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Uniformization and Index of Elliptic Operators Associated with Diffeomorphisms of a Manifold

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Abstract. We consider the index problem for a wide class of nonlocal elliptic operators on a smooth closed manifold, namely, differential operators with shifts induced by the action of a (not necessarily periodic) isometric diffeomorphism. The key to the solution is the method of uniformization. To the nonlocal problem we assign a pseudodifferential operator, with the same index, acting on the sections of an infinite-dimensional vector bundle on a compact manifold. We then determine the index in terms of topological invariants of the symbol, using the Atiyah–Singer index theorem.

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INTRODUCTION

Let M be a closed smooth Riemannian manifold and $g\colon M \to M$ a smooth isometric diffeomorphism. The powers of g generate an action of the group $\mathbb{Z}$ of integers on M. In this paper, we consider *differential operators with shifts*, i.e., operators of the form

$$D = \sum_k D_k T^k \colon C^\infty(M) \longrightarrow C^\infty(M), \tag{0.1}$$

where the sum is finite, the D_k are differential operators, and $T^k u(x) = u(g^k(x))$.

We introduce an ellipticity condition, which implies the Fredholm property for operators of this type, and compute the index. The method we use is pseudodifferential (actually even differential) uniformization. The idea is to study, instead of D, an elliptic differential operator with the same index on a suitably chosen manifold.

This procedure consists of two steps. We first replace the original manifold M by $M\times\mathbb{R}$ endowed with the diagonal action of $\mathbb{Z}$, which is free and proper (cf. [4] for ideas similar in spirit). On $M\times\mathbb{R}$ we define an elliptic operator as the external product of D with a special operator A of index one on the real line (actually, A is the annihilation operator of quantum mechanics).

In the second step, we interpret this operator on $M \times \mathbb{R}$ as a differential operator on sections of a Hilbert space bundle with fiber $l^2(\mathbb{Z})$ over the smooth quotient $(M \times \mathbb{R})/\mathbb{Z}$. The point of this construction is that the index is preserved while, for the resulting operator, which we call the *differential uniformization* of D, the action of g is simply a shift in the fiber, which is much easier to treat. We compute the symbol of the resulting operator, establish the Fredholm property (finiteness theorem), and express the index in terms of the symbol of D and topological invariants of the manifold.

Note that the index problem in a related situation (even for the case when, instead of $\mathbb{Z}$, we take an arbitrary discrete group of polynomial growth) was studied in [10]. The solution given there did not use the idea of uniformization and was rather complicated, while the method presented here is quite simple and natural.

One more remark. Of course, most of our constructions and proofs could be done at the symbolic level in the framework of noncommutative geometry [5, 4, 7]. For example, the differential uniformization, which we carry out for operators (see Theorem 1.3), corresponds in terms of *symbols* exactly to the product in KK-theory of Kasparov [7, 16]. However, in this paper, we give direct constructions and proofs in terms of *operators*, which, in our opinion, enhances the clarity of the constructions.

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1. STATEMENT OF THE MAIN RESULTS

1. Elliptic operators

Let M be a smooth Riemannian manifold and $g\colon M \to M$ a metric preserving diffeomorphism. We consider an operator of the form

$$D = \sum_k D_k T^k\colon C^\infty(M) \longrightarrow C^\infty(M) \tag{1.1}$$

with differential operators D_k and the shift operator T defined by $Tu(x) = u(g(x))$. While the operators D_k can have any order *a priori*, our construction is simplified if the order is equal to one. Hence, from now on, we *assume that the D_k are first-order differential operators.* The modifications for higher-order operators and for operators acting on vector bundles are left to the reader.

Definition 1.1. The *symbol* $\sigma(D)$ of D is the function on $T^*M \setminus 0$ taking values in operators on $l^2(\mathbb{Z})$ and defined by

$$(\sigma(D)(x,\xi))w(n) = \sum_k \sigma(D_k)(\partial g^n(x,\xi))\mathcal{T}^k w(n), \qquad w \in l^2(\mathbb{Z}), n \in \mathbb{Z}, \tag{1.2}$$

where $\mathcal{T}$ is the shift operator on $l^2(\mathbb{Z})$ given by $(\mathcal{T}w)(n) = w(n+1)$, $\sigma(D_k)$ is the principal symbol of D_k, and $\partial g = ({}^t dg)^{-1} : T^*M \to T^*M$ is the codifferential of g.

We say that D is *elliptic* if the operator in (1.2) is invertible for all $(x,\xi) \in T^*M \setminus 0$.

The elliptic operators are Fredholm (see [1, 2, 10]).

As we have already mentioned in the introduction, the aim of this paper is to obtain an index formula for elliptic operators of the form (1.1). The main idea used here is the pseudodifferential uniformization of an operator, i.e., a reduction of the operator to an elliptic pseudodifferential operator (in general, acting on sections of Hilbert bundles).

2. Uniformization of operators

The uniformization is carried out in two steps.

A. Reduction to a proper action, namely, the passage to the infinite cylinder. On the real line with coordinate t, consider the operator $A = \partial/\partial t + t\colon \mathcal{H}^s(\mathbb{R}) \to \mathcal{H}^{s-1}(\mathbb{R})$ (see Appendix for the definition of $\mathcal{H}^s(\mathbb{R})$ and more details on operators of this type). It is elliptic and has index one. Indeed, the cokernel is trivial, while the kernel is generated by the function $e^{-t^2/2}$.

We next extend the isometry $g\colon M \to M$ to an isometry $\widetilde{g}\colon M \times \mathbb{R} \to M \times \mathbb{R}$ of the infinite cylinder $M \times \mathbb{R}$ with local coordinates x, t by setting $\widetilde{g}(x,t) = (g(x), t+1)$. As above, we obtain an action of $\mathbb{Z}$ and a shift operator $\widetilde{T}$ by

$$(\widetilde{T}u)(x,t) = u(g(x), t+1). \tag{1.3}$$

Using the operators D_k of equation (1.1), we define the operator

$$\widetilde{D} = \sum_k D_k \widetilde{T}^k\colon \mathcal{H}^s(M \times \mathbb{R}) \longrightarrow \mathcal{H}^{s-1}(M \times \mathbb{R})$$

and consider the external product of $\widetilde{D}$ and A,

$$\widetilde{D} \# A = \begin{pmatrix} \widetilde{D} & A \\ -A^* & \widetilde{D}^* \end{pmatrix}\colon \mathcal{H}^s(M \times \mathbb{R}, \mathbb{C}^2) \longrightarrow \mathcal{H}^{s-1}(M \times \mathbb{R}, \mathbb{C}^2). \tag{1.4}$$

Here we take the adjoint with respect to the inner product in $L^2(M \times \mathbb{R})$.

