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On Rational Functions of First-Class Complexity

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Abstract. It is proved that, for every rational function of two variables $P(x,y)$ of analytic complexity one, there is either a representation of the form $f(a(x)+b(y))$ or a representation of the form $f(a(x)b(y))$, where $f(x), a(x), b(x)$ are nonconstant rational functions of a single variable. Here, if $P(x,y)$ is a polynomial, then $f(x)$, $a(x)$, and $b(x)$ are nonconstant polynomials of a single variable.

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The complexity problem for functions of several variables was studied by many authors (see, e.g., [1–7]). In the paper [5], Beloshapka suggested the following viewpoint on the complexity of analytic functions of two variables. An increasing sequence of complexity classes $Cl_0 \subset Cl_1 \subset \cdots \subset Cl_n \subset \cdots$ is constructed by induction: Cl_0 is the class of analytic functions of one variable (x or y), Cl_1 is the class of functions of the form $c(a(x)+b(y))$, where $a(x)$, $b(x)$, and $c(x)$ are analytic functions, and, further, Cl_{n+1} consists of the functions of the form $C(A_n(x,y)+B_n(x,y))$, where C is a function of one variable and A_n and B_n are functions in Cl_n. The representability in the form of superposition is treated as the local representability in a neighborhood of a point in general position. The number n is regarded as the complexity of a function if the function belongs to $Cl_n \backslash Cl_{n-1}$. Thus, the analytic functions of complexity one are the functions of the form $c(a(x)+b(y))$, where $a(x)$, $b(x)$, and $c(x)$ are analytic functions which are not identically equal to a constant. In the case of first-class complexity, for rational functions, it turns out that the functions $f(x)$, $a(x)$, and $b(x)$ can be described.

Theorem 1. *Let a local representation of the form*

$$f(a(x)+b(y)) = P(x,y) \tag{1}$$

be valid, where $P(x,y)$ is a rational function of first-class complexity and $f(x)$, $a(x)$, and $b(x)$ are analytic. Then there are rational functions $F(x)$, $A(x)$, and $B(x)$ such that either $f(x)=F(x)$, $a(x)=A(x)$, and $b(x)=B(x)$ or $f(x)=F(e^x)$, $a(x)=ln(A(x))$, and $b(x)=ln(B(x))$.

In other words, there is either an additive representation $F(A(x)+B(y))=P(x,y)$ or a multiplicative representation $F(A(x)B(y))=P(x,y)$ with rational $F(x)$, $A(x)$, and $B(x)$.

Proof. Let us take the partial derivatives with respect to x and y of both the sides of (1) and divide them by the other (we may assume that the derivative with respect to y is not identically equal to 0, because, otherwise, the function depends on x only, and hence, is a function of complexity class 0). We obtain

$$\frac{P_x(x,y)}{P_y(x,y)} = \frac{f(a(x)+b(y))_x}{f(a(x)+b(y))_y} = \frac{f'(a(x)+b(y))a'(x)}{f'(a(x)+b(y))b'(y)} = \frac{a'(x)}{b'(y)},$$

where $P_x(x,y)/P_y(x,y)$ is a rational function if $P(x,y)$ is. Therefore, integrating the equation $P_x(x,y)/P_y(x,y) = a'(x)/b'(y)$ for a chosen y, we obtain $a(x)$; integrating for a chosen x, we obtain $b(y)$:

$$a(x) = A(x) + \sum_{n=1}^{k} \alpha_n \ln(x-a_n), \qquad b(y) = B(y) + \sum_{n=1}^{l} \beta_n \ln(y-b_n),$$

where $A(x)$ and $B(y)$ are rational and α_i and β_i are constants.

The following cases are possible:

(1) all α_i and β_i vanish, i.e., $a(x)=A(x)$ and $b(y)=B(y)$;

(2) $A(x)$ and $B(y)$ are nonconstant, and there are nonzero coefficients among α_i and β_i;

(3) $A(x)$ is constant, $B(y)$ is nonconstant, and there are nonzero coefficients among α_i and β_i;

(3') $A(x)$ is nonconstant, $B(y)$ is constant, and there are nonzero coefficients among α_i and β_i;

(4) $A(x), B(y)$ are constant and there are incommensurable coefficients among α_i and β_i;

(5) $A(x), B(y)$ are constant and there are no incommensurable coefficients among α_i and β_i.

We claim that cases (2), (3), (3'), and (4) cannot be realized.

