ISSN 1061-9208, Russian Journal of Mathematical Physics, 2016, Vol. 23, No. 2, pp. 164–171. © Pleiades Publishing, Ltd., 2016.

Identities for Apostol-Type Frobenius–Euler Polynomials Resulting from the Study of a Nonlinear Operator

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Received August 10, 2013

Abstract. We introduce a special nonlinear differential operator and, using its properties, reduce higher-order Frobenius–Euler Apostol-type polynomials to a finite series of first-order Apostol-type Frobenius–Euler polynomials and Stirling numbers. Interesting identities are established.

DOI 10.1134/S1061920816020023

1. INTRODUCTION

For any complex numbers λ, u, where u differs from λ and 1, the Apostol type Frobenius–Euler polynomials $H_n(x;\lambda \mid u)$, in the variable x, are defined by the equation

$$F(t,x;\lambda|u) = \frac{1-u}{\lambda e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x;\lambda \mid u)\frac{t^n}{n!}. \tag{1.1}$$

The nth Apostol-type Frobenius–Euler number is given by

$$H_n(\lambda \mid u) := H_n(0;\lambda \mid u).$$

Note that

$$H_0(\lambda \mid u) = \begin{cases} 1 & \text{for } \lambda = 1, \\ \frac{1-u}{\lambda-u} & \text{for } \lambda \neq 1. \end{cases} \tag{1.2}$$

The ordinary Frobenius–Euler polynomials $H_n(x \mid u)$ correspond to the special case $\lambda = 1$, so that $H_n(x \mid u) = H_n(x;1 \mid u)$. Considering the following three relations,

$$\frac{1-u}{\lambda e^t - u} e^{xt} = \left(\sum_{n=0}^{\infty} H_m(\lambda \mid u)\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\frac{x^k t^k}{k!}\right), \quad \lambda\frac{1-u}{\lambda e^t - u}e^{xt} - u\frac{1-u}{\lambda e^t - u}e^{xt} = (1-u)e^{xt},$$
$$\frac{1-u}{\lambda e^t - u}e^{(x+y)t} = \left(\sum_{n=0}^{\infty} H_m(x;\lambda \mid u)\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\frac{y^k t^k}{k!}\right), \tag{1.3}$$

by equations (1.1) and (1.3), we obtain the following interesting identities satisfied by the polynomials $H_n(x;\lambda \mid u)$ and numbers $H_n(\lambda \mid u)$:

$$H_n(x;\lambda \mid u) = \sum_{l=0}^{n} \binom{n}{l} H_l(\lambda \mid u) x^{n-1}, \tag{1.4}$$

$$\lambda\big(H(\lambda \mid u)+1\big)^n - uH_n(\lambda \mid u) = \begin{cases} 1-u & \text{for } n=0, \\ 0 & \text{for } n>0, \end{cases} \tag{1.5}$$

$$H_n(x+y;\lambda \mid u) = \sum_{l=0}^{n} \binom{n}{l} H_l(x;\lambda \mid u) H_{n-l}(y;\lambda \mid u), \tag{1.6}$$

$$\lambda\big(H(\lambda \mid u)+x\big)^n - uH_n(x;\lambda \mid u) = \begin{cases} (1-u)x^n & \text{for } n=0, \\ 0 & \text{for } n>0, \end{cases} \tag{1.7}$$

with the usual convention about replacing $H(\lambda \mid u)^n$ by $H_n(\lambda \mid u)$.

The Apostol–Bernoulli polynomials $B_n(x;\lambda)$ and Apostol–Euler polynomials $E_n(x;\lambda)$ are given by the series

$$\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x;\lambda) \frac{t^n}{n!} \quad (|t| < |\log(\lambda)| \quad \text{if } \lambda \neq 1, \text{and } |t| < 2\pi \quad \text{if } \lambda = 1), \tag{1.8}$$

$$\frac{2}{\lambda e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x;\lambda) \frac{t^n}{n!} \quad \text{for } \lambda \neq -1, \text{ and } |t| < |\log(-\lambda)|. \tag{1.9}$$

Throughout the paper, for the complex logarithm, we consider the principal branch.