B. Reduction to the (smooth) orbit space. The isometry $\widetilde{g}$ defines a free proper action of the group $\mathbb{Z}$ on the cylinder. Hence, the corresponding orbit space is a smooth manifold. Let us consider functions on the cylinder as functions on the orbit space ranging in functions on the fiber of the projection[1] $M \times \mathbb{R} \longrightarrow (M \times \mathbb{R})/\mathbb{Z} = M_{\mathbb{Z}}$. The point of this representation is that the shift operator $\widetilde{T}$ on $M \times \mathbb{R}$ becomes an operator of multiplication, i.e., a local operator on the space $M_{\mathbb{Z}}$.

For L^2-spaces and Sobolev spaces, this representation of functions on $M \times \mathbb{R}$ in terms of functions on $M_{\mathbb{Z}}$ is described by the following proposition.

[1] The space $M_{\mathbb{Z}}$ is also called the *homotopy quotient* of M by the action of $\mathbb{Z}$.

Proposition 1.2. *Let $\mathcal{E}$ be the vector bundle $\mathcal{E} = (M \times \mathbb{R} \times l^2(\mathbb{Z}))/\mathbb{Z}$ over $M_\mathbb{Z}$ with fiber $l^2(\mathbb{Z})$. Here $n \in \mathbb{Z}$ acts by $(x, t, u) \mapsto (g^n(x), t+n, \mathcal{T}^{-n}u)$.*

(1) *We have the isomorphism*

$$\begin{array}{rcl} I\colon L^2(M\times\mathbb{R}) & \longrightarrow & L^2(M_\mathbb{Z}, \mathcal{E}), \\ (I\varphi)(x,t,n) & = & \varphi(g^n(x), t+n). \end{array} \tag{1.5}$$

(2) *The operator I in* (1.5) *induces an isomorphism of the Sobolev spaces*

$$I\colon \mathcal{H}^s(M\times\mathbb{R}) \to H^s(M_\mathbb{Z}, \mathcal{E}), \tag{1.6}$$

where the Sobolev space $H^s(M_\mathbb{Z}, \mathcal{E})$ of sections of the infinite-dimensional bundle $\mathcal{E}$ over $M_\mathbb{Z}$ is endowed with the norm

$$\|u\|^2 = \sum_n \|(1+\Delta_{x,t}+n^2)^{s/2}u(x,t,n)\|^2_{L^2(M\times[0,1])}. \tag{1.7}$$

Proof. A direct computation shows that the inverse mapping is equal to

$$(I^{-1}u)(x,t) = u(g^{-[t]}(x), \{t\}, [t]),$$

where $t = [t] + \{t\}$ is the decomposition into the integer and fractional part of t. This proves the isomorphism of the L^2-spaces. The statement for the Sobolev spaces is proved in Sec. 2.

Theorem 1.3 [Uniformization theorem]**.**

(1) *The operator $\mathcal{D} = I(\widetilde{D}\#A)I^{-1}$, which closes the commutative diagram*

$$\begin{array}{ccc} \mathcal{H}^s(M\times\mathbb{R}, \mathbb{C}^2) & \xrightarrow{\widetilde{D}\#A} & \mathcal{H}^{s-1}(M\times\mathbb{R}, \mathbb{C}^2) \\ {\scriptstyle I}\big\downarrow & & {\scriptstyle I}\big\downarrow \\ H^s(M_\mathbb{Z}, \mathcal{E}\otimes\mathbb{C}^2) & \xrightarrow{\mathcal{D}} & H^{s-1}(M_\mathbb{Z}, \mathcal{E}\otimes\mathbb{C}^2), \end{array} \tag{1.8}$$

is a differential operator. Its symbol in the sense of [11]*, Sec.* 3.3*, is the following* (*nonhomogeneous*) *operator-valued function on $T^*M_\mathbb{Z}$,*

$$\sigma(\mathcal{D})(x,\xi,t,\tau) = (\sigma(D)(x,\xi))\#(i\tau+t+n)\colon l^2(\mathbb{Z}, \mu_{\xi,\tau,s})\otimes\mathbb{C}^2 \longrightarrow l^2(\mathbb{Z}, \mu_{\xi,\tau,s-1})\otimes\mathbb{C}^2. \tag{1.9}$$

Here, $i\tau + t + n$ acts by multiplication, and $l^2(\mathbb{Z}, \mu_{\xi,\tau,s})$ has the norm

$$\|w\|^2_{\xi,\tau,s} = \sum_n (1+|\xi|^2+\tau^2+n^2)^s|w(n)|^2. \tag{1.10}$$

(2) *If D is elliptic, then the differential operator $\mathcal{D}$ is also elliptic, i.e., the operator in* (1.9) *is invertible for large $|(\xi,\tau)|$, say, for $|(\xi,\tau)| \geqslant R$, and the norm of the inverse is uniformly bounded.*

(3) *Under the assumptions of (b), the following equality holds:*

$$\operatorname{ind} D = \operatorname{ind}\mathcal{D}. \tag{1.11}$$

Theorem 1.3 will be proven in Secs. 2 and 3. We first establish assertions (1)–(2) and then show equality (3) for the indices.

3. Index formula

We denote by $S^*_R M_\mathbb{Z}$ the cosphere bundle of $M_\mathbb{Z}$ of radius R with R as in Theorem 1.3(2) and consider the space $\Lambda(S^*_R M_\mathbb{Z}, \operatorname{End}\mathcal{E})$ of differential forms on $S^*_R M_\mathbb{Z}$ ranging in endomorphisms of the Hilbert bundle $\mathcal{E}$. This space is endowed with the differential d (the differential is well defined, since $\mathcal{E}$ is flat). Taking the fiberwise trace of operators in $l^2(\mathbb{Z})$ gives the (partially defined) mapping $\operatorname{tr}\colon \Lambda(S^*_R M_\mathbb{Z}, \operatorname{End}\mathcal{E}) \longrightarrow \Lambda(S^*_R M_\mathbb{Z})$. We endow $M_\mathbb{Z}$ with the metric $h + dt^2$, where h is the g-invariant metric on M. Since g is an isometry, this metric is well defined, and we have the equality $\operatorname{Td}(T^*_\mathbb{C}M) = \operatorname{Td}(T^*_\mathbb{C}M_\mathbb{Z})$ of differential forms, which represent the Todd classes of the complexification of the cotangent bundles of M and $M_\mathbb{Z}$, respectively. With this identification, we introduce the following definition.

Definition 1.4. The *topological index* of an elliptic operator $\mathcal{D}$ is the number

$$\operatorname{ind}_t\mathcal{D} = \sum_j C_j \int_{S^*_R M_\mathbb{Z}} \operatorname{tr}\left[(\sigma(\mathcal{D})^{-1}d\sigma(\mathcal{D}))^{2j-1}\operatorname{Td}(T^*_\mathbb{C}M)\right], \tag{1.12}$$

where $C_j = (j-1)!/[(2\pi i)^j(2j-1)!]$.

