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Freudenthal–Weil Theorem for Pro-Lie Groups

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Abstract. An analog of the Freudenthal–Weil theorem holds for the discontinuous homomorphisms of a connected pro-Lie group into a compact group if and only if the radical of the pro-Lie group is amenable.

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1. INTRODUCTION

The famous Freudenthal–Weil theorem claims that a connected locally compact group admitting a continuous embedding in a compact group is the direct product of a connected compact group and a (finite-dimensional) vector group [1]. For connected Lie groups, this theorem was extended to arbitrary (not necessarily continuous) embeddings of a connected Lie group in a compact group [2] (for some generalizations, see [3]). In [4], the Freudenthal–Weil theorem was extended to (not necessarily continuous) embeddings of connected locally compact groups in compact groups and to some embeddings of connected pro-Lie groups, whose theory is presented in detail in [5]. In the present paper, we obtain necessary and sufficient conditions for a (not necessarily continuous) embedding of a connected pro-Lie group in a compact group to be the direct product of a connected compact group and a (finite-dimensional) vector group.

2. PRELIMINARIES

For the facts we need concerning pro-Lie groups and pro-Lie algebras, see Section 2 of [4]. The notions we need are: a *projective system D* of topological groups and the projective limit of this system, a pro-Lie group G and the set of all normal subgroups N such that G/N is a finite-dimensional Lie group, a topological Lie algebra, a pro-Lie algebra, a topological group with a Lie algebra and related notions, a pro-Lie group which is a projective limit of a system of solvable Lie groups, a prosolvable Lie algebra (a projective limit of a system of solvable finite-dimensional Lie algebras), the radical of a pro-Lie algebra, a semisimple Lie algebra, a closed subgroup of a pro-Lie group, the Levi–Mal'tsev decomposition for pro-Lie algebras, the radical of a pro-Lie group, a prosolvable Lie group, and the prosolvability of the radical of a pro-Lie group.

3. MAIN RESULTS

Theorem 1. *Let G be a pro-Lie group and let R be the radical of G. Let π be a homomorphic (not necessarily continuous) embedding of G in a compact topological group C. The following conditions are equivalent, namely,*

(1) *the group G is the direct product of a compact group K and the additive group of a weakly complete vector space V,*

(2) *the group R is amenable.*

Before proving the theorem, note that a connected locally compact group is a pro-Lie group whose radical is solvable, and thus amenable. Therefore, by the theorem, a (not necessarily continuous) homomorphism π into a compact group exists if and only if the group G is the direct product of a compact group K and the additive group of a finite-dimensional vector space V. Thus, the following corollary holds, which extends the original Freudenthal–Weil theorem to discontinuous embeddings of connected locally compact groups in compact groups.

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Corollary 1. *Let G be a connected locally compact group admitting a (not necessarily continuous) embedding in a compact group. Then G is the direct product of a torus group and a (finite-dimensional) vector group.*

Proof of Theorem 1. (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (1). Let π be an arbitrary continuous irreducible unitary representation of C, let ρ be the homomorphic embedding of G in C, and let $\theta = \pi \circ \rho$ be the corresponding unitary representation of G.

Since the representation space of θ is finite-dimensional, it follows that $\theta(R)$ is a linear group.

By assumption, R is amenable. It follows that $\theta(R)$ is amenable.

Hence, $\theta(R)$ does not contain a subgroup isomorphic to the free group with two generators F_2. According to the Tits alternative [6], this means that $\theta(R)$ is solvable-by-finite.

It follows from [5, Proposition 9.53] that a connected pro-Lie group has no nontrivial subgroups of finite index. Therefore, $\theta(R)$ has no nontrivial subgroups of finite index (otherwise the connected pro-Lie group R has such a subgroup), and thus $\theta(R)$ is solvable. The closure M of $\theta(R)$ is also solvable.

For this reason, M is a Lie group in the representation space of θ, and M is compact as a closed subgroup of $\pi(C)$. Therefore, the connected component of M is of finite index in M.

Applying [5, Proposition 9.53] again, we see that the component coincides with M, and therefore, M is a compact connected solvable subgroup of the compact group $\pi(C)$. Since every subgroup of this kind is commutative, it follows that $\theta(R)$ is commutative.

Thus, the image of $\rho(R)$ in every continuous irreducible unitary representation of C, which implies that $\rho(R)$ is commutative. Since ρ is an embedding, it follows that R is commutative.

The remaining part of the proof is similar to the end of the proof of the corresponding assertion in [4]; we present the corresponding consideration for the sake of completeness of our presentation.