In the special case $\lambda = 1$, $B_n(x;1) = B_n(x)$ is the well-known nth Bernoulli polynomial and $E_n(x;1) = E_n(x)$ is the nth Euler polynomial.

On the other hand, the interpretation of the Apostol–Bernoulli and Apostol–Euler polynomials in terms of Apostol type Frobenius–Euler polynomials is given as follows.

Lemma 1.1. *For any positive integer* n,

$$E_n(x;\lambda) = H_n(x;\lambda| -1) \quad \text{for } \lambda \neq -1, \tag{1.10}$$

and there are three equivalent ways to obtain the Apostol–Bernoulli polynomials $B_n(x;\lambda)$

$$B_n(x;\lambda) = \lim_{u\to 1} \frac{H'_n(x;\lambda|u)}{1-u} \quad \text{for any } \lambda, \tag{1.11}$$

where $H'_n(x;\lambda|u) = \frac{\partial}{\partial x} H_n(x;\lambda|u)$, *or*

$$B_n(x;\lambda) = n \lim_{u\to 1} \frac{H_{n-1}(x;\lambda|u)}{1-u} \quad \text{for any } \lambda, \tag{1.12}$$

or

$$B_{n+1}(x;\lambda) = -\frac{n+1}{2} H_n(x;-\lambda| -1) \quad \text{for } \lambda \neq 1. \tag{1.13}$$

The polynomials $B_n(x;\lambda)$ *and* $E_n(x;\lambda)$ *are related by the formula*

$$B_{n+1}(x;\lambda) = -\frac{n+1}{2} E_n(x;-\lambda) \quad \text{for } \lambda \neq 1. \tag{1.14}$$

The Bernoulli polynomials and Euler polynomials play fundamental roles in various branches of mathematics including combinatorics, number theory, special functions, and analysis, see for example [5, 8, 11]. One of the most important formulas satisfied by the Bernoulli polynomials is Carlitz's relation for the product of two Bernoulli polynomials [1, 3],

$$B_m(x)B_n(x)=\sum_{r=0}^{\infty}\left(\binom{m}{2r}n+\binom{n}{2r}m\right)\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r}\;+\;(-1)^{m+1}\frac{m!n!}{(m+n)!}B_{m+n},\qquad(1.15)$$

where $m+n\geqslant 2$. In [9], Nielson established similar formulas for $E_n(x)E_m(x)$ and $E_m(x)B_n(x)$.

Moreover, in view of (1.15), Carlitz considered the following identities for the Frobenius–Euler polynomials:

$$\begin{aligned}H_m(x\mid u)H_n(x\mid v)=&H_{m+n}(x\mid uv)\frac{(1-u)(1-v)}{1-uv}\\&+\frac{u(1-v)}{1-uv}\sum_{k=0}^{m}\binom{m}{k}H_k(u)H_{m+n-k}(x\mid uv)\\&+\frac{v(1-v)}{1-uv}\sum_{k=0}^{n}\binom{n}{k}H_k(v)H_{m+n-k}(x\mid uv),\end{aligned}$$

where $u,v\in\mathbb{C}$ with $u\neq 1$, $v\neq 1$, and $uv\neq 1$ (see [2]). In particular, if $u\neq 1$ and $uv=1$, then

$$\begin{aligned}H_m(x\mid u)H_n(x\mid u^{-1})=&-(1-u)\sum_{k=1}^{m}\binom{m}{k}H_k(u)\frac{B_{m+n-k+1}(x)}{m+n-k+1}\\&-(1-u^{-1})\sum_{k=1}^{n}\binom{n}{k}H_k(u^{-1})\frac{B_{m+n-k+1}(x)}{m+n-k+1}\\&+(-1)^{n+1}\frac{m!n!}{(m+n+1)!}(1-u)H_{m+n+1}(u).\end{aligned}\qquad(1.16)$$

From Lemma 1.1 and equation (1.16), we obtain Carlitz identity for Euler polynomials

$$\begin{aligned}E_m(x)E_n(x)=&-2\sum_{k=1}^{m}\binom{m}{k}E_k\frac{B_{m+n-k+1}(x)}{m+n-k+1}-2\sum_{k=1}^{n}\binom{n}{k}E_k\frac{B_{m+n-k+1}(x)}{m+n-k+1}\\&+2(-1)^{n+1}\frac{m!n!}{(m+n+1)!}E_{m+n+1}.\end{aligned}$$