The properties of the topological index are studied in Sec. 4.2 below.

Theorem 1.5. *For the elliptic operator $\mathcal{D}$ in* (1.8) *with symbol $\sigma(\mathcal{D})$, the topological index* (1.12) *is well defined provided that $\dim M > 1$, and then the index formula holds:*

$$\operatorname{ind}\mathcal{D} = \operatorname{ind}_t \mathcal{D}. \tag{1.13}$$

Equalities (1.11) and (1.13) give the index formula for the original operator D. Theorem 1.5 is proven in Sec. 4.

2. REDUCTION TO THE ORBIT SPACE

1. Isomorphism of Sobolev spaces

To see that the mapping (1.6) is an isomorphism, we use the two lemmas presented below.

Let γ be the one-dimensional complex vector bundle over $M_{\mathbb{Z}} \times \mathbb{S}^1$ defined by the equality $C^\infty(M_{\mathbb{Z}} \times \mathbb{S}^1, \gamma) = \{v \in C^\infty(M \times \mathbb{R} \times \mathbb{S}^1) \mid v(g(x), t+1, \varphi) = v(x,t,\varphi)e^{-i\varphi}\}$.

Lemma 2.1. *The mapping*

$$\begin{array}{rcc} K\colon H^s(M_{\mathbb{Z}}, \mathcal{E}) & \longrightarrow & H^s(M_{\mathbb{Z}} \times \mathbb{S}^1, \gamma) \\ \{u(x,t,n)\} & \longmapsto & \mathcal{F}_{n\to\varphi} u(x,t,\cdot) = \frac{1}{\sqrt{2\pi}} \sum_n u(x,t,n)e^{in\varphi}, \end{array}$$

which is just the fiberwise Fourier transform (series), is an isomorphism for all s.

Proof. This is an immediate consequence of the fact that the Fourier transform $\mathcal{F}_{n\to\varphi}$ takes the norm (1.7) defining $H^s(M_{\mathbb{Z}}, \mathcal{E})$ to the norm

$$\|u\|_s = \left\| \left(1 + \Delta_{x,t} - \partial^2/\partial\varphi^2\right)^{s/2} u(x,t,\varphi)\right\|_{L^2(M_{\mathbb{Z}} \times \mathbb{S}^1, \gamma)} \tag{2.1}$$

defining $H^s(M_{\mathbb{Z}} \times \mathbb{S}^1, \gamma)$ if we consider this space as the space of functions on $M_{\mathbb{Z}}$ taking values in functions on $\mathbb{S}^1$, cf. [10], Sec. 12.2.1.

Lemma 2.2. *The mapping $KI\colon \mathcal{H}^s(M \times \mathbb{R}) \longrightarrow H^s(M_{\mathbb{Z}} \times \mathbb{S}^1, \gamma)$ is an isomorphism for all s.*

Proof. First note that KI takes rapidly decreasing functions on $M \times \mathbb{R}$ to smooth sections of γ. The assertion then follows from the fact that this mapping takes the operator $\Delta_x + t^2 - \partial^2/\partial t^2$, which is the base for the norm in $\mathcal{H}^s(M \times \mathbb{R})$, to the Laplacian (modulo lower order terms) $\Delta_x - \partial^2/\partial\varphi^2 - \partial^2/\partial t^2$ that induces the norm in $H^s(M_{\mathbb{Z}} \times \mathbb{S}^1, \gamma)$.

So, $I = K^{-1}(KI)$ is an isomorphism, as the composition of two isomorphisms.

2. Computation of the operator-valued symbol

Let

$$B = \sum_k B_k\big(x, -i\partial/\partial x, t, -i\partial/\partial t\big)\widetilde{T}^k$$

be a differential operator with shifts on $M \times \mathbb{R}$. We have

$$\begin{aligned} \{u(x,t,n)\} &\overset{I^{-1}}{\longmapsto} u(g^{-[t]}(x), \{t\}, [t]) \\ &\overset{B}{\longmapsto} \sum_k B_k\big(x, -i\partial/\partial x, t, -i\partial/\partial t\big) u(g^{-[t]}(x), \{t\}, [t+k]) \\ &\overset{I}{\longmapsto} \Big\{ \sum_k B_k\big(g^n(x), -i\partial/\partial(g^n(x)), t+n, -i\partial/\partial t\big)\mathcal{T}^k u(x,t,n)\Big\}. \end{aligned}$$

Thus, the operator IBI^{-1} is indeed a differential operator on $M_{\mathbb{Z}}$ acting on the sections of the infinite-dimensional bundle $\mathcal{E}$. Its symbol in the sense of [11], Sec. 3.3, is equal to

$$\sigma(IBI^{-1})(x,\xi,t,\tau) = \sum_k \sigma(B_k)(\partial g^n(x,\xi), t+n, \tau)\mathcal{T}^k \colon l^2(\mathbb{Z}, \mu_{\xi,\tau,s}) \to l^2(\mathbb{Z}, \mu_{\xi,\tau,s-1}).$$

An easy computation shows that it smoothly depends on x, ξ, t, τ, satisfies the necessary estimates, and has the compact variation property. Namely, the derivatives

$$\partial\sigma(IBI^{-1})/\partial\xi, \partial\sigma(IBI^{-1})/\partial\tau\colon\ l^2(\mathbb{Z}, \mu_{\xi,\tau,s}) \to l^2(\mathbb{Z}, \mu_{\xi,\tau,s-1})$$

are compact operators for all x, ξ, t, τ.

For $B = \widetilde{D}\#A$ we obtain the symbol (1.9) and its properties stated in Theorem 1.3 (1).

3. Ellipticity

Lemma 2.3. *Let D be elliptic. Then the operator $\mathcal{D}$ is also elliptic, i.e., its symbol* (1.9) *is invertible provided that $|\xi|^2+\tau^2$ is large enough and the norm of the inverse is uniformly bounded.*