Consider case 2). Substitute the values $a(x)$ and $b(y)$ into (1). Making the change of variable $y = y(x) = B^{-1}(-A(x))$ in (1), we obtain

$$f\left(\ln\left(\prod_{n=1}^{k}(x-a_n)^{\alpha_n}\prod_{n=1}^{l}(y(x)-b_n)^{\beta_n}\right)\right) = P(x, y(x)). \tag{2}$$

Without loss of generality we may assume that $P(x, y(x))$ is not everywhere infinite (otherwise we add a constant to $y = y(x)$ and obtain a rational function $P(x, y(x) + \text{const})$; it is clear from the consideration below that this constant plays no role in the proof). The same stipulation holds in all similar situations below.

Making in (1) the change $y = y_0$ and writing $c = e^{b(y_0)} = const$, we obtain

$$f\left(\ln\left(ce^{A(x)}\prod_{n=1}^{k}(x-a_n)^{\alpha_n}\right)\right) = P(x, y_0). \tag{3}$$

Note that the functions $P(x, y(x)) = \Phi(x)$ and $P(x, y_0) = \Psi(x)$ are algebraic. Two cases are possible:

(2.1) $P(x, y(x)) = \Phi(x) \neq \text{const}$,

(2.2) $P(x, y(x)) = \Phi(x) = \text{const}$.

Consider case (2.1). Make the change $x = \Phi^{-1}(y)$ in (2) and the change $x = \Psi^{-1}(y)$ in (3). Then (2) and (3) are represented in the form

$$\prod_{n=1}^{k} q_n^{\alpha_n}(y)\prod_{n=1}^{l} r_n^{\beta_n}(y) = e^{f^{-1}(y)}, \tag{2'}$$

$$ce^{s(y)}\prod_{n=1}^{k} p_n^{\alpha_n}(y) = e^{f^{-1}(y)}, \tag{3'}$$

where $p_i(y)$, $q_i(y)$, $r_i(y)$, and $s(y)$ are algebraic functions obtained after the change (we can assign the constant c to any of these functions). The left-hand sides of $(2')$ and $(3')$ have finitely many singular points only, and therefore the function $e^{f^{-1}(y)}$ admits a continuation along all paths not passing through these points. One can choose equal analytic functions in the full analytic functions obtained by extending the left-hand sides of $(2')$ and $(3')$ (one should take the corresponding germs of $f^{-1}(y)$ on the right-hand sides of $(2')$ and $(3')$). Thus, for an appropriate choice of germs, the left-hand sides are equal. Equate the left-hand sides of (2') and (3'). Move all products to the left and include all functions into a single product by redenoting the functions $1/p_i(y), q_i(y), r_i(y)$ by $u_i(y)$ and the exponents by γ_n. We obtain

$$\prod_{n=1}^{m} u_n^{\gamma_n}(y) = e^{s(y)}. \tag{4}$$

However, equation (4) can hold only if $s(y) = \text{const}$. Indeed, otherwise, if all γ_n are rational, then the left-hand side of (4) is algebraic and the right-hand side is not. If there are irrational numbers among γ_n, then the left-hand side of (4) can take infinitely many values when going around the singular point, while the right-hand side can take finitely many values only. Therefore, $s(y) = \text{const}$, and hence, $A(y) = \text{const}$, which contradicts the initial assumption $A(y) \neq \text{const}$. Thus, case (2.1) is impossible.

Consider case (2.2). In this case, (2) becomes

$$f\left(\ln\left(\prod_{n=1}^{k}(x-a_n)^{\alpha_n}\prod_{n=1}^{l}(y(x)-b_n)^{\beta_n}\right)\right)=P(x,y(x))=\text{const},$$

which implies that

$$\prod_{n=1}^{k}(x-a_n)^{\alpha_n}\prod_{n=1}^{l}(y(x)-b_n)^{\beta_n}=\text{const}. \tag{5}$$

Let us keep $(y(x)-b_1)^{\beta_1}$ in the left-hand side of (5), move all other factors to the right-hand side, and take the equation thus obtained to the power $1/\beta_1$. We see that

$$(y(x)-b_1)=\text{const}\prod_{n=1}^{k}(x-a_n)^{-\frac{\alpha_n}{\beta_1}}\prod_{n=2}^{l}(y(x)-b_n)^{-\frac{\beta_n}{\beta_1}}. \tag{5'}$$