In this paper, we are interested in studying the sum of products of Apostol type Frobenius–Euler polynomials and in expressing them as linear combination of Apostol type Frobenius–Euler polynomials with simple coefficients. Our study is motivated by the following famous Euler formula. Namely, Euler proved the following identity on Bernoulli numbers.

$$\sum_{k=0}^{n}\binom{n}{k}B_kB_{n-k}=-nB_{n-1}-(n-1)B_n\quad(n\geqslant 1),\qquad(1.17)$$

and its polynomial version

$$\sum_{k=0}^{n}\binom{n}{k}B_k(x)B_{n-k}(y)=-n(1-x)B_{n-1}(x+y)-(n-1)B_n(x+y)\quad(n\geqslant 1)\qquad(1.18)$$

this was found by many authors, see [6].

For $N \in \mathbb{N}$, the nth Apostol type Frobenius–Euler polynomials of order N are defined by the generating function

$$\begin{aligned} F^N(t,x;\lambda|u) &= \underbrace{F(t,x;\lambda|u) \times F(t,x;\lambda|u) \times \cdots \times F(t,x;\lambda|u)}_{N-\text{ times}} \\ &= \underbrace{\left(\frac{1-u}{\lambda e^t - u}\right) \times \left(\frac{1-u}{\lambda e^t - u}\right) \times \cdots \times \left(\frac{1-u}{\lambda e^t - u}\right)}_{r-\text{ times}} e^{xt} \\ &= \sum_{n=0}^{\infty} H_n^{(N)}(x;\lambda \mid u)\frac{t^n}{n!} \quad \text{for} \quad u \in \mathbb{C} \quad \text{with} \quad u \neq \lambda, 1. \end{aligned} \tag{1.19}$$

In the special case, when $x = 0$, the polynomials $H_n^{(N)}(\lambda \mid u) := H_n^{(N)}(0;\lambda \mid u)$ are called the *nth Apostol type Frobenius–Euler numbers of order N*. Besides, the Apostol-Bernoulli $B_n^{(N)}(x;\lambda)$ and Apostol-Euler polynomials $E_n^{(N)}(x;\lambda)$ of higher order N are defined as follows:

$$\begin{gathered} \left(\frac{t}{\lambda e^t - 1}\right)^N e^{xt} = \sum_{n=0}^{\infty} B_n^{(N)}(x;\lambda)\frac{t^n}{n!}, \\ \text{for} \quad \lambda = 1, \quad |t| < 2\pi, \quad \text{and if} \quad \lambda \neq 1, \quad |t| < |\log(\lambda)|, \end{gathered} \tag{1.20}$$

and

$$\left(\frac{2}{\lambda e^t + 1}\right)^N e^{xt} = \sum_{n=0}^{\infty} E_n^{(N)}(x;\lambda)\frac{t^n}{n!}, \quad |t| < |\log(-\lambda)|, \quad \lambda \neq -1. \tag{1.21}$$

The purpose of this paper is to study Euler type formulas for Apostol type Frobenius–Euler numbers and polynomials. Our results can be wieved as a reformulation of Euler's identities in the framework of Apostol type of Frobenius–Euler numbers and polynomials. Naturally, from our results, we deduce easily the Euler's formulae (1.17) and (1.18). This gives a new way of proving it. To state and prove the main result of this paper, we introduce and investigate some special non-linear differential operators. In particular, we express them as a finite series of ordinary differential operators and Stirling numbers. From this study, we derive nonlinear differential equations. Then, we obtain some new and interesting identities for Apostol type Frobenius–Euler polynomials of higher order.

2. MAIN RESULTS AND THEIR PROOFS

We consider the function

$$f(t) := f(t,u) = \frac{1-u}{\lambda e^t - u}.$$

Theorem 2.1. *Let n, N be any positive integers. The function f is a solution to the following differential equation of order N*

$$y^{N+1}(t) = \left(\frac{u-1}{u}\right)^N \frac{1}{N!} \sum_{k=0}^{N} \mid s(N+1,k+1) \mid y^{(k)}(t),$$

where $f^{(k)}(t) = \dfrac{\partial^k}{\partial^k t} f(t,u)$ and $\mid s(N+1,k+1) \mid$ are the unsigned Stirling numbers of the second kind.