Consider an arbitrary one-parameter subgroup of G corresponding to an element ξ of $\mathfrak{s}$. This subgroup is contained in the minimal analytic subgroup $A(\mathfrak{s})$ of G ([4], Definition 9.9).

Consider the image L of the above one-parameter subgroup in the quotient group G/R. Since R is commutative, the complete preimage K of L in G is a solvable subgroup of G; hence, the closure of this subgroup is also solvable. Since G can be embedded in a compact group, we can prove as above that K is commutative.

Therefore, every one-parameter analytic subgroup of $A(\mathfrak{s})$ commutes with every element of R, and thus $A(\mathfrak{s})$ and its closure in G commute with R ([4], Proposition 9.10(iii)).

Since the closure of $A(\mathfrak{s})$ and R generate G ([4], Proposition 9.10(iii)), it follows that R is central in G. This implies that the Lie subalgebra $\mathfrak{s}$ is defined uniquely as the closure of the commutator subalgebra of the Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ ([4], Theorem 9.26 and Theorem 10.29(i)), and therefore, G is reductive ([4], Definition 10.27).

Recall that, by Theorem 10.48 of [4], the following statements are equivalent for a connected pro-Lie group G:

(1) G is reductive,

(2) $\mathfrak{g}$ is reductive,

(3) $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \dot{\mathfrak{g}}$ for a unique semisimple pro-Lie algebra $\dot{\mathfrak{g}}$ obtained as the closed commutator subalgebra of $\mathfrak{g}$ and the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$,

(4) $G = Z(G)S$ for the minimal analytic subgroup $S = A(\dot{\mathfrak{g}})$, and $\dot{\mathfrak{g}}$ is semisimple (and therefore G is semisimple). Here the connected component of the identity of the center $Z(G)$ of G is R (see [4], Definition 10.27).

By [4], Theorem 10.29(iii)(c), the group S is 'sandwiched' between two products via two morphisms

$$\prod_{j\in J} S_j \to G \to \prod_{j\in J} S_j/Z(S_j)$$

whose composition is just the quotient morphism obtained by passing to the quotient $S_j \to S_j/Z(S_j)$ in each simple factor S_j.

Since G admits an embedding in a compact group, it follows that S_j also admits such an embedding. This implies that S_j is a compact group, because a noncompact semisimple group admits only one finite-dimensional irreducible (automatically continuous, see [5] and [6]) unitary representation, namely, the identity representation.

Thus, all subgroups S_j in the above decomposition are compact. Therefore, the group S is compact. In particular, S is closed in G.

Let us use the notions of a compactly embedded Lie subalgebra and a procompact pro-Lie algebra as a pro-Lie algebra which is compactly embedded into itself (see [4], Definitions 12.1 and 12.10).

By [4], Theorem 12.12, a pro-Lie algebra $\mathfrak{g}$ is procompact if and only if $\mathfrak{g}$ is a direct product of its center and its commutator algebra, which is a product of simple compact Lie algebras.

The Lie subalgebra $\mathfrak{s}$ is compactly embedded in $\mathfrak{g}$ by [4], Lemma 12.39. Thus, the pro-Lie algebra $\mathfrak{g}$ is procompact by [4], Proposition 12.14.

It remains to apply Theorem 12.48 of [4] which claims that G contains a closed weakly complete central vector subgroup V and a maximal compact subgroup C which is characteristic, such that the function

$$\mu\colon V \times C \to G, \qquad \mu(v, c) = vc, \quad v \in V, \quad c \in C,$$

is an isomorphism of topological groups, as was to be proved.

Theorem 2. *For a connected pro-Lie group G with amenable radical, the following conditions are equivalent.*

(i) *G admits a (not necessarily continuous) embedding in a compact group such that the restriction of this embedding to the radical of G is finally continuous.*

(ii) *G admits a continuous embedding in a compact group.*

(iii) *The Bohr compactification morphism for G is injective.*

(iv) *G contains a closed weakly complete central vector subgroup V and a maximal compact subgroup C, which is characteristic, such that the function $\mu\colon V \times C \to G$, $\mu(v, c) = vc$, is an isomorphism of topological groups.*

Proof. (i)⇔(iv) is Theorem 1, (ii)⇔(iv) follows from Theorem 12.48 of [5], and (iii)⇔(iv) is immediate from Corollary 12.50 of [5].

Remark 1. If a finite-dimensional representation π of a pro-Lie group G is continuous, then the argument used in the proof of Theorem 1 shows that π is trivial on some $N \in \mathcal{N}$, and hence π can be regarded as a representation of the Lie quotient group G/N.

The corresponding fact fails to hold for discontinuous representations. For the corresponding observation, see the remark in [4], which uses [7].

Remark 2. The free group with two generators, F_2, is obviously countably solvable.

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