Proof. Let us first reduce our operator to the spaces $l^2(\mathbb{Z})$. Consider the commutative diagram

$$\begin{array}{ccc} l^2(\mathbb{Z},\mu_{\xi,\tau,s})\otimes\mathbb{C}^2 & \xrightarrow{\sigma(\mathcal{D})(x,\xi,t,\tau)} & l^2(\mathbb{Z},\mu_{\xi,\tau,s-1})\otimes\mathbb{C}^2 \\ \delta^s\Big\downarrow & & \delta^{s-1}\Big\downarrow \\ l^2(\mathbb{Z})\otimes\mathbb{C}^2 & \xrightarrow{\sigma_s(x,\xi,t,\tau)} & l^2(\mathbb{Z})\otimes\mathbb{C}^2, \end{array}$$

where $\delta=(1+|\xi|^2+\tau^2+n^2)^{1/2}$. In this diagram, the vertical mappings are isomorphisms, and the operator $\sigma_s(x,\xi,t,\tau)$ is defined by $\sigma_s(x,\xi,t,\tau)=\delta^{s-1}\sigma(\mathcal{D})(x,\xi,t,\tau)\delta^{-s}$. We claim that the operator $\sigma_s(x,\xi,t,\tau)\colon l^2(\mathbb{Z})\otimes\mathbb{C}^2\longrightarrow l^2(\mathbb{Z})\otimes\mathbb{C}^2$ is invertible for large $|\xi,\tau|$ and the norm of the inverse operator is uniformly bounded. To construct the inverse operator, we consider the following expression (the variables x,ξ,t,τ are omitted for brevity):

$$\sigma_s=\delta^{s-1}\big(\sigma(D)\#\sigma(A)\big)\delta^{-s}=\big(\delta^{s-1}\sigma(D)\delta^{-s}\big)\#\delta^{-1}\sigma(A)=(\sigma(D)\delta^{-1}\#\delta^{-1}\sigma(A))+Q. \qquad (2.2)$$

Here and below, $Q\colon l^2(\mathbb{Z})\otimes\mathbb{C}^2\to l^2(\mathbb{Z})\otimes\mathbb{C}^2$ stands for operator families of order -1 in the scale $l^2(\mathbb{Z},\mu_{\xi,\tau,s})$. The first equality in (2.2) is just the definition of the symbol σ_s; the second one holds, because A has no shifts, and the third one is valid because the commutator $[\mathcal{T},\delta^s]$ is an operator of order $\leqslant s-1$,

$$[\mathcal{T},\delta^s]=(\mathcal{T}\delta^s\mathcal{T}^{-1}-\delta^s)\mathcal{T}=[(1+|\xi|^2+\tau^2+(n+1)^2)^{s/2}-(1+|\xi|^2+\tau^2+n^2)^{s/2}]\mathcal{T}$$
$$=\delta^s\Big(\Big(1+\frac{2n+1}{1+|\xi|^2+\tau^2+n^2}\Big)^{s/2}-1\Big)\mathcal{T}=\delta^{s-1}P,$$

where P is a uniformly bounded operator family. Here we have used a Taylor expansion of $(1+\alpha)^{s/2}$ at $\alpha=0$ for the last equality. With a slight modification of the argument, we can also obtain the form $\delta^{-1}\sigma(D)\#\sigma(A)\delta^{-1}+Q$ on the right-hand side of (2.2).

To construct a left inverse of σ_s, consider the operator $\sigma_s^*\sigma_s$. A direct computation using (2.2) shows that

$$\sigma_s^*\sigma_s=\begin{pmatrix}\delta^{-1}(\sigma(D)^*\sigma(D)+(n^2+\tau^2))\delta^{-1} & 0\\ 0 & \delta^{-1}(\sigma(D)\sigma(D)^*+(n^2+\tau^2))\delta^{-1}\end{pmatrix}+Q.$$

Let us prove that the self-adjoint operators on the diagonal of this matrix are positive definite. Indeed, consider, for instance, the operator in the left upper corner. Using the ellipticity of D, we have

$$\langle\delta^{-1}(\sigma(D)^*\sigma(D)+n^2+\tau^2)\delta^{-1}w,w\rangle=\langle(\sigma(D)^*\sigma(D)+n^2+\tau^2)\delta^{-1}w,\delta^{-1}w\rangle$$
$$\geqslant c|\xi|^2\langle\delta^{-1}w,\delta^{-1}w\rangle+\langle(n^2+\tau^2)\delta^{-1}w,\delta^{-1}w\rangle\geqslant c'\langle\delta^2\delta^{-1}w,\delta^{-1}w\rangle=c'\langle w,w\rangle$$

with suitably small $c,c'>0$. This implies that $\delta^{-1}(\sigma(D)^*\sigma(D)+n^2+\tau^2)\delta^{-1}$ is positive definite, and the operator $\sigma_s^*\sigma_s$ is the sum of an invertible family with uniformly bounded inverse and an operator family which tends to zero as $(\xi,\tau)\to\infty$. Hence, the family $\sigma_s^*\sigma_s$ is uniformly invertible, and $(\sigma_s^*\sigma_s)^{-1}\sigma_s^*$ is a left inverse of σ_s.

To construct a right inverse for σ_s, we consider the family $\sigma_s\sigma_s^*$. A similar reasoning shows that it is also uniformly invertible. Therefore, a right inverse of σ_s is given by $\sigma_s^*(\sigma_s\sigma_s^*)^{-1}$.

The proof of the lemma is complete.

3. EQUALITY OF THE INDICES

Let us prove equality (1.11). Since I is an isomorphism, the equality of the indices is a corollary of the following proposition.

Proposition 3.1. *The operator $\widetilde{D}\#A$ in* (1.4) *is Fredholm for all s, its index does not depend on s, and we have the equality*

$$\operatorname{ind}(\widetilde{D}\#A)=\operatorname{ind}D. \qquad (3.1)$$

Proof. Step 1. The operator $\widetilde{D}\#A$ is an operator with shifts on the cylinder; its symbol is

$$\sigma(\widetilde{D}\#A)(x,\xi,t,\tau)=\sigma(D)(x,\xi)\#\sigma(A)(t,\tau): l^2(\mathbb{Z},\mathbb{C}^2)\longrightarrow l^2(\mathbb{Z},\mathbb{C}^2),$$

cf. the Appendix. Since D and A are elliptic, $\sigma(D)(x,\xi)$ is invertible whenever $\xi\neq 0$, and $\sigma(A)(t,\tau)$ is invertible whenever $t^2+\tau^2\neq 0$. Hence, the external product of these symbols is invertible whenever $|\xi|^2+t^2+\tau^2\neq 0$. Therefore, $\widetilde{D}\#A$ is elliptic, and thus a Fredholm operator.

Step 2. Consider the operator family

$$B_\varepsilon=\Big(\sum_k D_k\widetilde{T}_\varepsilon^k\Big)\#A:\mathcal{H}^s(M\times\mathbb{R},\mathbb{C}^2)\longrightarrow\mathcal{H}^{s-1}(M\times\mathbb{R},\mathbb{C}^2),\quad 0\leqslant\varepsilon\leqslant 1,$$

where $\widetilde{T}_\varepsilon u(x,t)=u(g(x),t+\varepsilon)$. By construction, $B_1=\widetilde{D}\#A$. The ellipticity condition in Definition 5.1 requires the invertibility of the symbol in the crossed product algebra and is therefore independent of ε, cf. Remark 5.4.