It is clear from (5') that either $(y(x)-b_1)=\text{const}$ (which is impossible, because $y(x)=B^{-1}(-A(x))$, where $A(x)$ and $B(y)$ are nonconstant) or all numbers α_i,β_i must be commensurable with β_1 (and hence, with one another). Thus, $\alpha_i=p_i\beta_1,\beta_i=q_i\beta_1$ with rational p_i and q_i. Then the expressions for $a(x)$ and $b(y)$ can be represented in the form

$$a(x)=A(x)+\beta_1\sum_{n=1}^{k}\ln(x-a_n)^{p_n}=\beta_1\left(\frac{A(x)}{\beta_1}+\ln\prod_{n=1}^{k}(x-a_n)^{p_n}\right), \tag{6}$$

$$b(y)=B(y)+\beta_1\sum_{n=1}^{l}\ln(y-b_n)^{q_n}=\beta_1\left(\frac{B(y)}{\beta_1}+\ln\prod_{n=1}^{l}(y-b_n)^{q_n}\right). \tag{7}$$

Let us substitute the values of $a(x)$ and $b(y)$ into (1). We have

$$f\left(\beta_1\left(\frac{A(x)}{\beta_1}+\ln\prod_{n=1}^{k}(x-a_n)^{p_n}\right)+\beta_1\left(\frac{B(y)}{\beta_1}+\ln\prod_{n=1}^{l}(y-b_n)^{q_n}\right)\right)=P(x,y). \tag{8}$$

For the convenience of the forthcoming considerations, we introduce the notation $\frac{A(x)}{\beta_1}=Q(x)$, $\prod_{n=1}^{k}(x-a_n)^{p_n}=R(x)$, $\frac{B(y)}{\beta_1}=S(y)$, $\prod_{n=1}^{l}(y-b_n)^{q_n}=T(y)$, $f(\beta_1 x)=V(x)$. Below we need only the fact that $Q(x)$, $R(x)$, $S(y)$, and $T(y)$ are algebraic. In the new notation, (8) becomes

$$V(Q(x)+S(y)+\ln(R(x)T(y)))=P(x,y). \tag{9}$$

We claim that the only realizable cases are

(2.2.1) $Q(x),S(y)\neq\text{const};\ R(x),T(y)=\text{const}$

and

(2.2.2) $Q(x),S(y)=\text{const};\ R(x),T(y)\neq\text{const}$.

Returning to the previous notation, one can readily see that cases (2.2.1) and (2.2.2) are related to cases (1) and (5), respectively. (We shall see below that only cases (1) and (5) can be realized indeed, and they are studied below.)

Suppose that a case which differs from (2.2.1) and (2.2.2) is possible. Make the change

$$p=Q(x)+S(y),\quad q=R(x)T(y).$$

If p and q are dependent (i.e., their image is an algebraic curve in $\mathbb{C}^2$), then, by the implicit function theorem, the left-hand side of (9) can be reduced to the form $g(p)=g(Q(x)+S(y))$, i.e., case (1) holds. If p and q are independent, then x and y are algebraic and can be expressed using p and q, $x = X(p, q)$, $y = Y(p, q)$. Therefore, successively (a) $q = q_0$ and (b) $p = p_0$ and assuming here, without loss of generality, that (a) $\ln(q_0) = 0$ and (b) $p_0 = 0$, respectively, we obtain

$$(a)\ V(p+\ln(q_0)) = P(X(p,q_0),Y(p,q_0)) = \widetilde{P}(p),\ (b)\ V(p_0+\ln(q)) = P(X(p_0,q),Y(p_0,q)) = \widetilde{\widetilde{P}}(q).$$

It follows from (a) that $V(p)$ algebraic, which then implies that $\ln(q) = V^{-1}(\widetilde{\widetilde{P}}(q))$ is algebraic. A contradiction. Thus, case (2.2) is impossible. Therefore, case (2) is impossible.

Consider case (3) (case (3') is similar). Substitute the values of $a(x)$ and $b(y)$ into (1). Making the change $y = x$ in (1) and writing $c_1 = e^{A(x)} = \text{const}$, we obtain

$$f\left(\ln\left(c_1 e^{B(x)} \prod_{n=1}^{k} (x-a_n)^{\alpha_n} \prod_{n=1}^{l} (x-b_n)^{\beta_n}\right)\right) = P(x,x). \tag{10}$$

Making in (1) the change $y = y_0$ and writing $c_2 = e^{b(y_0)} = \text{const}$, we obtain

$$f\left(\ln\left(c_1 c_2 \prod_{n=1}^{k} (x-a_n)^{\alpha_n}\right)\right) = P(x,y_0). \tag{11}$$

Thus, we arrive at a situation which is completely analogous to that considered in case (2.1). Thus, case (3) is impossible.