Proof. We prove this theorem by induction on N. For $N = 1$, we have the relation

$$\frac{\partial f}{\partial t} = -f + \frac{u}{u-1} f^2, \tag{2.1}$$

thus,

$$f^2 = \frac{u-1}{u}\left(f + f'\right), \tag{2.2}$$

and it is well-known that $|s(2,1)| = |s(2,1)| = 1$, see [4]. Thus theorem holds for $N = 1$. Let $N \geqslant 1$, we recall that the unsigned Stirling numbers $|s(N,k)|$ of second kind are given by

$$\langle x\rangle_N = \sum_{k=0}^{N} |s(N,k)|x^k, \quad \text{where} \quad \langle x\rangle_N = x(x+1)\cdots(x+N-1). \tag{2.3}$$

By the identity (2.3) and the relation $x\langle 1+x\rangle_N = \langle x\rangle_{N+1}$, the differential operator $\partial/\partial t$ satisfies the following equation

$$\left\langle 1 + \frac{\partial}{\partial t}\right\rangle_N = \sum_{k=0}^{N} |s(N+1,k+1)| \frac{\partial^k}{\partial t^k}. \tag{2.4}$$

Moreover, it is well known that the Stirling numbers $|s(n,k)|$ satisfy the following nice recursion formula

$$|s(N+1,k+1)| = |s(N,k)| + N|s(N,k+1)| \quad \text{if} \quad N \geqslant k \geqslant 0, \tag{2.5}$$

$$|s(N,N)| = 1, \quad |s(N,0)| = |s(0,k)| = 0 \quad \text{except} \quad |s(0,0)| = 1. \tag{2.6}$$

One can consult [4] for the properties of Stirling numbers. Now equations (2.1), (2.3), (2.4), (2.5), and (2.6) imply that the Theorem 2.1 holds.

Let us define

$$y^k(t,x) = y(t)^k e^{tx}, \quad y^{(k)}(t,x) = \frac{\partial^k}{\partial t^k}\left(y(t)e^{tx}\right).$$

Then, by Leibnitz formula, we obtain

$$y^{(k)}(t,x) = \sum_{m=0}^{k} \binom{k}{m} x^{k-m} y^{(m)}(t)e^{xt}.$$

Moreover, we can inverse this identity as follows

$$y^{(k)}(t) = \frac{\partial^k}{\partial t^k}\left(y(t,x)e^{-tx}\right)$$

again, we use the Leibnitz formula, obtaining

$$y^{(k)}(t)e^{xt} = \sum_{m=0}^{k} \binom{k}{m} (-x)^{k-m} y^{(m)}(t,x). \tag{2.7}$$

Hence, from equation (2.7), we can write

$$\begin{aligned}
y^{N+1}(t,x) &= y^{N+1}(t)e^{xt} \\
&= \left(\frac{u-1}{u}\right)^N \frac{1}{N!} \sum_{k=0}^{N} \mid s(N+1,k+1) \mid y^{(k)}(t)e^{xt} \\
&= \left(\frac{u-1}{u}\right)^N \frac{1}{N!} \sum_{k=0}^{N} \mid s(N+1,k+1) \mid \sum_{m=0}^{k} \binom{k}{m} (-x)^{k-m} y^{(m)}(t,x) \\
&= \left(\frac{u-1}{u}\right)^N \frac{1}{N!} \sum_{m=0}^{N} \left(\sum_{k=m}^{N} \binom{k}{m} (-x)^{k-m} |s(N+1,k+1)| \right) y^{(m)}(t,x).
\end{aligned}$$

Thus, we obtain the following theorem.