So it is clear that the operators B_ε are elliptic and hence are Fredholm operators for $\varepsilon\in[0,1]$. Since D and A commute, the technique of external products, cf. Sec. 9 in [3], shows that

$$\ker B_0=\ker B_0^*B_0=\ker D\otimes\ker A\simeq\ker D$$

and, similarly,

$$\operatorname{coker} B_0=\ker B_0B_0^*=\ker D^*\otimes\ker A\simeq\operatorname{coker} D.$$

Step 3. To prove equality (1.11), it remains to show that the index of B_ε does not depend on ε.

This is not trivial, because the mapping $\varepsilon\mapsto B_\varepsilon$ is not continuous in the operator norm. For the proof, we rely on a Hörmander theorem on the homotopy invariance of the index (see Theorem 19.1.10 in [8]). The cited result guarantees that the index does not change, provided that

(1) the family B_ε consists of Fredholm operators and the family of almost inverses B_ε^{-1} is strongly continuous in ε;

(2) the families

$$K_{1,\varepsilon}=1-B_\varepsilon B_\varepsilon^{-1}\quad\text{and}\quad K_{2,\varepsilon}=1-B_\varepsilon^{-1}B_\varepsilon \tag{3.2}$$

are uniformly compact, which means that the closures of the sets $\bigcup_{\varepsilon\in[0,1]}K_{1,\varepsilon}\mathbb{B}$ and $\bigcup_{\varepsilon\in[0,1]}K_{2,\varepsilon}\mathbb{B}$, where $\mathbb{B}\subset L^2(M\times\mathbb{R})$ is the unit ball, are compact.

Let us show that these two conditions are satisfied in our situation. Write $B_\varepsilon=\sum\widehat{b}_k\widetilde{T}_\varepsilon^k$, and denote the principal symbol of $\widehat{b}_k$ in the sense of the Appendix by b_k. While the symbol $\sigma(B_\varepsilon)$ defined in Remark 5.2 is in general an element of the crossed product algebra $\mathcal{B}\rtimes\mathbb{Z}$, where $\mathcal{B}$ consists of the continuous functions on the sphere $\{|\xi|^2+t^2+\tau^2=1\}$, the symbol lies in the case in question even in the subalgebra $\mathcal{A}\rtimes\mathbb{Z}$ of sequences with rapidly decreasing coefficients in the algebra $\mathcal{A}$ of all smooth functions on the sphere $\{|\xi|^2+t^2+\tau^2=1\}$; recall that the sum is finite and the $\widehat{b}_k$ are differential operators. According to a result of Schweitzer [15], this algebra is inverse closed. Hence there is a rapidly decreasing sequence $\{a_k\}_k\subset\mathcal{A}$ such that

$$\sigma(B_\varepsilon)^{-1}=\sum_k a_k\mathcal{T}^k.$$

We now choose a sequence $\{\widehat{a}_k\}_k$ of operators all of whose seminorms are rapidly decreasing in k. Let

$$B_\varepsilon^{-1}=\sum\widehat{a}_kT_\varepsilon^k.$$

A direct computation yields the estimate $\|\widetilde{T}_\varepsilon^k\|\leqslant C(1+|k|)^{|s|}$ for the norm of $\widetilde{T}_\varepsilon^k$ on $\mathcal{H}^s(M\times\mathbb{R})$. Hence the series for B_ε^{-1} converges. Moreover, the strong continuity of $\widetilde{T}_\varepsilon$ and the rapid decay of the $\widehat{a}_k$ imply the strong continuity of $\varepsilon\mapsto B_\varepsilon^{-1}$.

It remains to consider the families (3.2). Let us first consider $K_{1,\varepsilon}$. We infer from Cauchy's product formula that

$$1-B_\varepsilon B_\varepsilon^{-1}=1-\sum_{m+k=0}\widehat{b}_m\widetilde{T}_\varepsilon^m\widehat{a}_k\widetilde{T}_\varepsilon^{-m}-\sum_{l\neq 0}\Big(\sum_{m+k=l}\widehat{b}_m\widetilde{T}_\varepsilon^m\widehat{a}_k\widetilde{T}_\varepsilon^{-m}\Big)\widetilde{T}_\varepsilon^l.$$

By construction, the coefficient of each power of $\widetilde{T}_\varepsilon$ in this expression is an operator of order -1. As a consequence, the sum of the first two terms on the right-hand side is of order -1, and so is each of the terms $L_{l,\varepsilon}=\sum_{m+k=l}\widehat{b}_m\widetilde{T}_\varepsilon^m\widehat{a}_k\widetilde{T}_\varepsilon^{-m}$, $l\neq 0$. Now we observe that, for fixed m, $\varepsilon\mapsto\widetilde{T}_\varepsilon^m\widehat{a}_k\widetilde{T}_\varepsilon^{-m}$

is bounded. Since we have only finitely many summands for each l, the sum of the first two terms is a bounded function of ε with values in operators of order -1, and so is each $L_{l,\varepsilon}$. Using the boundedness of $\widetilde{T}_\varepsilon$, we conclude that all the mappings $\varepsilon \mapsto L_{l,\varepsilon}\widetilde{T}_\varepsilon^l$ take values in operators of order -1 and are bounded. Then the rapid decay of the sequence $\{\widehat{b}_k\}$ shows that $\varepsilon \mapsto K_{1,\varepsilon}$ is a bounded mapping in the space of operators of order -1. The family $K_{2,\varepsilon}$ can be considered similarly. Hence the conditions of Hörmander's theorem are fulfilled and $\operatorname{ind} B_\varepsilon$ is constant. This completes the proof.

4. INDEX OF OPERATORS IN HILBERT BUNDLES

In this section, we prove Theorem 1.5 on the index of the operator $\mathcal{D}$ with operator-valued symbol.

1. Reduction to a pseudodifferential operator with homogeneous symbol

Let $\mathcal{D}'$ be a pseudodifferential operator, of order zero on $M_\mathbb{Z}$, whose symbol is equal to $\sigma(\mathcal{D})$ for $|(\xi,\tau)| = R$, where R is a sufficiently large positive number, and is a homogeneous function of degree zero in the covariables (ξ,τ). Explicitly, this symbol is defined by

$$\sigma(\mathcal{D}')(x,\xi,t,\tau) = \sigma(\mathcal{D})\big(x, \frac{R\xi}{|(\xi,\tau)|}, t, \frac{R\tau}{|(\xi,\tau)|}\big).$$

Lemma 4.1. *The following equality holds*:

$$\operatorname{ind} \mathcal{D} = \operatorname{ind} \mathcal{D}'. \tag{4.1}$$

Proof. Let Λ be a first-order operator with the symbol $(|\xi|^2 + \tau^2 + n^2)^{1/2}$. The index of this operator is obviously equal to zero. We obtain

$$\operatorname{ind} \mathcal{D} = \operatorname{ind}(\mathcal{D}\Lambda^{-1}) = p_![\sigma(\mathcal{D}\Lambda^{-1})] = p_![\sigma(\mathcal{D})] = p_![\sigma(\mathcal{D}')] = \operatorname{ind} \mathcal{D}',$$

where $p\colon T^*M_{\mathbb{Z}^N} \to pt$ is the projection and $[\sigma(\mathcal{D})] \in K^0(T^*M_\mathbb{Z})$ is the class of the symbol in K-theory. Here the first equality and from the fact that Λ has index zero, and the second and last equalities follow from the index formula in K-theory, see [9] or Theorem 3.89 in [11].