Consider case (4). Here $a(x)$ and $b(y)$ are of the form

$$a(x) = \sum_{n=1}^{k} \alpha_n \ln(x-a_n) + \text{const}, \qquad b(y) = \sum_{n=1}^{l} \beta_n \ln(y-b_n) + \text{const}.$$

Let us group the summands with commensurable coefficients (the summands can be assigned to any term) and redenote the coefficients by A_n and B_n. We obtain the sum of logarithms of algebraic functions with incommensurable coefficients,

$$a(x) = \sum_{n=1}^{k'} A_n \ln(a_n(x)), \qquad b(y) = \sum_{n=1}^{l'} B_n \ln(b_n(y)).$$

Substitute the values of $a(x)$ and $b(y)$ into (1). Making in (1) the change $x = y$, we see that

$$f\left(A_1\left(\ln\left(a_1(y) \prod_{n=2}^{k'} a_n^{\frac{A_n}{A_1}}(y) \prod_{n=1}^{l'} b_n^{\frac{B_n}{A_1}}(y)\right)\right)\right) = P(y,y). \tag{12}$$

Making the substitution $x = x_0$ in (1) and writing $c_3 = e^{\frac{a(x_0)}{A_1}} = \text{const}$, we obtain

$$f\left(A_1\left(\ln\left(c_3 \prod_{n=1}^{l'} b_n^{\frac{B_n}{A_1}}(y)\right)\right)\right) = P(x_0,y). \tag{13}$$

Let us now proceed in the same way as in the consideration of case (2). Writing $P(y, y) = \Theta(y)$ and $P(x_0, y) = \Omega(y)$, we note that $\Theta^{-1}(y)$ and $\Omega^{-1}(y)$ are algebraic. Make the change $y = \Theta^{-1}(x)$ in (12) and the change $y = \Omega^{-1}(x)$ in (13). Then (12) and (13) acquire the form

$$g_1(x) \prod_{n=2}^{k'} g_n^{\frac{A_n}{A_1}}(x) \prod_{n=1}^{l'} h_n^{\frac{B_n}{A_1}}(x) = e^{f^{-1}(x)/A_1}, \tag{12'}$$

$$c_3 \prod_{n=1}^{l'} t_n^{\frac{B_n}{A_1}}(x) = e^{f^{-1}(x)/A_1}, \tag{13'}$$

where $g_i(x)$, $h_i(x)$, and $t_i(x)$ are algebraic functions obtained after the change (we assign the constant c_3 to any of them). Let us equate the left-hand sides of (12') and (13'). Let us move all products to the left-hand side and include all functions into a single product. Redenoting A_i/A_1 and B_i/A_1 by C_i and the functions $1/g_i(x)$, $1/h_i(x)$, and $t_i(x)$ by $v_i(x)$, we obtain

$$\prod_{n=1}^{j} v_n^{C_i}(x) = g_1(x). \tag{14}$$

However, if there are irrational exponents C_i, then equation (14) is impossible, because the left-hand side of (14) takes infinitely many values when going around the singular point and the right-hand side only finitely many values, because $g_1(x)$ is algebraic. Thus, case (4) is impossible; however, if all exponents C_i are rational, then we arrive at case (5).

Thus, only cases (1) and (5) are possible. Consider case 1). For the rational function $P(x,y)$, by (1), we have $f(A(x)+B(y)) = P(x,y)$, where $A(x)$ and $B(y)$ are rational functions and $f(x)$ is an algebraic function. Let

$$f(z) = \sum_{n=-N}^{\infty} c_n z^{n/k}. \tag{15}$$

Then, for $k=1$, the function $f(x)$ is rational, and everything is proved, and, for $k \neq 1$, we have the equation

$$(A(x)+B(y))^{1/k} = R(x,y), \tag{16}$$

where $R(x,y)$ is a rational function, because, otherwise, the left-hand side of $f(A(x)+B(y)) = P(x,y)$ has branching points and the right-hand side does not have any. Take (16) to the power k and differentiate the result with respect to x and to y. We obtain $0 = (R(x,y)^k)_{xy}$, which is impossible for $k \neq 1$. Thus, $k=1$, and this completes the proof of the theorem in case (1).