Theorem 2.2. *Let n, N be any positive integers. The function* $y(t,x) = \dfrac{1-u}{\lambda e^t - u} e^{xt}$ *satisfies the differential equation*

$$y^{N+1}(t,x) = \left(\frac{u-1}{u}\right)^N \frac{1}{N!} \sum_{m=0}^{N} \left(\sum_{k=m}^{N} \binom{k}{m} (-x)^{k-m} |s(N+1,k+1)| \right) y^{(m)}(t,x).$$

From (1.1) and (1.19), we see that

$$f(t) = \frac{1-u}{\lambda e^t - u} = \sum_{n=0}^{\infty} H_n(\lambda|u) \frac{t^n}{n!}, \tag{2.8}$$

and

$$f^N(t) = \underbrace{\left(\frac{1-u}{\lambda e^t - u}\right) \times \left(\frac{1-u}{\lambda e^t - u}\right) \times \cdots \times \left(\frac{1-u}{\lambda e^t - u}\right)}_{N-\text{ times}} = \sum_{n=0}^{\infty} H_n^{(N)}(\lambda|u) \frac{t^n}{n!}, \tag{2.9}$$

where $H_n^{(N)}(\lambda|u)$ are called the nth Apostol-type Frobenius–Euler numbers of order N.

From (2.8), we have

$$f^{(k)}(t) = \sum_{l=0}^{\infty} H_{l+k}(u) \frac{t^l}{l!}.$$

Therefore, by Theorem 2.1, (2.8), and (2), we obtain the following theorem.

Theorem 2.3. *For any positive integers n and N and any complex number λ, the following identity holds*

$$H_n^{(N+1)}(\lambda \mid u) = \left(\frac{u-1}{u}\right)^N \frac{1}{N!} \sum_{k=0}^{N} \mid s(N+1,k+1) \mid H_{n+k}(\lambda \mid u).$$

For Apostol–Euler numbers, we have a reduction formula.

Corollary 2.4. *Let n, N be a positive integers and λ a complex number, $\lambda \neq -1$. Then, the following identity holds*

$$E_n^{(N+1)}(0;\lambda) = \frac{2^N}{N!} \sum_{k=0}^{N} \mid s(N+1,k+1) \mid E_{n+k}(0;\lambda).$$

Using Lemma 1.1, for any nonnegative integer N, we obtain

$$E_n^{(N+1)}(x;-\lambda) = (-1)^{N+1} \frac{n!}{(n+N+1)!} B_{n+N+1}^{(N+1)}(x;\lambda) \quad \text{for} \quad \lambda \neq 1. \tag{2.10}$$

Now Theorem 2.3 and Corollary 2.4 yield

Corollary 2.5. *Let n and N be positive integers. Then*

$$B_{n+N+1}^{(N+1)}(0;\lambda) = (-1)^N (n+N+1) \binom{n+N}{N} \sum_{k=0}^{N} |s(N+1,k+1)| \frac{B_{n+k+1}(0;\lambda)}{n+k+1}. \tag{2.11}$$

From (2.8), we can derive the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty} H_n^{(N)}(\lambda|u)\frac{t^n}{n!} &= \underbrace{\left(\frac{1-u}{\lambda e^t-u}\right)\times\left(\frac{1-u}{\lambda e^t-u}\right)\times\cdots\times\left(\frac{1-u}{\lambda e^t-u}\right)}_{N-\text{ times}}\\ &=\left(\sum_{l_1=0}^{\infty} H_{l_1}(\lambda|u)\frac{t_1^l}{l_1!}\right)\times\cdots\times\left(\sum_{l_N=0}^{\infty} H_{l_N}(\lambda|u)\frac{t^{l_N}}{l_N!}\right)\\ &=\sum_{n=0}^{\infty}\left(\sum_{l_1+\cdots+l_N=0}\frac{H_{l_1}(\lambda|u)H_{l_2}(u)\cdots H_{l_N}(\lambda|u)n!}{l_1!l_1!\cdots l_N!}\right)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{l_1+\cdots+l_N=0}\binom{n}{l_1,\ldots,l_N!}H_{l_1}(\lambda|u)\cdots H_{l_N}(\lambda|u)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following corollary.