2. The topological index of operator-valued symbols

We prove the index formula for pseudodifferential operators with operator-valued symbols in the following class.

Definition 4.2. An element $\sigma \in C^\infty(S_R^*M_\mathbb{Z}, \operatorname{End}\mathcal{E})$ is a *symbol* if its derivatives

$$\frac{\partial\sigma}{\partial\xi}, \frac{\partial\sigma}{\partial t}, \frac{\partial\sigma}{\partial\tau}\colon l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z}) \tag{4.2}$$

are operators of order -1 in the scale of spaces $l^2(\mathbb{Z},\mu_s)$ defined using the weight $\mu_s(n) = (1+n^2)^{s/2}$. A symbol is said to be *elliptic* if it is invertible at each point of $S_R^*M_\mathbb{Z}$.

The symbol of the operator $\mathcal{D}'$ is an example of a symbol of this kind. Note also that, due to the assumptions on the derivatives, these are symbols in the sense of Luke [9], i.e., they have compact variation in the covariables.

We shall now study the topological index of Definition 1.4 for symbols of this class. The following lemma establishes essential properties of the integrand.

Lemma 4.3. *Let $\omega \in \Lambda^{2k}(M)$ be a closed $\mathbb{Z}$-invariant form of degree $2k$. Then*

(1) *the expression*

$$\operatorname{tr}\left[(\sigma^{-1}d\sigma)^{2j-1}\omega\right] \in \Lambda^{2j-1+2k}(S_R^*M_\mathbb{Z}) \tag{4.3}$$

is a well-defined smooth closed form if $2j-1+2k > \dim M + 1$;

(2) *the cohomology class of* (4.3) *is invariant with respect to homotopies of the elliptic symbol σ if $2j-1+2k > \dim M + 2$.*

Proof. Write $m=\dim M$ for simplicity.

(a) Since ω is $\mathbb{Z}$-invariant, it follows that we can consider ω as a smooth form on $S^*_R M_{\mathbb{Z}}$. It suffices to prove the statement in local coordinates, which we denote by x,ξ,t,τ. Decompose the form $(\sigma^{-1}d\sigma)^{2j-1}$ into a sum of monomials,

$$(\sigma^{-1}d\sigma)^{2j-1}=\sum_{\alpha,\beta,\gamma,\delta} f_{\alpha\beta\gamma\delta}(x,\xi,t,\tau)dx^\alpha dt^\beta d\xi^\gamma d\tau^\delta, \tag{4.4}$$

where the summation ranges over the multi-indices $\alpha,\beta,\gamma,\delta$. Denote $\mu=|\beta|+|\gamma|+|\delta|$. Then $|\alpha|+\mu=2j-1$ and $|\alpha|+2k\leqslant m$. This inequality means simply that a nonzero monomial can have at most n differentials dx. It follows that $\mu\geqslant 2k-m+2j-1$. By (4.2), this implies that the product $(\sigma^{-1}d\sigma)^{2j-1}\omega$ is an operator of order $-\mu$. Therefore, if $2k-m+2j-1>1$, then this product is a form which ranges in trace-class operators, and the form (4.3) is well defined. The fact that the form is closed holds because the even form ω commutes with $\sigma^{-1}d\sigma$ follows from the last two equalities in the chain

$$\begin{aligned} d\big(\operatorname{tr}((\sigma^{-1}d\sigma)^{2j-1}\omega)\big)&=(2j-1)\operatorname{tr}\big(d(\sigma^{-1}d\sigma)(\sigma^{-1}d\sigma)^{2j-2}\omega\big)=-(2j-1)\operatorname{tr}((\sigma^{-1}d\sigma)^{2j}\omega)\\ &=(2j-1)\operatorname{tr}((\sigma^{-1}d\sigma)^{2j-1}\omega(\sigma^{-1}d\sigma))=(2j-1)\operatorname{tr}((\sigma^{-1}d\sigma)^{2j}\omega). \end{aligned}$$

(b) Let $\sigma=\sigma_\varepsilon$ be a homotopy with parameter $\varepsilon\in[0,1]$ of invertible elements. Then the corresponding forms $\operatorname{tr}((\sigma^{-1}d\sigma)^{2j-1}\omega)$ define the same cohomology class. This follows from the transgression formula

$$(\frac{\partial}{\partial\varepsilon})\operatorname{tr}\big((\sigma^{-1}d\sigma)^{2j-1}\omega\big)=(2j-1)d\operatorname{tr}\big(\sigma^{-1}(\frac{\partial\sigma}{\partial\varepsilon})(\sigma^{-1}d\sigma)^{2j-2}\omega\big),$$

which holds for $2k+2j-1>m+2$.

The integral of expression (4.3) over $S^*_R M_{\mathbb{Z}}$ can be nonzero only if $\beta+|\gamma|+\delta\geqslant m+1$ in (4.4), and hence $2j-1\geqslant m+1$. Moreover, if $m>1$, then $m+2<2m+1$, and we necessarily have $k>0$ if $2j-1\leqslant m+2$. We thus conclude, from Lemma 4.3, that the following assertion holds.

Corollary 4.4. *The topological index is a well-defined homotopy invariant if* $\dim M>1$.

2. Index theorem

We next establish a cohomological index formula in the spirit of Rozenblum [12, 13] for our situation.

Theorem 4.5. *The Fredholm index of an elliptic pseudodifferential operator*

$$\widehat{\sigma}\colon L^2(M_{\mathbb{Z}},\mathcal{E})\to L^2(M_{\mathbb{Z}},\mathcal{E})$$

with symbol σ is equal to

$$\operatorname{ind}\widehat{\sigma}=\operatorname{ind}_t\widehat{\sigma}, \tag{4.5}$$

where the topological index $\operatorname{ind}_t\widehat{\sigma}$ is defined in (1.12).