Consider case (5). Due to the commensurability, the coefficients are of the form δh_i with rational h_i. Let LCM be the least common multiple of the denominators h_i and let $c_4 = e^{\frac{LCM}{\delta}(A(x)+B(y))} =$ const. Then $P(x,y)$ can be represented in the form

$$f\left(\frac{\delta}{LCM}\ln(c_4 C(x)D(y))\right) = P(x,y),$$

where $C(x)$ and $D(y)$ are rational functions and $f(\frac{\delta}{LCM}\ln(c_4 x))$ is an algebraic function. Setting $H(x) = f(\frac{\delta}{LCM} ln(c_4 x))$, let us prove the rationality of $H(x)$, which will imply the desired assertion. Thus, we have the equation $H(C(x)D(y)) = P(x,y)$. Let $\deg = \{\max n : (C(x)D(y))^{\frac{1}{n}} \text{ is rational}\}$. Let $H(x^{\deg}) = \tilde{H}(x)$, $C(x)^{\frac{1}{\deg}} = \tilde{C}(x)$, and $D(y)^{\frac{1}{\deg}} = \tilde{D}(y)$. Arguing as in the consideration of case (1), we represent $\tilde{H}(x)$ by a series of the form (15) and obtain an equation similar to (16),

$$(\tilde{C}(x)\tilde{D}(y))^{1/k} = Q(x,y), \tag{16'}$$

where $Q(x,y)$ is a rational function. However, then we have $k=1$ by the choice of deg and by the construction of $\tilde{C}(x)$ and $\tilde{D}(y)$. This completes the proof of the theorem.

Theorem 2. *Let $P(x,y)$ be a polynomial of analytic complexity one. Then there is either an additive representation $F(A(x)+B(y)) = P(x,y)$ or a multiplicative representation $F(A(x)B(y)) = P(x,y)$, where $F(x)$, $A(x)$, and $B(x)$ are polynomials.*

Proof. By Theorem 1, we have either an additive or a multiplicative representation with rational functions $F(x)$, $A(x)$, and $B(x)$. Consider the case of additive representation (the multiplicative case can be treated in a similar way). Thus, in the case of a polynomial, we have $f(A(x)+B(y)) = P(x,y)$, where $A(x)$, $B(y)$, and $f(x)$ are rational functions.

Let $A(x)$ have a pole. Then, varying y, we obtain infinitely many poles $A(x)+B(y)$. However, $f(x)$ has only finitely many poles, and hence, $P(x,y)$ has a pole, which is impossible, because $P(x,y)$ is a polynomial. Therefore, $A(x)$ is a polynomial. We similarly see that $B(y)$ is a polynomial. This implies clearly that $f(x)$ is a polynomial, because $(A(x)+B(y))$ and $P(x,y)$ have no poles. This completes the proof of the theorem.

Proposition 3. *An additive and a multiplicative representations cannot be realized simultaneously, i.e., if there are simultaneously two representations of a rational function $P(x, y) = F(A(x) + B(y)) = G(C(x)D(y))$ with rational $A(x)$, $B(x)$, $C(x)$, $D(x)$, $F(x)$, and $G(x)$, then the complexity of the function P is equal to zero. In particular, this holds if $P(x, y)$ and $A(x)$, $B(x)$, $C(x)$, $D(x)$, $F(x)$, and $G(x)$ are polynomials.*

Proof. Let $P(x, y)$ be a rational function admitting representations of both types,

$$F(A(x) + B(y)) = G(C(x)D(y)) = P(x, y). \tag{17}$$

Differentiate (17) with respect to x and to y and divide one equation by the other (the derivatives with respect to y are not identically zero, because, otherwise, (17) depends only on x and therefore belongs to Cl_0). We have

$$\frac{P_x(x,y)}{P_y(x,y)} = \frac{F(A(x)+B(y))_x}{F(A(x)+B(y))_y} = \frac{F'(A(x)+B(y))A'(x)}{F'(A(x)+B(y))B'(y)} = \frac{A'(x)}{B'(y)},$$
$$\frac{P_x(x,y)}{P_y(x,y)} = \frac{G(C(x)D(y))_x}{G(C(x)D(y))_y} = \frac{G'(C(x)D(y))C'(x)D(y)}{G'(C(x)D(y))C(x)D'(y)} = \frac{(\ln(C(x)))'}{(\ln(D(y)))'},$$

and hence, choosing some $y = y_0$ and integrating with respect to x, we obtain

$$\frac{A(x)}{B'(y_0)} = \frac{\ln(C(x))}{\ln(D(y_0))'} + \text{const}. \tag{18}$$

The left-hand side of (18) is rational, and therefore, the right-hand side must also be rational, which is possible only if $C(x)$ is constant. However, in this case, $P(x, y)$ depends only on y, and therefore, belongs to Cl_0. If $P(x, y)$ is a polynomial, then we arrive at the same conclusion. This completes the proof of the proposition.

Several questions remain open.

(1) Does a similar theorem hold for algebraic functions?

(2) Let $P(x, y)$ be a rational or an algebraic function of complexity two, i.e., $P(x, y) = F(f(a(x)+ b(y)) + g(c(x) + d(y)))$. What can be said about the seven analytic functions entering the composition?

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