Corollary 2.6. *For $N\in\mathbb{N}$ and $n\in\mathbb{Z}_+$,*

$$\sum_{l_1+\cdots+l_N=n}\binom{n}{l_1,\ldots,l_N!}H_{l_1}(\lambda|u)H_{l_2}(\lambda|u)\cdots H_{l_N}(\lambda|u) = \left(\frac{u-1}{u}\right)^{N-1}\frac{1}{(N-1)!}\sum_{k=0}^{N-1}\mid s(N,k+1)\mid H_{n+k}(\lambda\mid u).$$

By Corollary 2.2, we obtain the next result.

Theorem 2.7. *Let n be a positive integer. Then*

$$H_n^{(N+1)}(x;\lambda\mid u) = \left(\frac{u-1}{u}\right)^N\frac{1}{N!}\sum_{j=0}^{N}\left(\sum_{k=j}^{N}(-x)^{k-j}\mid s(N+1,k+1)\mid\binom{k}{j}\right)H_{n+j}(x;\lambda\mid u).$$

Therefore, from the above theorem, we obtain a recurrence formula for the Apostol–Euler polynomials of higher order.

Corollary 2.8. *Let n be a positive integer. Then*

$$E_n^{(N+1)}(x;\lambda) = \frac{2^N}{N!}\sum_{j=0}^{N}\left(\sum_{k=j}^{N}\binom{k}{j}(-x)^{k-j}\mid s(N+1,k+1)\mid\right)E_{n+j}(x;\lambda).$$

From relation (2.10) and Corollary 2.8, we have also similar recurrence formula for the Apostol–Bernoulli polynomials of higher order.

Corollary 2.9.

$$B_{n+N+1}^{(N+1)}(x;\lambda) = (-1)^N(n+N+1)\binom{n+N}{N}\sum_{j=0}^{N}\left(\sum_{k=j}^{N}(-x)^{k-j}|s(N+1,k+1)|\right)\frac{B_{n+j+1}(x;\lambda)}{n+j+1}. \tag{2.13}$$

From Theorem 2.8, we deduce the following corollaries. For $\lambda = 1$ and $x = 0$, we have the following.

Corollary 2.10. *Let n be a positive integer. Then*

$$H_n^{(N+1)}(u) = \left(\frac{u-1}{u}\right)^N \frac{1}{N!} \sum_{k=0}^{N} \mid s(N+1, k+1) \mid H_{n+k}(u).$$

For $\lambda = 1$, we obtain a polynomial version.

Corollary 2.11. *Let n be a positive integer. Then*

$$H_n^{(N+1)}(x \mid u) = \left(\frac{u-1}{u}\right)^N \frac{1}{N!} \sum_{j=0}^{N} \left(\sum_{k=j}^{N} \binom{k}{j} (-x)^{k-j} \mid s(N+1, k+1) \mid \right) H_{n+j}(x \mid u).$$

From the above corollaries, we also obtain the following nonlinear recurrence formulas for the Euler numbers and polynomials:

Corollary 2.12. *Let n be a positive integer. Then*

$$E_n^{(N+1)}(0) = \left(\frac{u-1}{u}\right)^N \frac{1}{N!} \sum_{k=0}^{N} \mid s(N+1, k+1) \mid E_{n+k}(0),$$

$$E_n^{(N+1)}(x) = \frac{2^N}{N!} \sum_{j=0}^{N} \left(\sum_{k=j}^{N} \binom{k}{j} (-x)^{k-j} \mid s(N+1, k+1) \mid \right) E_{n+k}(x).$$

We also deduce the following nonlinear recurrence formulas for the Bernoulli numbers and polynomials.

Corollary 2.13. *Let n be a positive integer. Then*

$$B_{n+N+1}^{(N+1)} = (-1)^N (n+N+1) \binom{n+N}{N} \sum_{k=0}^{N} |s(N+1, k+1)| \frac{B_{n+k+1}}{n+k+1}.$$

The polynomial version of this corollary is given by

Corollary 2.14. *Let n be a positive integer. Then*

$$B_{n+N+1}^{(N+1)}(x) = (-1)^N (n+N+1) \binom{n+N}{N} \sum_{j=0}^{N} \left(\sum_{k=j}^{N} (-x)^{k-j} |s(N+1, k+1)| \right) \frac{B_{n+j+1}(x)}{n+j+1}.$$

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