Proof. Step 1. Let us reduce the symbol to the form of the identity plus a compact-valued symbol. To this end, we choose a covector $\tau_0\neq 0$. Let σ' be the elliptic symbol

$$\sigma'(x,\xi,t,\tau)=\sigma(x,\xi,t,\tau)\sigma^{-1}(x,0,t,\tau_0). \tag{4.6}$$

We have

$$\operatorname{ind}\widehat{\sigma}=\operatorname{ind}\widehat{\sigma'},\quad \operatorname{ind}_t\widehat{\sigma}=\operatorname{ind}_t\widehat{\sigma'}. \tag{4.7}$$

Both equalities follow from the logarithmic property of the (topological) index and the fact that $\sigma(x,0,t,\tau_0)$ is the symbol of a multiplication operator.

Step 2. Let us now replace the symbol (4.6), which we shall denote for brevity by σ, by a symbol of the form of the identity plus a finite rank symbol. To this end, we take a sequence $\{P_N\}$, $N\geqslant 1$ of smooth projections $P_N\colon\mathcal{E}\longrightarrow\mathcal{E}$ of ranks tending to infinity as $N\to\infty$ and $\operatorname{Im}P_N\subset\operatorname{Im}P_{N+1}$. Below we take this family to be equal to the orthogonal projection to the subspace in $l^2(\mathbb{Z})$ spanned by the vectors $e_{-N+1},e_{-N+2},\dots,e_{N-1},e_N$, $\cos\varphi(t)e_{-N}+\sin\varphi(t)e_{N+1}$, where $\{e_j\}$ is the standard base in $l^2(\mathbb{Z})$ and $\varphi(t)$ is a smooth monotone function such that $\varphi(t)=0$ if $t<1/3$ and $\varphi(t)=\pi/2$ if $t>2/3$. Obviously, these vectors are pairwise orthogonal for each t.

Lemma 4.6. *If N is large enough, then σ is linearly homotopic in the class of elliptic symbols to the symbol*
$$\sigma_0 = (1 - P_N) + P_N \sigma P_N, \tag{4.8}$$
which is equal to the direct sum of the identity and an elliptic symbol $P_N \sigma P_N$ of finite rank.

Proof. 1. A direct computation shows that P_N is continuous in $m \in S^*_R M_\mathbb{Z}$ uniformly for $N \geqslant 1$. Hence, the symbol $\sigma_{0,N} = (1 - P_N) + P_N \sigma P_N$ is also uniformly continuous in m and N. Given $\varepsilon > 0$ (a specific choice of this number will be given below), by uniform continuity there exists a finite subset $\{m_j\} \subset S^*_R M_\mathbb{Z}$ with the following property: given $m \in S^*_R M_\mathbb{Z}$, we have
$$\|\sigma(m) - \sigma(m_j)\| \leqslant \varepsilon, \quad \|\sigma_{0,N}(m) - \sigma_{0,N}(m_j)\| \leqslant \varepsilon \tag{4.9}$$
for some m_j and all N.

2. Since $\sigma(m_j)$ is equal to the sum of an identity operator and a compact operator, there exists an N_0 such that for all $N \geqslant N_0$ we have
$$\|\sigma(m_j) - \sigma_{0,N}(m_j)\| \leqslant \varepsilon \quad \text{for all } m_j. \tag{4.10}$$

3. Now (4.9) and (4.10) imply the estimate
$$\|\sigma(m) - \sigma_{0,N}(m)\| \leqslant 2\varepsilon + \|\sigma(m_j) - \sigma_{0,N}(m_j)\| \leqslant 2\varepsilon + \varepsilon = 3\varepsilon \tag{4.11}$$
for all m and $N \geqslant N_0$. Let us now choose $\varepsilon = \left(3\|\sigma^{-1}\|\right)^{-1}$. Then (4.11) implies the inequality
$$\|\sigma - \sigma_{0,N}\| \leqslant \|\sigma^{-1}\|^{-1}.$$
Hence, the linear homotopy $(1 - \alpha)\sigma + \alpha\sigma_{0,N}$, $\alpha \in [0, 1]$, consists of invertible elements (this is proved by using Neumann series).

The proof of the lemma is complete.

Step 3. To prove the theorem, it suffices to check (using the Atiyah–Singer index formula) that (4.5) holds for the symbol σ_0 (see (4.8)). For the analytic index, we have
$$\operatorname{ind} \widehat{\sigma}_0 = \operatorname{ind} P_N \widehat{\sigma} P_N, \tag{4.12}$$
since σ_0 is the direct sum of the identity and the symbol $P_N \sigma P_N$. Further, the index of the operator
$$P_N \widehat{\sigma} P_N \colon L^2(M_\mathbb{Z}, P_N \mathcal{E}) \longrightarrow L^2(M_\mathbb{Z}, P_N \mathcal{E})$$
acting on the sections of the finite-dimensional vector bundle $P_N \mathcal{E}$ is calculated using the Atiyah–Singer formula. Hence, we see that, to prove the theorem, it remains to show that the topological index of $\widehat{\sigma}_0$ is equal to the Atiyah–Singer topological index of $P_N \widehat{\sigma} P_N$. To this end, we choose, in $\mathcal{E}$, the connection:
$$\nabla_0 = P_N d P_N + (1 - P_N) d (1 - P_N) \colon \Lambda(S^*_R M_\mathbb{Z}, \mathcal{E}) \longrightarrow \Lambda(S^*_R M_\mathbb{Z}, \mathcal{E}),$$
which is the direct sum of flat connections in the bundles $P_N \mathcal{E}$ and $(1 - P_N)\mathcal{E}$ (the connections are flat, since P_N depends only on t and d is flat).

Lemma 4.7. *The topological index of the symbol σ_0 can be computed using either of the connections $\nabla = d$ or ∇_0. More precisely, for each $j \geqslant 1$, we have the equality*
$$\int_{S^*_R M_\mathbb{Z}} \operatorname{tr}\left[(\sigma_0^{-1} d\sigma_0)^{2j-1} \operatorname{Td}(T^*_\mathbb{C} M)\right] = \int_{S^*_R M_\mathbb{Z}} \operatorname{tr}\left[(\sigma_0^{-1} \nabla_0 \sigma_0)^{2j-1} \operatorname{Td}(T^*_\mathbb{C} M)\right]. \tag{4.13}$$

Proof. The proof of this lemma is quite standard (see [6]). More precisely, first, one reduces the integrals (4.13) over the odd-dimensional manifold $S^*_R M_\mathbb{Z}$ to integrals of Chern character forms on the even-dimensional manifold $T^* M_\mathbb{Z}$. Second, the de Rham cohomology class of Chern character forms on $T^* M_\mathbb{Z}$ does not depend on the choice of the connection. Hence, the integrals on $T^* M_\mathbb{Z}$ (and hence on $S^*_R M_\mathbb{Z}$) are equal.

