sigma^4),$$

we get

$$G_k=-\frac{|D_{1k}|^2}{D_0}\left(\Big\langle\frac{\Theta_x k}{|\Theta_x k|},\,p\Big\rangle^2+\frac{1}{\lambda^2|\Theta_x k|^2}\Big(2\Big\langle\frac{\Theta_x k}{|\Theta_x k|},\,p\Big\rangle^2-p^2\Big)^2\right)+O(\lambda^0)+O(p^6). \tag{6.14}$$

Dropping the small corrections, we can write the desired formula (2.12) for symbol of operator for small λ. This ends the proof of Theorem 1.

CONCLUSION

In this paper, we have developed a scheme of research of fast changing solutions in the linear theory of water waves in a basin with rapidly oscillating depth. In particular, we have derived Boussinesq type equations for the long waves occurring for this type of the bottom oscillations and show that they imply effects similar to dispersion ones and depending on the depth, height, and width. We compared the influence of two different type of (weak) dispersions on the wave profile: the standard water dispersion and dispersion implied by fast oscillations of the bottom.

We show also that the effects of mode conversion (or intersection of characteristics) arise in this problem, which, however, do not play any role in the case of long waves. Nevertheless, the study of such “resonance” effects for short waves seems to be very interesting and remains open.

Finally we say that the class of possible depth function mentioned in the very beginning of the paper is natural to generalize for more wide class similar to one introduced in [23].

REFERENCES

1. A. Bensoussan and J.-L. Lions, *Papanicolaou G, Asymptotic Analysis for Periodic Structures. Studies in Mathematics and Its Applications.* Vol. 5 (North-Holland Publ. Company, Amsterdam–New York–Oxford, XXIV).
2. V. V. Zhikov, S. M. Kozlov, and O. A. Olejnik, *Homogenization of Differential Operators* (Moscow, Fizmatlit, 1993).
3. N. S. Bakhvalov and G. P. Panasenko, *Averaging Processes in Periodic Media. Mathematical Problems of the Mechanics of Composite Materials* (Nauka, Moscow, 1984).
4. J. J. Stoker, *Water Waves* (Interscience, New York, 1957).
5. C. C. Mei, *The Applied Dynamics of Ocean Surface Waves* (World Scientific, Singapore, 1989).
6. E. N. Pelinovski, *Hydrodynamics of Tsunami Waves* (Nizhnii Novgorod, 1996).
7. V. V. Kucherenko, "Asymptotics of the Solution of the System $A(x, -ih\partial/\partial x)u = 0$ as $h \to 0$ in the Case of Characteristics of Variable Multiplicity," Izv. Math. **8** (3), 631 (1974).
8. J. Bruening, V. V. Grushin, and S. Yu. Dobrokhotov, "Averaging of Linear Operators, Adiabatic Approximation, and Pseudodifferential Operators," Math. Notes **92** (2), 151–165 (2012).
9. V. V. Grushin and S. Yu. Dobrokhotov, "Homogenization in the Problem of Long Water Waves over Bottom Site with Fast Oscillations," Math. Notes **95** (3), 324–337 (2014).
10. V. V. Belov, S. Yu. Dobrokhotov, and T. Ya. Tudorovskiy, "Operator Separation of Variables for Adiabatic Problems in Quantum and Wave Mechanics," J. Engry. Math. **55** (1–4), 183–237 (2016).
11. S. Yu. Dobrokhotov, S. A. Sergeev, and B. Tirozzi, "Asymptotic Solutions of the Cauchy Problem with Localized Initial Conditions for Linearized Two-Dimensional Boussinesq-Type Equations with Variable Coefficients," Russ. J. Math. Phys. **20** (2), 155–171 (2013).
12. J. Bruening, V. V. Grushin, and S. Yu. Dobrokhotov, "Approximate Formulas for the Eigenvalues of a Laplace Operator, on a Torus, Which Arises in Linear Problems with Oscillating Coefficients," Russ. J. Math. Phys. **19** (3), 1–10 (2012).
13. V. V. Grushin, S. Yu. Dobrokhotov, and S. A. Sergeev, "Homogenization and Dispersion Effects in the Problem of Propagation of Waves Generated by a Localized Source," Proc. Steklov Inst. Math. **281**, 161–178 (2013).
14. S. Yu. Dobrokhotov, "Application of the Maslov Theory to Two Problems for Equations with Operator-Valued Symbol: Electron–Phonon Interaction and the Schrodinger Equation with Rapidly Oscillating Potential," Uspekhi Mat. Nauk **39** (4), 125 (1984).
15. V. S. Buslaev, "Semiclassical Approximation for Equations with Periodic Coefficients," Uspekhi Mat. Nauk **42** (6), 77–98 (1987) [Russ. Math. Surv. **42** (6), 97–125 (1987)].
16. V. P. Maslov, *Operational Methods* (Mir, Moscow, 1973).
17. V. P. Maslov and M. V. Fedoryuk, *Semiclassical Approximaion for Equations of Quantum Mechanics* (Nauka, Moscow, 1978).
18. V. P. Maslov and M. V. Karasev, *Nonlinear Poisson Brakets. Geometry and Quantization* (Nauka, Moscow, 1991).
19. S. Yu. Dobrokhotov, "Maslov's Methods in Linearized Theory of Gravitational Waves on the Liquid Surface," Dokl. AN SSSR **269** (1), 76–80 (1983) [Sov. Phys. Doklady **28**, 229–231 (1983)].
20. S. Yu. Dobrokhotov and P. N. Zhevandrov, "Asymptotic Expansions and the Maslov Canonical Operator in the Linear Theory of Water Waves, I," Russ. J. Math. Phys. **10** (1), 1–31 (2003).
21. S. Yu. Dobrokhotov and P. N. Zhevandrov, "Maslov's Operational Method in Problems of Water Waves Generated by a Source Moving Over Uneven Bottom," Izv. AN SSSR, Fiz. Atmos. Okeana **21** (7), 744–751 (1985) [Atmos. Ocean. Phys. **21** (7), 572–577 (1985)].
22. J. Bruening, V. Grushin, S. Yu. Dobrokhotov, and T. Tudorovskii, "Generalized Foldy–Wouthuysen Transformation and Pseudodifferential Operators," Teoret. Mat. Fiz. **167** (2), 171–192 (2011) [Theoret. and Math. Phys. **167** (2), 2011].
23. S. Yu. Dobrokhotov, V. E. Nazaikinskii, B. Tirozzi, "On a Homogenization Method for Differential Operators with Oscillating Coefficients," Doklady Mathematics **91** (2), 227–231 (2015).