By this lemma, we can calculate the topological index of $\widehat{\sigma}_0$ using the connection ∇_0. Since this is a direct sum connection and the symbol is also a direct sum, we see that the topological index is equal to the sum of the topological index of the identity (which is zero) and the topological index of the operator $P_N\widehat{\sigma}P_N$,

$$\operatorname{ind}_t \widehat{\sigma}_0 = \operatorname{ind}_t(P_N\widehat{\sigma}P_N) = \sum_j C_j \int_{S^*_R M_{\mathbb{Z}}} \operatorname{tr}\left[(\sigma_N^{-1}\nabla_N\sigma_N)^{2j-1}\operatorname{Td}(T^*_{\mathbb{C}}M)\right], \tag{4.14}$$

where $\nabla_N = P_N dP_N$ is a flat connection in $P_N\mathcal{E}$ and $\sigma_N = P_N\sigma P_N$. However, the last expression coincides with the Atiyah–Singer topological index of the symbol σ_N (see [6] again). Hence, (4.12) and (4.14) give the desired equality $\operatorname{ind}\widehat{\sigma}_0 = \operatorname{ind}_t \widehat{\sigma}_0$. This completes the proof of Theorem 4.5.

Now the index theorem, Theorem 1.5, follows from equation (4.1) and Theorem 4.5.

APPENDIX. OPERATORS ON THE INFINITE CYLINDER

Symbol classes

We consider a class of pseudodifferential operators, on the cylinder $M\times\mathbb{R}$, similar to that introduced by Shubin in [14]. To this end, we cover $M\times\mathbb{R}$ with coordinate patches of the form $U_j\times\mathbb{R}$, where the $U_j\subseteq\mathbb{R}^{\dim M}$ are bounded coordinate charts for M. Denote the variables on $M\times\mathbb{R}$ by (x,t), the covariables by (ξ,τ), and say that $P\colon C^\infty(M\times\mathbb{R})\to C^\infty(M\times\mathbb{R})$ is an operator of order m on $M\times\mathbb{R}$ if its local symbols $a=a(x,\xi,t,\tau)$ with respect to these coordinates satisfy the estimates

$$D^\alpha_\xi D^\beta_\tau D^\gamma_x D^\delta_t a(x,\xi,t,\tau) = O\big((1+|\xi|^2+|t|^2+|\tau|^2)^{(m-|\alpha|-|\beta|-|\delta|)/2}\big).$$

These operators naturally act on the scale of Sobolev spaces $\mathcal{H}^s(M\times\mathbb{R})$ of all tempered distributions u on $M\times\mathbb{R}$ such that $(1+t^2+\Delta_x-\partial_t^2)^{s/2}u\in L^2(M\times\mathbb{R})$. For arbitrary s, an operator P of order m defines a Fredholm operator in $\mathcal{L}(\mathcal{H}^s(M\times\mathbb{R}),\mathcal{H}^{s-m}(M\times\mathbb{R}))$ if and only if it is elliptic, i.e., $p(x,\xi,t,\tau)$ is invertible for large $|(\xi,t,\tau)|$, and

$$p(x,\xi,t,\tau)^{-1} = O((1+|\xi|^2+|t|^2+|\tau|^2)^{-m/2}).$$

We have the subclass of all 'classical' operators whose symbols in these coordinates have an asymptotic expansion of the form $p\sim\sum p_{m-j}$, where p_{m-j} is homogeneous of degree $m-j$ in (ξ,t,τ) for large $|(\xi,t,\tau)|$, say, for $|(\xi,t,\tau)|\geqslant 1$. Clearly, a classical operator of order m is elliptic if and only if the principal symbol p_m is invertible for $|(\xi,t,\tau)|=1$.

Operators with shifts

We next consider operators of the form

$$\widetilde{D} = \sum D_k\widetilde{T}^k, \tag{5.1}$$

where the sum is finite, the D_k are operators of order m on the cylinder as described above, and $\widetilde{T}$ was defined in (1.3).

The conjugation $P\mapsto \widetilde{T}P\widetilde{T}^{-1}$ induces the action $p\mapsto p(\partial g(x,\xi),t,\tau)$ on the principal symbol. Following the approach of Antonevich and Lebedev, we give the following definition:

Definition 5.1. The *symbol* of the operator $\widetilde{D}$ is the operator

$$\sigma(\widetilde{D})(x,\xi,t,\tau) = \sum_k \sigma(D_k)(\partial g^n(x,\xi),t,\tau)\mathcal{T}^k : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z}), \tag{5.2}$$

where $\sigma(D_k)$ is the principal symbol of order m of D_k.

We say that $\widetilde{D}$ is elliptic if $\sigma(\widetilde{D})$ is invertible for all (x,ξ,t,τ) with $|\xi|^2+t^2+\tau^2=1$.

Remark 5.2. One can also define the symbol of the operator (5.1) following the approach of [1, 2] as an element of the crossed product $\mathcal{B}\rtimes\mathbb{Z}$, where $\mathcal{B}$ is the algebra of continuous functions on the cospheres $\{|\xi|^2+t^2+\tau^2=1\}$. Then the invertibility of the symbol in the crossed product algebra is equivalent to the fact that, for all (x,ξ,t,τ) with $|\xi|^2+t^2+\tau^2=1$, the operator (5.2) is invertible on $l^2(\mathbb{Z})$ (see Theorem 21.2 of [1]).

Theorem 5.3. *If $\widetilde{D}$ is elliptic, then it is a Fredholm operator in $\mathcal{L}(\mathcal{H}^s(M\times\mathbb{R}), \mathcal{H}^{s-m}(M\times\mathbb{R}))$ for each s. The index, kernel, and cokernel of $\widetilde{D}$ do not depend on s.*

Proof. On the spaces $\mathcal{H}^s(M\times\mathbb{R})$, the shift $\widetilde{T}$ is unitary modulo a compact operator, and the principal symbol is independent of s. The Fredholm property therefore follows from the approach in [1, 2]. The independence of the index of s is proved using the fact that D almost commutes with order reduction operators. The independence of the kernel and the cokernel of s follows from the fact that, on one hand, the dimension of the kernel is a nonincreasing function of s, while the dimension of the cokernel is nondecreasing and, on the other hand, the difference of these dimensions is equal to the index, which is constant in s.

Remark 5.4. Instead of the shift $\widetilde{T}$, we could have considered the shift $\widetilde{T}_\varepsilon$, $0\leqslant\varepsilon\leqslant 1$ induced by the diffeomorphism $(x,t)\mapsto(g(x),t+\varepsilon)$ on $M\times\mathbb{R}$. Replacing $\widetilde{T}$ in (5.1) by $\widetilde{T}_\varepsilon$ leads to the same symbol in (5.2), since the action on the principal symbol is the same.

Remark 5.5. Theorem 5.3 could also be proved using a reduction to a compact manifold, as it was done for Connes operators on the real line in [10], Sec. 12.2.1.

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